SOLUTION FOR ACOUSTIC PRESSURE ON VIBRATING RECTANGULAR PLATE USING VARIATIONAL PRINCIPLE AND RAYLEIGH-RITZ METHOD

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1. INTRODUCTION

The prediction of acoustic radiation from vibrating bodies is an important problem in engineering. An example relevant for civil engineering might be the vibrations of structural elements of a highway bridge that lead to annoying low-frequency noise emission. Analytical solutions for acoustic pressure can only be obtained for simple geometries conforming to separable coordinate systems, e.g. a sphere or an infinite cylinder. In other cases, variational principles can be used to obtain optimal results. The variational principles have been used so far to calculate the pressure on vibrating circular discs or axisymmetric shells of finite length. However their application for rectangular plates has not been studied yet. Therefore, the objective of this paper is to derive a variational principle-based solution for acoustic pressure on a vibrating rectangular plate using the Rayleigh-Ritz method.

2. VARIATIONAL PRINCIPLE AND RAYLEIGH-RITZ SOLUTION

A variational principle for acoustic pressure was derived by Wu et al.[1]. It states that quantity $J[p]$ is stationary to small variations in pressure. Quantity $J[p]$ is defined as

$$\begin{aligned} J[p] = \frac{1}{2}\iint_S \iint_{S'} \left\{ k^2 \mathbf{n}(\mathbf{x}_S) \bullet \mathbf{n}(\mathbf{x}'_S) p(\mathbf{x}_S) p(\mathbf{x}'_S) - [\mathbf{n}(\mathbf{x}_S) \times \nabla p(\mathbf{x}_S)] \bullet [\mathbf{n}(\mathbf{x}'_S) \times \nabla' p(\mathbf{x}'_S)] \right\} G(\mathbf{x}_S \mid \mathbf{x}'_S) dS dS' \\ - 4\pi i \omega \rho \iint_S p(\mathbf{x}_S) \left\{ v_n(\mathbf{x}_S) + \frac{1}{4\pi} \lim_{\varepsilon \to 0} \iint_{S'} v_n(\mathbf{x}'_S) \times \mathbf{n}(\mathbf{x}_S) \bullet \nabla G[\mathbf{x}_S + \varepsilon \mathbf{n}(\mathbf{x}_S) \mid \mathbf{x}'_S] dS' \right\} dS \end{aligned} \tag{1}$$

where $p(\mathbf{x}_S)$ is pressure at location $\mathbf{x}_S$, $v_n(\mathbf{x}_S)$ is surface normal velocity component, and $\mathbf{n}(\mathbf{x}_S)$ is outward unit normal vector. The parameters are: k is the wave number, ω is the frequency of surface oscillations, ρ is the fluid density, and c is the speed of sound. $G(\mathbf{x}_S \mid \mathbf{x}'_S)$ is the free space Green function defined as

$$G(\mathbf{x}_S \mid \mathbf{x}'_S) = \exp\left(ik\left|\mathbf{x}_S - \mathbf{x}'_S\right|\right) / \left|\mathbf{x}_S - \mathbf{x}'_S\right| \tag{2}$$

It should be noted that evaluation of $J[p]$ in Eq. (1) requires double surface integration.

For a rectangular plate of width $2a$, height $2b$, aspect ratio $t=b/a$ and negligible thickness, Eq. (1) can be formulated in the following non-dimensional form:

$$\begin{aligned} J_0[p_0] = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \left\{ (ka)^2 p_0(\xi,\eta) p_0(\xi',\eta') - \left[\frac{\partial p_0(\xi,\eta)}{\partial \xi}\frac{\partial p_0(\xi',\eta')}{\partial \xi'} + \frac{1}{t^2}\frac{\partial p_0(\xi,\eta)}{\partial \eta}\frac{\partial p_0(\xi',\eta')}{\partial \eta'} \right] \right\} \\ \times G_0(\xi,\eta,\xi',\eta') d\xi d\eta d\xi' d\eta' - 2\pi i ka \int_{-1}^{1}\int_{-1}^{1} p_0(\xi,\eta) v_{n0}(\xi,\eta) d\xi d\eta \end{aligned} \tag{3}$$

$$G_0(\xi,\eta,\xi',\eta') = \exp\left[ika\sqrt{(\xi-\xi')^2 + t^2(\eta-\eta')^2}\right] \Big/ \sqrt{(\xi-\xi')^2 + t^2(\eta-\eta')^2} \tag{4}$$

where the non-dimensional quantities are defined as follow: $J_0=J/4\rho^2c^4a$, $p_0=p/\rho c^2$, $v_{n0}= v_n/c$, $\xi=x/a$, and $\eta=y/b$. Eq. (3) permits application of various approximation techniques such as FEM, in which the pressure trial functions are defined only over a limited part of the plate, or the Rayleigh-Ritz method, in which they span the entire plate. This latter approach is used in this research.

