Green's function of a pair of heat source and sink in a mixed boundary value problem

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1. Introduction

In two-dimensional thermoelasticity, the Green's functions of heat source, which embeds in an infinite plane with a hole, inclusion or debonded inclusion, have found extensive applications in thermal stress analysis, as well as in boundary element method as a fundamental solution. Any newly developed Green's function, especially that derived with respect to a complicated geometry, will be of significant importance in applications. Thus the analytic works concerning point heat sources have received considerable attention. Fukui et al., [1, 2] derived the solutions of circular hole and inclusion problem under a heat source. Zhang and Hasebe [3] solved the problem of adiabatic crack under a heat source. Recently, Yoshikawa and Hasebe [4] studied the problem of an elliptic rigid inclusion or a hole under a pair of heat source and sink. In [5, 6], they studied the problems of arbitrarily shaped hole and rigid inclusion under a pair of heat source and sink. However, those references generally treated only the external force or displacement boundary value problem. Few references on the mixed boundary value problem are available.

In this paper, the authors situate the discussion on the Green's function of a mixed boundary value problem, which models an infinite plane containing both a debonded rigid arbitrarily shaped inclusion and a pair of heat source and sink.

2. Formulation

Consider a thermoelastic problem of a debonded rigid arbitrary shaped inclusion under a pair of heat source and sink with intensity M. N debondings are assumed to occur on the interface between the rigid inclusion and the elastic matrix. Also the boundary of the inclusion is assumed either adiabatic or isothermal.

Using a rational mapping function [7, 5]

$$z=\omega(\zeta)=E_0\zeta+\sum_{k=1}^{n}\frac{E_k}{\zeta_k-\zeta}+E_{-1} \tag{1}$$

where E_0, E_k, E_{-1} and ζ_k are constants, the physical plane outside the arbitrarily shaped inclusion in the z-plane can be mapped onto exterior of a unit circle in the ζ-plane. The bonded and debonded inclusion boundaries are mapped onto the L_i and S_i segments on the unit circle, respectively. α_i and β_i represent the coordinates of both ends of S_i. The temperature function for the problem can be given as [6]:

$$\mathrm{Y}(\zeta)=-\frac{\mathrm{M}}{2\pi\mathrm{k}}\left\{\log\left(\frac{\zeta-\zeta_a}{\zeta-\zeta_b}\right)+\Gamma\log\left(\frac{\zeta-\zeta'_a}{\zeta-\zeta'_b}\right)\right\}+\mathrm{const.} \tag{2}$$

where $\zeta'_a\equiv 1/\overline{\zeta_a}$, $\zeta'_b\equiv 1/\overline{\zeta_b}$, ζ_a and ζ_b represent the coordinate of the points of heat source and sink on the ζ-plane, k denotes the conductivity of the material, and the value of the constant term can be determined by the temperature at a standard point. By employing the complex stress functions $\varphi(\zeta)$ and $\psi(\zeta)$, the boundary conditions on the unit circle can be written as

$$\varphi(\sigma)\{1-\delta(\sigma)(1+\kappa)\}+\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi'(\sigma)}+\overline{\psi(\sigma)}=f(\sigma) \tag{3}$$

where

$$f(\sigma)=\begin{cases}C_i & \text{On } L_i\\ 2\mathrm{G}\alpha'\int\mathrm{Y}(\sigma)\omega'(\sigma)d\sigma\equiv H(\sigma) & \text{On } S_i\end{cases} \tag{4}$$

C_i denotes the resultant force applied to the boundaries L_1 to L_i, $\delta(\sigma)=0$ for σ on the segments L_i, whereas $\delta(\sigma)=1$ for σ on the segments S_i. $\alpha'=\alpha(1+\nu)$, $\kappa=3-4\nu$ for plane strain, while $\alpha'=\alpha$, $\kappa=(3-\nu)/(1+\nu)$ for plane stress. ν, α and G represent the Poisson's ratio, the linear thermal expansion coefficient and the shear modulus of the material, respectively.

