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Stress Intensity Factors of a Crack Outside an Arbitrary Shaped Rigid Inclusion in Thin Plate Bending Problem

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1. INTRODUCTION

The characterization of crack in a plate has been a subject of considerable research interest for several decades due to its importance in evaluation of structural safety. When an inclusion is present in a matrix, it will affect the stress distribution depending on its hardness, shape and location. Considering both of them, the studies on the problem of the interaction between cracks and inclusions are more practically important in providing a good understanding of mechanical behaviors of structures with defects. Hitherto most of the analytical solutions on the inclusion-crack problem are often restricted to very simple geometries, and considered the effect in plane, which neglects the bending mode deformation of crack. The present paper is concerned with the interaction of a line crack and an arbitrary shaped rigid inclusion from plate bending point of view.

2. ANALYSIS OF THE PROBLEM

The considered problem is shown in Fig. 1, which is an infinite plate with a rigid inclusion and a line crack under the action of the remote uniform bending moments M_x^∞, M_y^∞ and torque M_{xy}^∞. The crack is considered to satisfy the free surface conditions. To simplify the analytical procedures, we use the principle of superposition to convert the original problem into two particular inclusion problems I and II. Problem I can be described as the infinite plate with only the inclusion subjected to the remote loadings; while problem II is the infinite plate with the inclusion subjected to a continuous distribution of dislocations along line AB, in which the induced traction along the line are opposite to those obtained in problem I. In this paper, the infinite region outside the square inclusion is transformed into the infinite region (S^+) outside the unit circle (in ζ plane) by the following rational mapping function:

$$z = \omega(\zeta) = E_o\zeta + \sum_{k=1}^{2n}\frac{E_k}{\zeta_k - \zeta} + E_{-1} \qquad (1)$$

where E_0, E_k and E_{-1} are constants, poles ζ_k are located in the unit circle in ζ plane. Constant n is selected as 24 in this paper. The detail of formulation (1) has been stated in reference [1], which is therefore omitted here.

2.1 Solution of problem I

As mentioned above, the remote loadings M_x^∞, M_y^∞ and M_{xy}^∞ are applied to the infinite plate with the rigid inclusion in problem I. In ζ plane, the whole stress functions can be expressed in the form of a summation of two parts as follows[2]:

$$\varphi(\zeta) = \varphi_1(\zeta) + \varphi_2(\zeta),\ \psi(\zeta) = \psi_1(\zeta) + \psi_2(\zeta) \qquad (2)$$

in which $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ are obtained from the uniform field:

$$\varphi_1(\zeta) = P\cdot\omega(\zeta),\qquad \psi_1(\zeta) = Q\cdot\omega(\zeta) \qquad (3a)$$

where $$P = -\frac{(M_x^\infty + M_y^\infty)}{4D(1+\nu)},\ Q = \frac{(M_y^\infty - M_x^\infty + 2iM_{xy}^\infty)}{2D(1-\nu)} \qquad (3b)$$

$\varphi_2(\zeta)$ and $\psi_2(\zeta)$ in (2) represent the regular functions defined in S^+, which denotes the outside region of the unit circle in ζ plane. Then, through the regularity of $\psi_2(\zeta)$ [3], which is obtained from the free boundary condition [2,4], $\varphi_2(\zeta)$ can be determined as:

$$\varphi_2(\zeta) = \sum_{k=1}^{2n}\frac{B_k\cdot\overline{A}_{1k}}{n(\zeta-\zeta_k)} - \sum_{k=1}^{2n}\frac{(n+1)PE_k}{n(\zeta_k-\zeta)} - \frac{Q\cdot\overline{E}_o}{n\zeta}\quad(\zeta\in S^+) \qquad (4a)$$

where $B_k = \dfrac{E_k}{\omega'(\zeta_k')}$, $A_{1k} = \varphi_2'(\zeta_k')$, $\zeta_k' = 1/\overline{\zeta}_k$ (k=1, 2, ... 2n) (4b)

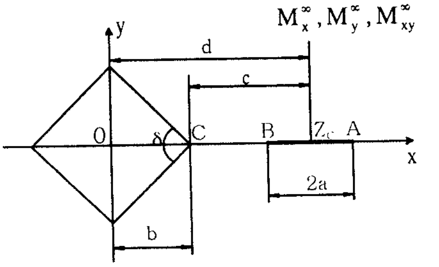

Fig. 1 Infinite plate with a rigid inclusion and a line crack under remote bendings

in which undetermined values A_{1k} can be evaluated by solving a series of algebraic equations obtained by letting $\zeta = \zeta_k'$ (k =1, 2, ..., 2n) individually in its first derivative $\varphi_2'(\zeta)$.

2.2 Solution of problem II

In problem II, a continuous distribution of dislocations is assumed along line AB in the infinite plate outside the rigid inclusion. This can be obtained by the superposition of the fundamental solutions of the point dislocations. Then the Green's function for a point dislocation needs to be determined. Expression (2) is also used in this section, but here, the two parts of stress functions represent a singular and a regular part, respectively. We have known the stress potentials $\varphi_1(\zeta)$ and $\psi_1(\zeta)$ for a point dislocation located in an infinite plate as:

$$\varphi_1(\zeta) = \frac{L}{2\pi i}\frac{1-\nu}{4}\log(\zeta-\zeta_o),\quad \psi_1(\zeta) = \frac{\overline{L}}{2\pi i}\frac{3+\nu}{4}\log(\zeta-\zeta_o) - \frac{L}{2\pi i}\frac{1-\nu}{4}\frac{\overline{\omega(\zeta_o)}}{\omega'(\zeta_o)(\zeta-\zeta_o)} \qquad (5)$$

