Green's Function for the Bending Problem of Dissimilar Semi-infinite Plates with a Debonded Elliptical Hole at Interface

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1. Introduction

The study on the behavior of a hole at the bimaterial interface is of practically importance in providing a good understanding of the debonding phenomenon and for determining appropriate factors that affect the mechanical properties of composite elements of structures. Earlier studies on the interfacial crack primarily dealt with local stress fields and energy release rates associated with pre-existing cracks. In recently years, the effects of voids and inclusions spaced at an interface attract many researchers. However, there is still a lack of analytical work on fundamental solutions or Green's functions for this type of problems. And very few studies are concerned with the bending of thin plate. The objective in the present study is to derive the Green's function of a point dislocation in a composite plate with debondings emanating from an elliptical hole.

2. Statement of the Problem

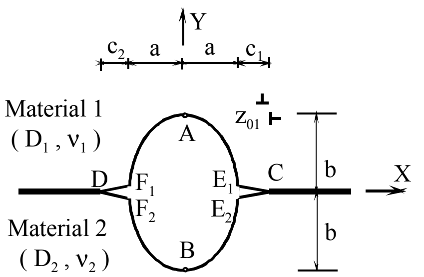

Fig.1 Bonded dissimilar semi-infinite plates

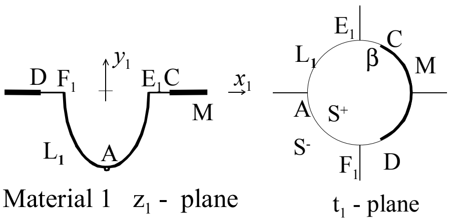

Fig.2 z_1-plane and the unit circle

The bending problem of elasticity is considered for a point dislocation occurring in one of the dissimilar semi-infinite plates (*XY*-plane) as shown in Fig.1, where an elliptical hole is spaced at the interface with debondings emanating at both horizontal vertices. Herein, the point dislocation in the plate is defined as [1]:

$$L=\left\{\frac{\partial W}{\partial X}+i\frac{\partial W}{\partial Y}\right\}_L \tag{1}$$

where $\{\ \}_L$ denotes the increment of the braced expression when moving around the dislocation point in the counter-clockwise direction. The semi-axes of the ellipse are denoted with *a* (the *x*-axis) and *b* (the *y*-axis); the z_1 and z_2 planes are occupied with materials 1 and 2, respectively. As specified in Fig.2 (shows the z_1-plane and the related unit circle), the mapping function [2] that maps a half-plane with a semi-ellipse to a unit circle can be expressed as

$$z_j=\frac{E_o}{1-t_j}+\sum_{k=1}^{N}\frac{E_k}{\zeta_k-t_j}+E_c \qquad (j=1,2) \tag{2}$$

where E_0, E_k and E_c are complex constants, poles ζ_k are located outside the unit circle in the t_1 and t_2 planes. N=28. We denote the bonded arcs with M, and the unbonded arcs with L_j (see Fig.2). This boundary divides the t_j-planes into two regions S_j^+ and S_j^-.

To proceed with the solution of the described problem, we decompose the original formulation into two parts A and B. In part A, we assume that a point dislocation exists in material 1. Part B is a complementary one, and is stated in such a way that the original boundary condition is satisfied under consideration of stresses and displacements induced by part A. This yields, for $\varphi_1(t_1)$ and $\psi_1(t_1)$:

$$\varphi_1(t_1)=\varphi_1^A(t_1)+\varphi_1^B(t_1), \qquad \psi_1(t_1)=\psi_1^A(t_1)+\psi_1^B(t_1) \tag{3}$$

where $\varphi_1^B(t_1)$ and $\psi_1^B(t_1)$ are the holomorphic functions defined in S^+. $\varphi_2(t_2)$ and $\psi_2(t_2)$ are originally holomorphic in S^+.

