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A Fundamental Solution of Thin Plate Bending Problem Containing a Point Dislocation

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1. Introduction

Because of the intimating relationship between crack and dislocation, it is convenient to make use of a fundamental solution of a dislocation to solve the crack problems. Then how to construct the solution of a point dislocation becomes the basic problem. It is evident that the complex stress functions approach developed by Muskhelishvili et al. is a valid tool to deal with it[1]. Hasebe et al. also have done a lot advances in this field[2,3]. In this paper, considering the thin plate bending problem with a hole edge crack, we define a kind of point dislocation which is concerned with the deflection of plate, and its solution can naturally be adopted as the Green's function of the distributed dislocation with respect to crack problem. From the single valued condition satisfied by the stress state, a general formula of stress functions where point dislocation is expressed is obtained, and then by utilizing a rational mapping function which makes the hole edge crack transform into a regular configuration, and through the analytical continuation over the boundary, we obtain its analytical solution. Finally, the stress distribution and stress intensity factors at the crack tip are presented.

2. General formula of plate under point dislocation

In the thin plate bending problem, we define the dislocation of a point as $L=\left\{\frac{\partial w}{\partial x}+i\frac{\partial w}{\partial y}\right\}_L$, where $\{\ \}_L$ denotes the increase in value of the expression in the braced brackets when moving round the dislocation point thoroughly in the counter-clockwise direction, and w is the deflection of thin plate. Then based on the single valuedness of the stress state, the stress functions can be expressed in terms of the point dislocation applied to the contour:

$$\varphi(z)=\frac{1}{2\pi i}\frac{(1-\nu)L}{4}\ln z+\varphi*(z),\qquad \psi(z)=\frac{1}{2\pi i}\frac{(3+\nu)\bar{L}}{4}\ln z+\psi*(z)$$

Where $\varphi*(z)$ and $\psi*(z)$ are the holomorphic single-valued functions. The stress boundary condition is:

$$n\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)}=\frac{1}{D(1-\nu)}\int_0^s\left[m(s)+i\int_0^s p(s)ds\right](dx+idy)+iCz+C_1$$

where C is a real constant and C_1 is a complex constant. D and ν is the rigidity and the Poisson's ratio of the plate, respectively.

3. Derivation of the stress functions

If there is a point dislocation in an infinite plate which contains a hole edge crack, through conformal transformation performed by the rational mapping function $z=\omega(\zeta)=E_o\zeta+\sum_{k=1}^{m}\frac{E_k}{\zeta_k-\zeta}+E_{-1}$, the stress functions can be transformed into ζ plane. The functions of the dislocation are:

$$\varphi_1(\zeta)=\frac{L}{2\pi i}\frac{1-\nu}{4}\ln(\zeta-\zeta_o),\qquad \psi_1(\zeta)=\frac{\bar{L}}{2\pi i}\frac{3+\nu}{4}\ln(\zeta-\zeta_o)-\frac{L}{2\pi i}\frac{1-\nu}{4}\frac{\overline{\omega(\zeta_o)}}{\omega'(\zeta_o)(\zeta-\zeta_o)}$$

By the analytical continuation obtained from the free boundary condition, we can represent the function $\psi(\zeta)$ by the function $\varphi(\zeta)$ in the whole region:

$$\psi(\zeta)=-n\overline{\varphi(1/\bar{\zeta})}-\frac{\overline{\omega(1/\bar{\zeta})}}{\omega'(\zeta)}\varphi'(\zeta),\quad \zeta\in S^+\qquad n=-\frac{3+\nu}{1-\nu}$$

Here the functions $\varphi(\zeta)$ and $\psi(\zeta)$ can be divided into 2 parts which represent the irregular and holomorphic single valued terms, respectively.

$$\varphi(\zeta)=\varphi_1(\zeta)+\varphi_2(\zeta),\qquad \psi(\zeta)=\psi_1(\zeta)+\psi_2(\zeta)$$

