# Analytical solution of consolidation of double-layered ground with vertical drains

Saga University SM.JSCE Xiao-Wu TANG
Saga University M.JSCE Katustada ONITUSKA

## INTROUCTION

Vertical drains are usually installed in subsoil consisting of several layers. By now, only the problem for double-layered ground with ideal vertical drains has be solved by analytical method (Tang, 1996). If well resistance and smear action are considered, the problem solved only by numerical methods (Onoue,1988; Amirebrahimi,1993).The analytical solution for double-layered ground with vertical drains is presented.

## MATHEMATICAL MODELLING

$$\frac{k_{si}}{m_{vi}\gamma_w}\left(\frac{1}{r}\frac{\partial u_{si}}{\partial r}+\frac{\partial^2 u_{si}}{\partial r^2}\right)+\frac{k_{vi}}{m_{vi}\gamma_w}\frac{\partial^2 \bar{u}_i}{\partial z^2}=\frac{\partial \bar{u}_i}{\partial t} \quad (1)$$

$$\frac{k_{hi}}{m_{vi}\gamma_w}\left(\frac{1}{r}\frac{\partial u_{ni}}{\partial r}+\frac{\partial^2 u_{ni}}{\partial r^2}\right)+\frac{k_{vi}}{m_{vi}\gamma_w}\frac{\partial^2 \bar{u}_i}{\partial z^2}=\frac{\partial \bar{u}_i}{\partial t} \quad (2)$$

$$\frac{\partial^2 u_{wi}}{\partial z^2}=-\frac{2}{r_w}\frac{k_{si}}{k_w}\left(\frac{\partial u_{si}}{\partial r}\right)\bigg|_{r=r_w} \qquad i=1,2 \quad (3)$$

$$\bar{u}_i=\frac{1}{\pi(r_e^2-r_w^2)}\left(\int_{r_w}^{r_s}2\pi u_{si}dr+\int_{r_s}^{r_e}2\pi u_{ni}dr\right) \quad (4)$$

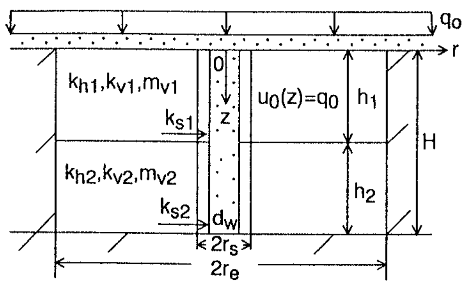

Fig. 1. Analysis scheme

Continuous conditions: $z=h_1$ , $u_{w1}=u_{w2}$ , $\bar{u}_1=\bar{u}_2$ , $k_{v1}\frac{\partial \bar{u}_1}{\partial z}=k_{v2}\frac{\partial \bar{u}_2}{\partial z}$ , $\frac{\partial u_{w1}}{\partial z}=\frac{\partial u_{w2}}{\partial z}$

## SOLUTION OF SYSTEM

The solutions of $u_{w1}, u_{w2}, \bar{u}_1$ and $\bar{u}_2$ are:

$$u_{w1}=\sum_{m=0}^{\infty}A_m w_{m1}e^{-\beta_m t} \ (5) \qquad u_{w2}=\sum_{m=0}^{\infty}A_m w_{m2}e^{-\beta_m t} \ (6) \qquad \bar{u}_1=\sum_{m=0}^{\infty}A_m g_{m1}e^{-\beta_m t} \ (7) \qquad \bar{u}_2=\sum_{m=0}^{\infty}A_m g_{m2}e^{-\beta_m t} \ (8)$$

where: $w_{m1}=a_{m1}\sin\left(\lambda_{m1}\frac{z}{H}\right)+c_{m1}\sinh\left(\xi_{m1}\frac{z}{H}\right)$, $w_{m2}=b_{m2}\cos\left[\lambda_{m2}\left(1-\frac{z}{H}\right)\right]+d_{m2}\cosh\left[\xi_{m2}\left(1-\frac{z}{H}\right)\right]$

$$g_{m1}=a_{m1}\left(1+\frac{1}{\varphi_1}\lambda_{m1}^2\right)\sin\left(\lambda_{m1}\frac{z}{H}\right)+c_{m1}\left(1-\frac{1}{\varphi_1}\xi_{m1}^2\right)\sinh\left(\xi_{m1}\frac{z}{H}\right)$$