3. Analysis

Based on the stress boundary condition and the continuity of rotations along the bonded interface, we can decouple the stress functions $\varphi_1(t_1)$ and $\varphi_2(t_2)$ in the formulation by using the principle of analytical continuation. The problem of obtaining $\varphi_1^B(t_1)$ is reduced to a Riemann-Hilbert problem as follows:

$$\varphi_1^{B+}(\sigma)-\varphi_1^{B-}(\sigma)=0 \qquad \text{on } L_1 \tag{4a}$$

$$\varphi_1^{B+}(\sigma)+P_1\cdot\varphi_1^{B-}(\sigma)=Q_1\cdot g_{1A}(\sigma)+(Q_1-1-P_1)\varphi_1^A(\sigma) \qquad \text{on M} \tag{4b}$$

with

$$P_1=\frac{\kappa_1\kappa_2D_2(1-\nu_2)-\kappa_1D_1(1-\nu_1)}{\kappa_1\kappa_2D_1(1-\nu_1)-\kappa_2D_2(1-\nu_2)},\quad Q_1=\frac{\kappa_1D_1(1-\nu_1)(1-\kappa_2)}{\kappa_2D_2(1-\nu_2)-\kappa_1\kappa_2D_1(1-\nu_1)} \tag{4c}$$

where ν_j is the Poisson's ratio of the materials, $\kappa_j=-(3+\nu_j)/(1-\nu_j)$, D_j is the flexural rigidity of the plate. The function $\varphi_1^A(t_1)$ in (4b) can be obtained in problem A (the procedure is omitted here). Function $g_{1A}(\sigma)$ satisfies the equation

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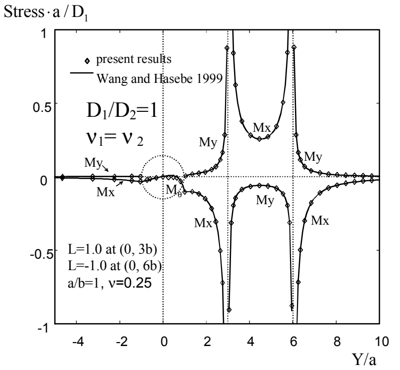

Fig.3 Stress distribution along Y-axis

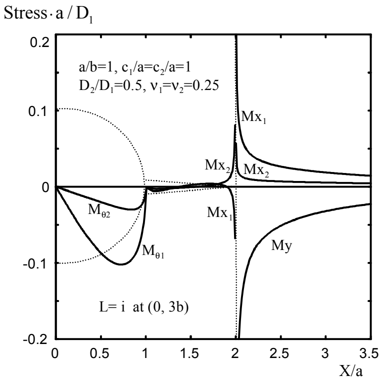

Fig.4 Stress distribution along X-axis

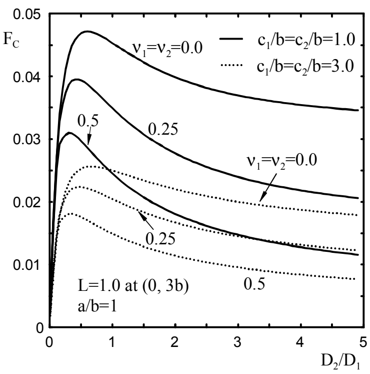

Fig.5 Stress intensity of debonding at point C versus D_2/D_1

$$g_{1A}(t_1)=\varphi_1^B(t_1)-\frac{\kappa_2 D_2(1-\nu_2)}{\kappa_1 D_1(1-\nu_1)}\overline{\varphi_2(1/\bar{t}_1)} \tag{5}$$

It can be assumed to be a sum of fractional expressions that are irregular in S^+ and S^- respectively, as follows:

$$g_{1A}(t_1)=\sum_k \frac{c_{1k}}{\xi_{1k}-t_1}+\sum_k \frac{d_{1k}}{\eta_{1k}-t_1} \tag{6}$$

where ξ_{1k}, η_{1k}, c_{1k} and d_{1k} are complex constants to be determined, and $|\xi_{1k}|>1$ and $|\eta_{1k}|<1$. Then the solution of the problem stated with (4a,b) can be given as [3]:

$$\varphi_1^B(t_1)=\frac{Q_1\cdot\chi_1(t_1)}{2\pi i}\int_M \frac{g_{1A}(\sigma)}{\chi_1^+(\sigma)(\sigma-t_1)}d\sigma$$