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where ζ_0 denotes the dislocation point in ζ plane corresponding to z_o in z plane. Similar to the previous investigation in problem I, using the regularity of $\psi_2(\zeta)$, $\varphi_2(\zeta)$ can be obtained as follows:

$$\varphi_2(\zeta)=\sum_{k=1}^{2n}\left[-\frac{\bar{L}}{2\pi i}\frac{1-\nu}{4}\frac{1}{(\bar{\zeta}'_k-\bar{\zeta}_o)}+\overline{A_{2k}}\right]\cdot\frac{B_k}{n(\zeta-\zeta_k)}-\frac{\bar{L}}{2\pi i}\frac{1-\nu}{4n}\cdot\frac{\omega(\zeta_o)-\omega(\zeta'_o)}{\overline{\omega'(\zeta_o)}}\cdot\frac{\zeta_o'^2}{\zeta'_o-\zeta}+\frac{L}{2\pi i}\frac{3+\nu}{4n}\log(\frac{1}{\zeta}-\bar{\zeta}_o)\quad(\zeta\in S^+)\quad(6)$$

where $\zeta'_o\equiv 1/\bar{\zeta}_o,\quad A_{2k}\equiv\varphi'_2(\zeta'_k)$ (7)

Then undetermined values A_{2k} in (6) can be determined from the algebraic equations that are obtained by letting $\zeta=\zeta'_k$ (k =1, 2,..., 2n) individually in the first derivative $\varphi'_2(\zeta)$.

Using the Green's function obtained previously, we can formulate problem II in an integration form. The modified dislocation density can be expressed as follows:

$$H(t)=\sqrt{a^2-t^2}\frac{d}{dt}L(t),\qquad |t-z_c|<a \qquad (8)$$

where t and z_c denote the points along the crack and the center of crack AB, respectively. Based on the crack surface condition (remain traction free superposed by problem I, and the single-valuedness of the deflection angle), we can obtain the following equations by means of the Gauss-Chebyshev method:

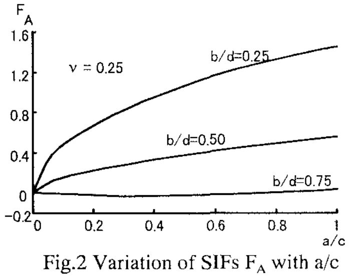

Fig.2 Variation of SIFs F_A with a/c

$$\sum_{j=1}^{M}\frac{\pi}{M}\left\{H_n(t_j)\left[m_n(t_j,s_k)+i\cdot p_n(t_j,s_k)\right]+H_\tau(t_j)\left[m_\tau(t_j,s_k)+i\cdot p_\tau(t_j,s_k)\right]\right\}+m_I(s_k)+i\cdot p_I(s_k)=0\ \ (k=1,\ldots,M-1)\qquad(9a)$$

$$\sum_{j=1}^{M}\frac{\pi}{M}\left\{H_n(t_j)+i\cdot H_\tau(t_j)\right\}=0\qquad(9b)$$

where $t_j=z_0+a\cos\left(\frac{(2j-1)\pi}{2M}\right)$ (j = 1, 2, ..., M) (10a)

$s_k=z_0+a\cos\left(\frac{k\pi}{M}\right)$ (k = 1, 2, ..., M-1) (10b)

$m_j(t,s)$ and $p_j(t,s)$ (j=n, τ) are the bending moments and the bending forces obtained from the unit point dislocation in j direction to the crack surface, respectively. $m_I(s_k)$ and $p_I(s_k)$ are the bending moments and the bending forces at the points on the crack AB, which are obtained in problem I. From the above 2M equations with the real and imaginary parts separated, unknowns $H_n(t_j)$ and $H_\tau(t_j)$ can be obtained.

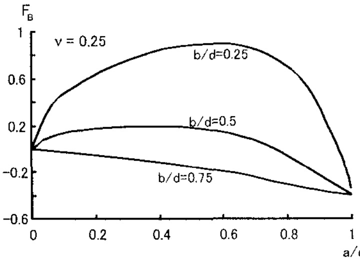

Fig. 3 Variation of SIFs F_B with a/c

3. NUMERICAL RESULTS AND DISCUSSION

Several numerical examples are carried out for the problem described in Fig.1, in which the remote loading is typically taken as M_y^∞. For convenience, the nondimensional stress intensity factors at crack tips A, B of opening mode are used in this paper:

$$\begin{Bmatrix}F_A\\F_B\end{Bmatrix}=\frac{\nu+3}{\nu+1}\frac{1}{M_y^\infty\sqrt{b}}\begin{Bmatrix}K_A\\K_B\end{Bmatrix}\qquad(11)$$

where b is the scale of the rigid inclusion as shown in Fig.1, and shearing mode is vanished because of the symmetry of the configuration and the applied load. The calculated nondimensional stress intensity factors of crack tips A and B are shown in Figs. 2 and 3. We can find that, the influence of the rigid inclusion on crack tip B becomes larger when the crack approach it. In this case, the stress intensity factor of crack tip B gradually decreases, even to a negative value, while crack tip A is not influenced so much by the rigid inclusion.

It should be noted that, the rational mapping function technique used in this paper is valuable to arbitrary shaped boundary in principle. It provides a powerful tool in treating problems with complicated configuration. And in solving both problems I and II, the analytical continuation over the fixed boundary based on the fundamental theory of the complex variable function, can directly determine the stress functions without involving in the integral calculation. This is a particular advantage of the proposed approach.

5. REFERENCES

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