The stress functions can be broken down into two parts

$$\varphi(\zeta)=\varphi_1(\zeta)+\varphi_2(\zeta) \tag{5a}$$

$$\psi(\zeta)=\psi_1(\zeta)+\psi_2(\zeta) \tag{5b}$$

The first two components denote the Green's function of a traction-free arbitrarily shaped hole in an infinite plane under the pair of heat source and sink, which can be given [6]

$$\varphi_1(\zeta)=\varphi_{11}(\zeta)+\varphi_{12}(\zeta) \tag{6a}$$

$$\psi_1(\zeta)=-\overline{\varphi_1(1/\overline{\zeta})}-\frac{\overline{\omega(1/\overline{\zeta})}}{\omega'(\zeta)}\varphi'_1(\zeta) \tag{6b}$$

where

$$\varphi_{11}(\zeta)=\frac{\alpha\mathrm{MGR}}{4\pi\mathrm{k}}\left[\{\omega(\zeta)-\omega(\zeta_a)\}\{\log(\zeta-\zeta_a)\}-\{\omega(\zeta)-\omega(\zeta_b)\}\{\log(\zeta-\zeta_b)\}+A\log(\zeta)\right] \tag{6c}$$

$$\varphi_{12}(\zeta)=\frac{\alpha\mathrm{MGR}}{4\pi\mathrm{k}}\left[\{\omega(\zeta)-\omega(\zeta_a)\}\log\left(\frac{\zeta}{\zeta-\zeta'_a}\right)-\{\omega(\zeta)-\omega(\zeta_b)\}\log\left(\frac{\zeta}{\zeta-\zeta'_b}\right)-\sum_{k=1}^{n}\frac{E_kB_k}{\zeta_k-\zeta}\right]-\sum_{k=1}^{n}\frac{E_k}{\overline{\omega'(\zeta'_k)}}\frac{g_{1k}}{\zeta_k-\zeta} \tag{6d}$$

$$A=-E_0\Gamma(\zeta'_a-\zeta'_b)+\sum_{k=1}^{n}E_k\left(\frac{1}{\zeta_k-\zeta_a}-\frac{1}{\zeta_k-\zeta_b}\right) \tag{6e}$$

$$B_k=\log\frac{\zeta_k-\zeta_a}{\zeta_k-\zeta_b}-\log\frac{\zeta'_a}{\zeta'_b}+\frac{1}{\overline{\omega'(\zeta'_k)}}\left\{\overline{A}\zeta_k+\sum_{j=1}^{n}\frac{\overline{E_k}}{(\overline{\zeta_j}-\overline{\zeta'_k})}\left(\frac{1}{(\overline{\zeta_j}-\overline{\zeta_a})}-\frac{1}{(\overline{\zeta_j}-\overline{\zeta_b})}\right)\right\} \tag{6f}$$

$$g_{1k}=\overline{\varphi'_{12}(\zeta'_k)} \tag{6g}$$

where R=(1+ν)/(1−ν) for plane strain and R=(1+ν) for plane stress. The value of the unknown constant g_{1k} can be determined by solving a system of linear algebraic equations derived from (6g). The second functions in (5) are unknown, however, they are holomorphic outside the unit circle.

Introducing a Plemelj function as

$$\chi(\zeta)=\prod_{j=1}^{N}(\zeta-\alpha_j)^m(\zeta-\beta_j)^{1-m} \tag{7}$$

where

$$m = 0.5 - i\ln\kappa/2\pi \tag{8}$$

Substituting (5) and (6) into (3), a relation is obtained

$$\varphi_2(\sigma)\{1-\delta(\sigma)(1+\kappa)\} + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\varphi_2'(\sigma)} + \overline{\psi_2(\sigma)} = f(\sigma) + \varphi_1(\sigma)\delta(\sigma)(1+\kappa) \tag{9}$$

Multiplying both sides of (9) with a factor $d\sigma/[2\pi i(\sigma-\zeta)\chi^{-}(\sigma)]$, and carrying out the Cauchy integration along the unit circle in clockwise direction, a closed form solution of $\varphi_2(\zeta)$ can be derived

