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Analysis of Primary Instability of Flow past A Circular Cylinder Using Finite Element Method

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1. Introduction

Numerical methods have been applied to theoretical studies of instability and transition to turbulence since shortly after the advent of digital computer [1]. A rational asymptotic framework was developed for treating the linear and weakly nonlinear stability of nonparallel flows by means of the finite difference [2]and spectral methods [3]. By means of three-dimensional disturbance with the primary instability wave, the secondary instability or the three-dimensionality of flow also can be analysed. The purpose of this study is to analyze the linear stability of incompressible flow by employing the finite element method. As an example, we focus on the primary instability of a circular cylinder flow in which the three-dimensional disturbances are allowed. To overcome the singularity of eigenvalue problem in the linear analysis of incompressible flows, we employ a slight compressibility to eliminate the singularity. The basic flow of circular cylinder is two-dimensional which is computed by means of an improved velocity correction method by which the constraint of continuity can be satisfied [4].

2. Mathematical Theory

Considering the three-dimensional cylinder that is infinitely long in the axial direction, in which the diameter of cylinder is D and the uniform inflow velocity is U_0, let us assume that the fluid is slightly compressible, isothermal, and Newtonian, the nondimensional scales for length, velocity, time, density, and kinematic pressure are D, U_0, D/U_0, ρ_0, and CU_0, where C is acoustic speed in fluid. The dimensionless form of the continuity for the Newtonian fluid flow is

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0 \; in \; \Omega, \tag{1}$$

where ρ and $\mathbf{v}$ denote the nondimensional density and velocity vector, respectively. Given the pressure only as the function of density, the modified continuity equation can be written as follows

$$\frac{Dp}{Dt} + \frac{1}{M_a}\nabla\cdot\mathbf{v} = 0 \; in \; \Omega, \tag{2}$$

where $M_a = U_0/C$ denotes the Mach number in the fluid. To derive the equation of motion, considering the constitutive relation of the Newtonian fluid with Stokes's hypothesis, the stress tensor $\mathbf{T}$ is

$$\mathbf{T} = -\frac{1}{M_a}p\mathbf{I} + \frac{1}{Re}[(\nabla\mathbf{v} + (\nabla\mathbf{v})^T) - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{v}], \tag{3}$$

where $\mathbf{I}$ is the identity, $Re = U_0 D/\nu$ indicates the Reynolds number in which ν is the kinematic viscosity of fluid. Under the condition of slight compressibility, considering ν of fluid a constant and $\mathbf{f}$ a constant body force (e.g., gravitation), the nondimensional momentum equations can be shown as

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{M_a}\nabla p + \frac{1}{Re}[\nabla^2\mathbf{v} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{v})] + \mathbf{f} \; in \; \Omega. \tag{4}$$

Provided $\Gamma = \Gamma_N \cup \Gamma_S$ in which Γ_S denotes the Dirichlet boundaries, Γ_N is the Neumann boundaries. No-slip conditions are imposed on all well-boundaries. On the inflow and outflow, the boundary conditions are

$$u = 1 \;\; and \;\; v = 0 \quad on \; \Gamma_S, \tag{5}$$

$$\mathbf{T}\cdot\mathbf{n} = \hat{t} \quad on \; \Gamma_N, \tag{6}$$

where $\mathbf{n}$ denotes the boundary normal unit vector, $\hat{t}$ indicates the specified value on the boundary.

As the cylinder is assumed as infinitely long in the axial direction z, the base flow whose stability is being examined is two-dimensional and steady, with the result that the equations for this flow simplify to

$$\nabla\cdot\mathbf{V} = 0 \quad in \;\; \Omega, \tag{7}$$

$$\mathbf{V}\cdot\nabla_*\mathbf{V} = -\nabla_* P + \frac{1}{Re}\nabla_*^2\mathbf{V} \quad in \;\; \Omega, \tag{8}$$

where $\mathbf{V}$ and P are the velocity and kinematic pressure in the base flow, respectively. ∇_* represents the two-dimensional gradient operator.