$$g_{m2}=b_{m2}\left(1+\frac{1}{\varphi_2}\lambda_{m2}^2\right)\cos\left[\lambda_{m2}\left(1-\frac{z}{H}\right)\right]+d_{m2}\left(1-\frac{1}{\varphi_2}\xi_{m1}^2\right)\cosh\left[\xi_{m2}\left(1-\frac{z}{H}\right)\right]$$

$$\varphi_i=(n^2-1)\frac{2}{F_i}\frac{k_{hi}}{k_w}\frac{H^2}{r_e^2}, \quad \lambda_{mi}=H\sqrt{\frac{\Xi_{mi}+\sqrt{\Xi_{mi}^2-4\Lambda_i\Theta_{mi}}}{2\Lambda_i}}, \quad \xi_{mi}=H\sqrt{\frac{-\Xi_{mi}+\sqrt{\Xi_{mi}^2-4\Lambda_i\Theta_{mi}}}{2\Lambda_i}}$$

$$\Lambda_i=\frac{k_{vi}}{m_{vi}\gamma_w}, \quad \Xi_{mi}=-\left\{\frac{k_{hi}}{m_{vi}\gamma_w}\frac{2}{r_e^2F_i}\left[1+\frac{k_v}{k_w}(n^2-1)\right]-\beta_m\right\}, \quad \Theta_{mi}=-(n^2-1)\frac{2}{r_e^2F_i}\frac{k_{hi}}{k_w}\beta_m$$

$$F_i=\left(\ln\frac{n}{s}+\frac{k_{hi}}{k_{si}}\ln s-\frac{3}{4}\right)\frac{n^2}{n^2-1}+\frac{s^2}{n^2-1}\left(1-\frac{k_{hi}}{k_{si}}\right)\left(1-\frac{s^2}{4n^2}\right)+\frac{k_{hi}}{k_{si}}\frac{1}{n^2-1}\left(1-\frac{1}{4n^2}\right), \quad n=\frac{r_e}{r_w}, \quad s=\frac{r_s}{r_w}$$

Substituting Eq.(5), Eq.(6) , Eq.(7) and Eq.(8) into continuous conditions, and changing to matric form:

Key word: consolidation, vertical drain, anisotropic consolidation, stratification, preloading
Department of Civil Engineering, Saga University, Honjo-machi 1, Saga 840, Tel:0952-288690, Fax:0952-288699

$$\mathbf{S}\mathbf{X}^{T}=0 \quad (9) \qquad \text{where: } \mathbf{S}=\begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix} \quad (10) \qquad \mathbf{X}=\begin{bmatrix} a_{m1} & c_{m1} & b_{m2} & d_{m2} \end{bmatrix} \quad (11)$$

$$s_{11}=\sin(\lambda_{m1}\rho)\ ,\ s_{12}=\sinh(\xi_{m1}\rho)\ ,\qquad s_{13}=-\cos[\lambda_{m2}(1-\rho)]\ ,\qquad s_{14}=-\cosh[\xi_{m2}(1-\rho)]$$

$$s_{21}=\left(1+\frac{1}{\varphi_1}\lambda_{m1}^2\right)\sin(\lambda_{m1}\rho)\ ,\qquad s_{22}=\left(1-\frac{1}{\varphi_1}\xi_{m1}^2\right)\sinh(\xi_{m1}\rho),$$

$$s_{23}=-\left(1+\frac{1}{\varphi_2}\lambda_{m2}^2\right)\cos\left[\lambda_{m2}(1-\rho)\right]\ ,\qquad s_{24}=-\left(1-\frac{1}{\varphi_2}\xi_{m2}^2\right)\cosh\left[\xi_{m2}(1-\rho)\right]\ ,$$

$$s_{31}=\lambda_{m1}\cos(\lambda_{m1}\rho)\ ,\ s_{32}=\xi_{m1}\cosh(\xi_{m1}\rho),\ s_{33}=-\lambda_{m2}\sin\left[\lambda_{m2}(1-\rho)\right]\ ,\ s_{34}=\xi_{m2}\sinh\left[\xi_{m2}(1-\rho)\right]$$

$$s_{41}=k_{v1}\left[\left(1+\frac{1}{\varphi_1}\lambda_{m1}^2\right)\lambda_{m1}\cos(\lambda_{m1}\rho)\right]\ ,\qquad s_{42}=k_{v1}\left[\left(1-\frac{1}{\varphi_1}\xi_{m1}^2\right)\xi_{m1}\cosh(\xi_{m1}\rho)\right]$$

$$s_{43}=-k_{v2}\left\{\left(1+\frac{1}{\varphi_2}\lambda_{m2}^2\right)\lambda_{m2}\sin\left[\lambda_{m2}(1-\rho)\right]\right\},\ s_{44}=k_{v2}\left\{\left(1-\frac{1}{\varphi_2}\xi_{m2}^2\right)\xi_{m2}\sinh\left[\xi_{m2}(1-\rho)\right]\right\},\ \rho=\frac{h_1}{H}$$