$$+\frac{(Q_1-1-P_1)\cdot\chi_1(t_1)}{2\pi i}\int_M \frac{\varphi_1^A(\sigma)}{\chi_1^+(\sigma)(\sigma-t_1)}d\sigma+\chi_1(t_1)\cdot R_1(t_1) \tag{7}$$

where $R_1(t_1)$ is a rational function, and Plemelj function $\chi_1(t_1)$ is given by

$$\chi_1(t_1)=(t_1-\alpha)^{m_1}(t_1-\beta)^{1-m_1}, \qquad m_1=0.5+i\cdot\frac{\ln|P_1|}{2\pi} \tag{8}$$

with α and β being the points on the unit circle (t_1 and t_2 planes) corresponding to the tips of debonding. Since $\chi_1(t_1)$ is a multi-valued function, the branch $\chi_1(t_1)/t_1\to 1$ is chosen for $t_1\to\infty$. The value of $R_1(t_1)$ in (7) can be determined by using the regular characteristics of $\psi_1^B(t_1)$ in S_1^+ and the formulae of analytical continuation as

$$R_1(t_1)=\frac{1}{\kappa_1}\sum_{k=1}^{N}\frac{E_k\cdot\overline{A_{1k}}}{\overline{\omega'(\zeta_k')}\cdot\chi_1(\zeta_k)(t_1-\zeta_k)} \tag{9}$$

where $A_{1k}\equiv\varphi_1^{B'}(\zeta_k)$. Similarly, the formulations for the function $\varphi_2(t_2)$ can be obtained. Then substituting the expressions of $\varphi_1^B(t_1)$ and $\varphi_2(t_2)$ into (5), the coefficients in (6) are determined as follows:

$$\xi_{1k}=\zeta_k, \qquad \eta_{1k}=\zeta_k' \tag{10a}$$

$$c_{1k}=-\frac{B_k\overline{A_{1k}}}{\kappa_1}, \qquad d_{1k}=-\frac{\kappa_2 D_2(1-\nu_2)}{\kappa_1 D_1(1-\nu_1)}\frac{\zeta_k'^2\cdot\overline{B}_k A_{2k}}{\kappa_2} \tag{10b}$$

where $B_k\equiv E_k/\overline{\omega'(\zeta_k')}$. The complex constants A_{1k} and A_{2k} can be determined by using the relations: $A_{1k}=\varphi_1^{B'}(\zeta_k')$ and $A_{2k}=\varphi_2'(\zeta_k')$, (k =1, 2,…, N), and by solving the resulting system of 4N simultaneous linear algebraic equations in the real and imaginary parts of A_{1k} and A_{2k}. Then both stress functions $\varphi_1(t_1)$ and $\varphi_2(t_2)$ are determined.

4. Numerical results and Discussion

When two opposite dislocations $L=1$ and $L=-1$ occur at points (0, 3b) and (0, 6b), respectively, the distribution of moments M_x and M_y on the *Y*-axis and M_θ on the hole's boundary are depicted in Fig.3, where we assumed the materials 1 and 2 are the same with $\nu=0.25$, a/b=1. It is noted that our results in this case agree very well with those obtained in [1]. Fig.4 shows the stress distribution along the X-axis and the hole's boundary for point dislocation: L=1 initiated at point (0, 3b). The following parameters are chosen for the actual computations: $D_2/D_1=0.5$, $\nu_1=\nu_2=0.25$, a/b=1, $c_1/a=c_2/a=1$. The effect of the rigidity ratio D_2/D_1 on the stress intensity of debonding tip C (F_C) was also explored in this paper as shown in Fig.5.

The Green's function obtained in this study is potentially suitable for analyzing a variety of thin plate problems. It can be used as a kernel for boundary integral representations in BEM analysis, where it can notably simplify the procedure of the standard boundary integral equations. Moreover, since a crack can be treated as a dislocation distribution, this solution can also be used to study the interaction problem between a crack and bimaterial interface.

5. References:

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