To investigate the stability of the base flow to disturbances, we need the equations that govern the evolution of these perturbations. To this end, we perturb the base flow by disturbance velocities $\mathbf{v}'$ and the kinematic pressure by p'. Substituting the perturbed velocity into (2) and (4), subtracting the base flow equations (7) and (8) and linearising, the following equations for p' and $\mathbf{v}'$ subject to no-slip conditions on all boundaries are

$$\frac{Dp'}{Dt} + (\mathbf{v}'\cdot\nabla)P + \frac{1}{M_a}\nabla\cdot\mathbf{v}' = 0 \;\; in \; \Omega, \tag{9}$$

$$\frac{D\mathbf{v}'}{Dt} + (\mathbf{v}'\cdot\nabla)\mathbf{V} = -\frac{1}{M_a}\nabla p' + \frac{1}{Re}[\nabla^2\mathbf{v}' + \frac{1}{3}\nabla(\nabla\cdot\mathbf{v}')] \;\; in \; \Omega, \tag{10}$$

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where the two-dimensional operator $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla_*)$. The boundary conditions of the disturbances are

$$\mathbf{v}' = 0 \quad on\ \Gamma_S, \tag{11}$$

$$[-\frac{1}{M_a}p'\mathbf{I} + \frac{1}{Re}((\nabla\mathbf{v}' + (\nabla\mathbf{v}')^T) - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{v}')]\cdot\mathbf{n} = 0 \quad on\ \Gamma_N. \tag{12}$$

In terms of the normal mode, we represent the disturbances of velocity in the symmetry plane and spanwise direction of cylinder as

$$u' = i\hat{u}(x,y)e^{i\kappa z + \omega t}, \tag{13}$$

$$w' = \hat{w}(x,y)e^{i\kappa z + \omega t}, \tag{14}$$

where i is the imaginary unit, κ is the spanwise wave number, and $\omega = \omega_r + i\omega_i$ denotes the complex growth rate. The normal modes of pressure p' and velocity v' are the same as that form of (13). The assumed form of the eigenvector is completely general and allows for both steady and oscillatory modes, depending on whether the eigenvalue ω is real or complex, respectively. According to the linear stability theory, if ω is complex, the neutral condition is $\omega_r = 0$, and the onset of instability is oscillatory with dimensionless Strouhal number $St = \frac{\omega_i}{2\pi}$. Substituting these normal modes into (9) and (10), presents an eigenproblem with the growth rate being the eigenvalue

$$\omega\hat{\phi} = \mathcal{L}(\mathbf{V}, P, \kappa)\hat{\phi} \tag{15}$$

subject to no-slip boundary conditions on all boundaries, where $\hat{\phi} = \{\hat{u}, \hat{v}, \hat{w}, \hat{p}\}^t$, $\mathcal{L}$ is the linear operator including the convection, pressure gradient, viscosity, and compressibility terms.

As for the discretization of (15) by means of the finite element method, the linear interpolation function based on the 3-node triangular element for eigenfunctions is employed in this case. After the superposition of element matrices, the temporal mode of the generalized eigenproblem is expressed as follows

$$\mathbf{A\Phi} = \omega\mathbf{B\Phi}, \tag{16}$$

where $\mathbf{A}$ is the discretized matrix of operator $\mathcal{L}$, $\mathbf{B}$ is the assembling consistent mass matrix, $\mathbf{\Phi} = \{\hat{u}, \hat{v}, \hat{w}, \hat{p}\}^t$ is an assembling vector of eigenfunction.

3. Numerical Results

The primarily stable state of circular cylinder flow is that the symmetric eddies exist behind the cylinder over a range of $Re = 5 \sim 49$. The Kármán vortex street happens at the onset of the primary instability. The accurately critical parameters are $Re_c = 46.389 \pm 0.01$ for Reynolds number and $St_c = 0.126$ for Strouhal number at the onset of primary instability. Meanwhile, we obtained the critical wave number κ_c is equal to zero, it means that the flow at the primary instability is two-dimensional, the same results were reported in [5]. From the perturbed vorticity $\hat{\omega}_{zi}$ of eigenfunction in (16), the saddle-like vorticity is shedding alternatively behind the cylinder (see Fig. 1). By using the spectra of (16), we can reconstruct the Kármán vortex street near the critical state. Figure 2 shows the linear analysis results by means of reconstruction during a half of shedding period.

FIG. 1. Perturbed vorticity $\hat{\omega}_{zi}$

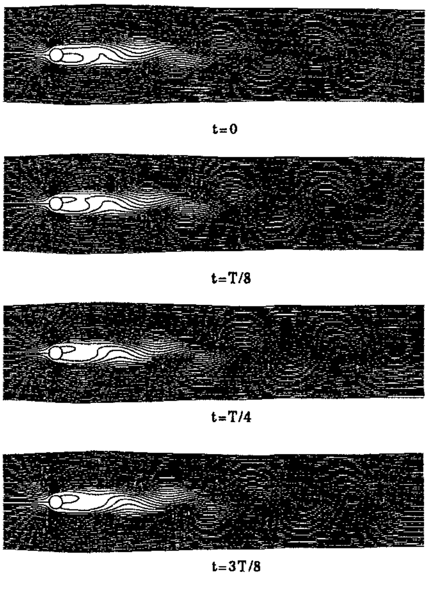

FIG. 2. Instantaneous Streamlines at Re=50

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