In order to get unequal zero solutions of $\mathbf{X}$, we order:

$$\mathbf{S}=0 \qquad (12)$$

so, $\beta_m$ is obtained.And then, substituting $\beta_m$ into Eq.(9), and order $a_{m1}=1$, $c_{m1}$, $b_{m2}$ and $d_{m2}$ are obtained.

By the virtue of orthogonality of system, we get:

$$A_m=\frac{m_{v1}\int_0^h u_0 g_{m1}dz+m_{v2}\int_h^H u_0 g_{m2}dz}{m_{v1}\int_0^h g_{m1}^2dz+m_{v2}\int_h^H g_{m2}^2dz} \qquad (13)$$

The average consolidation degree at any depth is:

$$U_i(z)=1-\frac{\bar{u}_i}{u_0}=1-\frac{1}{u_0}\sum_{m=0}^{\infty}A_m g_{mi}(z)e^{-\beta_m t} \qquad (14)$$

The overall average consolidation degree for any layer are:

$$\bar{U}_1=1-\frac{1}{h_1}\int_0^{h_1}\frac{\bar{u}_1}{u_0}dz=1-\frac{1}{u_0}\frac{1}{h_1}\sum_{m=0}^{\infty}A_m\left(\int_0^{h_1}g_{m1}dz\right)e^{-\beta_m t} \quad (15)$$

$$\bar{U}_2=1-\frac{1}{h_2}\int_{h_1}^{H}\frac{\bar{u}_2}{u_0}dz=1-\frac{1}{u_0}\frac{1}{h_2}\sum_{m=0}^{\infty}A_m\left(\int_{h_1}^{H}g_{m2}dz\right)e^{-\beta_m t} \quad (16)$$

The overall average consolidation degree for whole soil layer is:

$$\bar{U}=1-\frac{\int_0^{h_1}\bar{u}_1dz+\int_{h_1}^{H}\bar{u}_2dz}{u_0H}=\frac{1}{H}\left(h_1\bar{U}_1+h_2\bar{U}_2\right) \qquad (17)$$

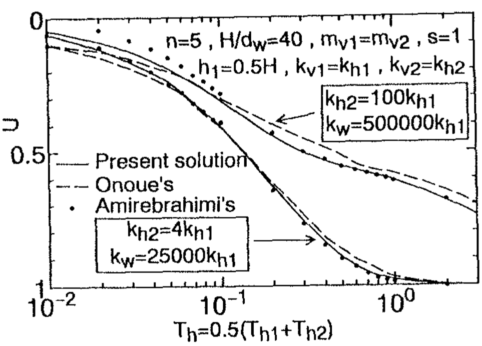


Fig. 2 Comparison of overall average consolidation degree for whole soil layer (No smear action)

In figure, $T_{hi}=\frac{c_{hi}}{4r_e^2}t$ , $c_{hi}=\frac{k_{hi}}{m_{vi}\gamma_w}$

## COMPARING WITH NUMERICAL SOLUTIONS

Comparisons of results by present solution to Onoue's results by FDM and Amirebrahimi's by FEM are shown in Fig.2.

## CONCLUSIONS

1). The analytical solution for double-layered ground with vertical drains is obtained.

2). The difference between the results by the present analytical solution and by numerical solutions is small.

## REFERENCES

1). Amirebrahimi, A.M. (1993): " Continuum model and analysis of wick-drained systems, " Int. J. for Numerical and Analytical methods in Geomechanics", Vol. 17 2).Onoue, A.(1988): " Consolidation of multi-layered anistropic soils by vertical drains with well resistance", Soils and Foundations, Vol.28, No.3. 3). Tang, X.W., Onitsuka, K.(1996): " New development of consolidation theory for double-layered ground with vertical ideal drains,"West Branch Ann. Conf. of JSCE.