In this study, steady-state rigid body vibrations of the plate with the velocity changing as $v_{n0}(\xi,\eta,t)= v_{n0}\exp(-i\omega t)$ are considered. One of the possible sets of trial functions that satisfy the geometric boundary conditions of the problem (i.e. vanishing pressure at the edges of the plate) are cosinusoidal functions. The pressure distribution may be then assumed as:

$$p_0(\xi,\eta) = \sum_{m,n=1}^{N} P_{mn} \cos\frac{(2m-1)\pi\xi}{2} \cos\frac{(2n-1)\pi\eta}{2} \tag{5}$$

Using the assumed pressure distribution of Eq. (5), the formula for quantity J_0 can be evaluated as

$$J_0 = \frac{1}{2}\sum_{m,n,p,q=1}^{N} A_{mnpq} P_{mn} P_{pq} - \sum_{m,n}^{N} B_{mn} P_{mn} \tag{6}$$

Terms A_{mnpq} in Eq. (6) require computationally expensive double surface integration. The change of variables: $u=(\xi-\xi')/2$, $v=\xi'$, $u'=(\eta-\eta')/2$, $v'=\eta'$, reduces computations to a single surface integral:

$$A_{mnpq} = 2\int_0^1\int_0^1 \left\{ (ka)^2 F_{mp}(u) F_{nq}(u') - \frac{\pi^2}{4}\left[(2m-1)(2p-1) H_{mp}(u) F_{nq}(u') + \frac{(2n-1)(2q-1)}{t^2} F_{mp}(u) H_{nq}(u') \right] \right\} \times \frac{\exp\left[2ika\sqrt{u^2+t^2u'^2}\right]}{\sqrt{u^2+t^2u'^2}} du du' \tag{7}$$

where functions $F_{mn}(u)$ and $H_{mn}(u)$ are defined as follow:

$$F_{mn}(u) = \frac{(-1)^{m+n}\left[(2m-1)\sin(2n-1)\pi u - (2n-1)\sin(2m-1)\pi u\right]}{(m-n)(m+n-1)\pi}, \quad m \neq n \tag{8}$$

$$F_{mm}(u) = 2\left[(1-u)\cos(2m-1)\pi u + \frac{\sin(2m-1)\pi u}{(2m-1)\pi}\right] \tag{9}$$

$$H_{mn}(u) = \frac{(-1)^{m+n}\left[(2n-1)\sin(2n-1)\pi u - (2m-1)\sin(2m-1)\pi u\right]}{(m-n)(m+n-1)\pi}, \quad m \neq n \tag{10}$$

$$H_{mm}(u) = 2\left[(1-u)\cos(2m-1)\pi u - \frac{\sin(2m-1)\pi u}{(2m-1)\pi}\right] \tag{11}$$

Finally, the unknown pressure coefficients, P_{mn}, can be found through solution of the matrix equation

$$\sum_{p,q=1}^{N} A_{mnpq} P_{pq} = B_{mn}, \quad m,n = 1,\ldots,N \tag{12}$$

3. NUMERICAL SIMULATIONS

The analyzed plate is square (t=1). Fig. 1 shows the results of dimensionless surface pressure, p_0, versus distance from the center point along the line $0<\xi<1$, $\eta=\tilde{0}$, for various ka values. The results were obtained using N=10 trial functions. It has been noticed that a small number of trial functions is sufficient to obtain a good convergence in-the-mean, but the pointwise convergence is much slower. This suggests that a careful convergence study should be conducted, including various sets of trial functions.

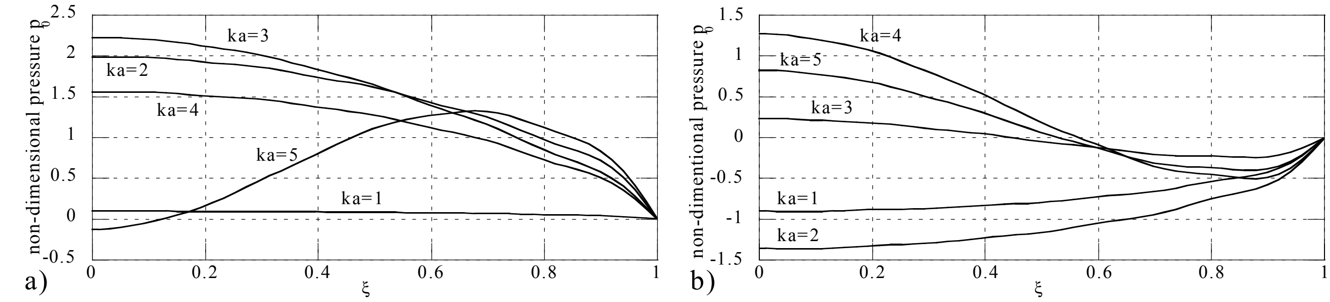

Fig. 1 Dimensionless surface pressure versus distance from the center point: a) real part; b) imaginary part.

4. CONCLUSIONS

In this paper the problem of acoustic pressure on vibrating rectangular plate is studied. The variational principle that relates surface pressure and velocity is used. The stationary solution for the pressure on vibrating rigid plate is obtained using the Rayleigh-Ritz method and cosinusoidal trial functions. A good convergence in-the-mean can be obtained using a small number of trial functions, however the pointwise convergence is much slower. A more careful convergence study should be conducted.

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REFERENCE

1) Wu X. F., Pierce A. D., Ginsberg J. H.: Variational Method for Computing Surface Acoustic Pressure on Vibrating Bodies, Applied to Transversely Oscillating Discs. IEEE Journal of Oceanic Engineering, Vol. OE-12, No. 2, 1987, pp. 412-418.