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Noticing the regularity of the function $\psi_2(\zeta)$ in S^+, the function $\varphi_2(\zeta)$ can be obtained by making the irregular terms in $\psi_2(\zeta)$ be naught. Then we get:

$$\varphi_2'(\zeta)=\frac{1}{n}\sum_{k=1}^{m}\left[-\frac{\bar{L}}{2\pi i}\frac{1-\nu}{4}\frac{1}{\overline{(\zeta'_k-\zeta_o)}}+\overline{\varphi_2'(\zeta'_k)}\right]\cdot\frac{-E_k}{\overline{\omega'(\zeta'_k)}(\zeta-\zeta_k)^2}$$

$$-\frac{\bar{L}}{2\pi i}\frac{1-\nu}{4n}\cdot\frac{\omega(\zeta_o)-\omega(\zeta'_o)}{\overline{\omega'(\zeta_o)}}\cdot\frac{\zeta_o'^2}{(\zeta'_o-\zeta)^2}+\frac{L}{2\pi i}\frac{3+\nu}{4n}\left(\frac{\bar{\zeta}_o}{\bar{\zeta}_o\zeta-1}-\frac{1}{\zeta}\right),\qquad \text{where } \zeta'_k=1/\bar{\zeta}_k,\ \zeta'_o=1/\bar{\zeta}_o$$

From this formula, let ζ be ζ'_k (k=1,...m) individually, the constants $\varphi_2'(\zeta'_k)$ can be determined by these linear equations. The stress intensity factor at the crack tip can be calculated by[2]:

$$K_C=K_{1C}-iK_{2C}=-2D(1+\nu)\frac{\varphi'(\zeta_t)}{\sqrt{\omega''(\zeta_t)}}$$

where ζ_t is the value of ζ on the unit circle corresponding to the crack tip. K_{1C} is the stress intensity factor for bending and K_{2C} is the one for twisting.

4. Results

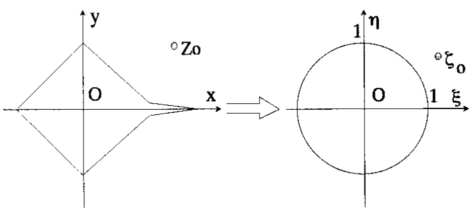

Fig.1 Conformal transformation

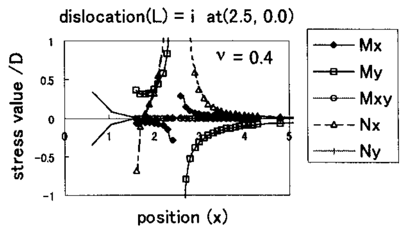

Fig.2 Stress distribution along x axis

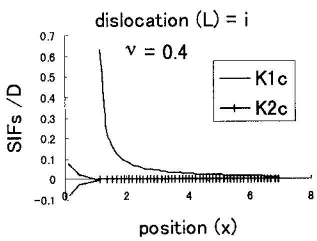

Fig.3 SIFs when dislocation moves along x axis

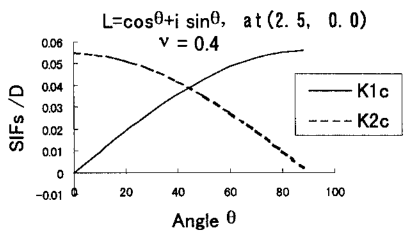

Fig.4 SIFs when dislocation rotates from $0°$ to $90°$

For the configuration shown in Fig. 1, the stress distribution and the stress intensity factors are presented in Figs.2, 3, 4 respectively when a point dislocation moves along x axis and its direction rotates 90 degrees.

5. Conclusions

For a point dislocation, which is defined as the difference of the deflection angle existing in an infinite plate with any form of hole edge crack, the closed form solution is obtained. The rational mapping function enable us to analyze complicated configuration, and make the analytical solution feasible. To dislocation problems, analytical continuation of stress function over the boundary avoids calculating integration, is more convenient and valid than other usual methods. This solution can be used as the Green's function of the distributed dislocation with respect to crack problems. So it provides a useful fundamental solution for thin plate bending problems with crack.

6. References

[1]Muskhelishvili, N. I., 1963, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff
[2]N. Hasebe and S. Inohara, 1980, '*Stress Analysis of a Semi-Infinite Plate with an Oblique Edge Crack*', Ingenieur-Archiv, v49, pp.51-62.
[3]N. Hasebe and Y. Z. Chen, 1996, '*Stress intensity solution for the interaction between a hole edge crack and a line crack*', International Journal of Fracture, v77, pp.351-366.