$$\begin{aligned}\varphi_2(\zeta) = \frac{\alpha\mathrm{MGR}}{4\pi\mathrm{k}}\Bigg[& \{\omega(\zeta_a)+\Gamma\omega(\zeta_a')\}\log(\frac{\zeta}{\zeta-\zeta_a'}) \\ & -\{\omega(\zeta_b)+\Gamma\omega(\zeta_b')\}\log(\frac{\zeta}{\zeta-\zeta_b'}) \\ & +(1+\Gamma)\omega(\zeta)\log(\frac{\zeta-\zeta_a'}{\zeta-\zeta_b'}) - \sum_{k=1}^{n}C_{1k}\log(\frac{\zeta}{\zeta-\zeta_k}) + \sum_{k=1}^{n}\frac{E_kB_k}{\zeta_k-\zeta} \\ & +\chi(\zeta)\Bigg\{\int_0^{\zeta_a'}\frac{\{\omega(\zeta_a)+\Gamma\omega(\zeta_a')\}}{\chi(\sigma)(\sigma-\zeta)}d\sigma - \int_0^{\zeta_b'}\frac{\{\omega(\zeta_b)+\Gamma\omega(\zeta_b')\}}{\chi(\sigma)(\sigma-\zeta)}d\sigma \\ & -\sum_{k=1}^{n}\int_0^{\zeta_k}\frac{C_{1k}}{\chi(\sigma)(\sigma-\zeta)}d\sigma + \int_{\zeta_a'}^{\zeta_b'}\frac{(1+\Gamma)\omega(\sigma)}{\chi(\sigma)(\sigma-\zeta)}d\sigma + \sum_{k=1}^{n}\frac{C_{2k}}{\zeta_k-\zeta}\Bigg\}\Bigg] \\ & +\sum_{k=1}^{n}\frac{E_kg_{1k}}{\overline{\omega'(\zeta_k')}(\zeta_k-\zeta)} \\ & +\chi(\zeta)\left[\frac{1}{2\pi i}\sum_{j=1}^{N}\int_{L_i}\frac{C_jd\sigma}{\chi(\sigma)(\sigma-\zeta)} - \sum_{k=1}^{n}\frac{E_k(g_{1k}+g_{2k})}{\chi(\zeta_k)\overline{\omega'(\zeta_k')}(\zeta_k-\zeta)}\right]\end{aligned} \tag{10a}$$

where

$$C_{1k} = E_k(\frac{1}{\zeta_k-\zeta_a} + \frac{\Gamma}{\zeta_k-\zeta_a'} - \frac{1}{\zeta_k-\zeta_b} - \frac{\Gamma}{\zeta_k-\zeta_b'}) \tag{10b}$$

$$C_{2k} = \frac{E_k}{\chi(\zeta_k)}\left\{(1+\Gamma)\log\frac{\zeta_k-\zeta_b'}{\zeta_k-\zeta_a'} - B_k\right\} \tag{10c}$$

$$g_{2k} = \overline{\varphi_2'(\zeta_k')} \tag{10d}$$

the unknown constants g_{2k} and C_j can be obtained by solving a system of linear algebraic equations derived from both the relation (10d) and the holomorphic condition of function $\varphi_2(\zeta)$ at infinity.

Another stress function $\psi(\zeta)$ can be derived by analytic continuation on the traction-free boundary as

$$\psi(\zeta) = -\overline{\varphi(1/\overline{\zeta})} - \frac{\overline{\omega(1/\overline{\zeta})}}{\omega'(\zeta)}\varphi'(\zeta) \tag{11}$$

3. Stress distribution

Numerical example of stress distribution is considered for the problem of the pair of heat source and sink accompanying with a rectangular rigid inclusion. Two debondings are assumed to generate symmetrically on the interface between the inclusion and the matrix. The Poisson's ratio is taken to be 0.3, and the plane strain state is considered. For the case when the heat source and sink locate at points ($2a$, 0) and ($-2a$, 0), respectively, on the x-axis, dimensionless stress distribution along the x-axis and inclusion boundary (in upper half-plane) under adiabatic condition is shown in Fig.1. It can be seen that the normal and tangential stresses on the debonded boundaries are zero, which indicate that the traction-free condition is satisfied. All the stress components have singularities at debonding tips, and have concentrations at the corners of the inclusion. The stress components of σ_x and σ_y have singularities at the points of heat source and sink. The tangential stress on the x-axis is zero due to the symmetry.

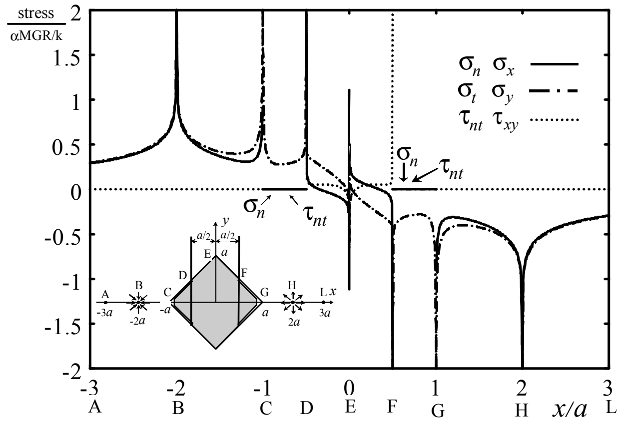

Figure 1. Stress distribution along the inclusion boundary and the x-axis.

4. Conclusion

A closed form solution, the Green's function, of a heat source and sink located at any two points in an infinite plane containing an either adiabatic or isothermal debonded arbitrarily shaped rigid inclusion, is derived.

The basic point in the derivation procedure of the Green's functions for the mixed boundary value problem is the use of the Green's function of the external force boundary value problem and the Cauchy integration method.

References

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