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PENNY-SHAPED CRACK AT THE INTERFACE OF TWO BONDED DISSIMILAR ELASTIC HALF-SPACES UNDER GENERAL SURFACE LOADINGS

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1. INTRODUCTION

In the present paper, the problem of determining the distribution of stress in the neighbourhood of a penny-shaped crack situated at the interface of two bonded dissimilar elastic half spaces under general surface loadings is considered. The surface loads applied on the upper surface of the crack is different from the lower surface, but axisymmetric. Basic equations of elasticity have been solved using Hankel transforms and Abel operator of the first kind. General expressions are obtained for stress and displacement components in terms of stress and displacement discontinuities at the interface plane. Using the limiting values of stress, displacements on the crack plane and boundary conditions of the crack, the problem is reduced to that of solving Riemann-Hilbert problem. Explicit expressions are obtained for stress and displacement discontinuities. Expressions for stress components on the crack plane are not easy to obtain in terms of the loading terms. But in terms of stress and displacement discontinuities stress components are derived in its simplest form

2. STRESS DISTRIBUTION AT THE INTERFACE OF TWO BONDED DISSIMILAR ELASTIC HALF SPACES

Let two elastic solids of infinite extent are bonded along a plane. In terms of cylindrical coordinates (r,θ,z), interface plane is z=0. The upper half space ($z > 0$) is occupied by an elastic material having elastic constants λ_1, μ_1 and the lower half space ($z < 0$) by a material with constants λ_2, μ_2, where λ_k, μ_k (k=1, 2) are Lame's elastic constants. Stress and displacement fields in the solid of bonded dissimilar elastic half-spaces are assumed to be torsionless axisymmetric. The non-zero displacement and stress components $u_r^{(k)}$, $u_z^{(k)}$, $\sigma_{rr}^{(k)}$, $\sigma_{rz}^{(k)}$, $\sigma_{\theta\theta}^{(k)}$ and $\sigma_{zz}^{(k)}$ (k = 1, 2) are independent of θ. Where the stress and displacement fields for k = 1, 2 are in the upper half space ($z > 0$) and lower half space ($z < 0$) respectively. Let the limiting values as $z \to 0+$ of stress and displacement components be denoted by $\sigma_{rz}^{(1)}(r,0)$, $\sigma_{zz}^{(1)}(r,0)$, $u_r^{(1)}(r,0)$, $u_z^{(1)}(r,0)$. Similarly, let the limiting values as $z \to 0-$ of stress and displacement components be denoted by $\sigma_{rz}^{(2)}(r,0)$, $\sigma_{zz}^{(2)}(r,0)$, $u_r^{(2)}(r,0)$, $u_z^{(2)}(r,0)$ We set

$$\frac{\partial}{\partial\rho}\int_\rho^\infty \frac{\rho[u_r^{(1)}(r,0)-u_r^{(2)}(r,0)]}{\sqrt{r^2-\rho^2}}dr = A(\rho)\,,\ \rho>0 \tag{1}$$

$$\frac{\partial}{\partial\rho}\int_\rho^\infty \frac{r[u_z^{(1)}(r,0)-u_r^{(2)}(r,0)]}{\sqrt{r^2-\rho^2}}dr = B(\rho)\,,\ \rho>0 \tag{2}$$

$$\int_\rho^\infty \frac{\rho[\sigma_{rz}^{(1)}(r,0)-\sigma_{rz}^{(2)}(r,0)]}{\sqrt{r^2-\rho^2}}dr = C(\rho)\,,\ \rho>0 \tag{3}$$

$$\int_\rho^\infty \frac{r[\sigma_{zz}^{(1)}(r,0)-\sigma_{zz}^{(2)}(r,0)]}{\sqrt{r^2-\rho^2}}dr = D(\rho)\,,\ \rho>0 \tag{4}$$

The expressions obtained for stress and displacement components in terms of A, B, C, D can be used to solve class of problems in which boundary conditions are prescribed at the interface plane z = 0.

3. PENNY SHAPED CRACK IN THE INTERFACE AND SOLUTION

Let a penny shaped crack be located in the interface of a bonded dissimilar elastic half spaces, and crack occupies the region $0 \le r \le a, (z=0)$. The crack is subjected to axisymmetric surface tractions which are not necessarily self-equilibrating. That is, shear and normal stresses applied on the upper face of the crack $(0<r<a, z\to 0+)$ is different from the lower face $(0<r<a, z\to 0-)$ of the crack. System is in equilibrium since the body is very large compared to the radius of the crack. Perfect bonding is assumed outside the crack region $r>a, z=0$. That is, the stress and displacement components are continuous outside the crack region in the interface plane. At the rim of the crack displacement components are assumed to be continuous. The continuity and boundary conditions on the interface plane z = 0 are given by

$$u_r^{(1)}(r,0)=u_r^{(2)}(r,0);\ u_z^{(1)}(r,0)=u_z^{(2)}(r,0),\ (r>a) \tag{5}$$

$$\sigma_{rz}^{(1)}(r,0)=\sigma_{rz}^{(2)}(r,0);\ \sigma_{zz}^{(1)}(r,0)=\sigma_{zz}^{(2)}(r,0),\ (r>a) \tag{6}$$

$$\sigma_{zz}^{(1)}(r,0)+\sigma_{zz}^{(2)}(r,0)=H^*(r),\ (0<r<a) \tag{7}$$

$$\sigma_{zz}^{(1)}(r,0)-\sigma_{zz}^{(2)}(r,0)=Q^*(r),\ (0<r<a) \tag{8}$$

$$\sigma_{rz}^{(1)}(r,0)+\sigma_{rz}^{(2)}(r,0)=G^*(r),\ (0<r<a) \tag{9}$$

$$\sigma_{rz}^{(1)}(r,0)-\sigma_{rz}^{(2)}(r,0)=P^*(r),\ (0<r<a) \tag{10}$$

where $H^*(r)$, $Q^*(r)$, $G^*(r)$, $P^*(r)$ are known functions representing the surface tractions. Using the conditions (5)-(10), we can determine the unknown functions A, B. C. D in the expressions for stress and displacement components in terms of prescribed functions $H^*(r)$, $G^*(r)$, $P^*(r)$, $Q^*(r)$. Continuity conditions (5)-(6) and equations (1)-(4) give rise

$$B(t)=A(t)=C(t)=D(t)=0\quad t>a \tag{11}$$

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Key words: *Penny-shaped crack, Riemann-Hilbert Problem, Axisymmetric, Dissimilar, Self-equilibrating*

The boundary conditions (8), (10), equations (3), (4) together with (11) determine the unknown functions C, D and are given by

$$C(t) = \int_t^a \frac{tP^*(r)}{\sqrt{r^2 - t^2}} dr \; ; D(t) = \int_t^a \frac{rQ^*(r)}{\sqrt{r^2 - t^2}} dr \, , 0 < t < a \quad (12)$$

Using limiting values of stress components and boundary conditions (7) and (9) we can reduce the problem to a set of singular integral equations for A, B. They can be written as

$$\Lambda(t) = A(t) + iB(t) \quad (13)$$

$$\Lambda(t) - \frac{\delta}{\pi i} \int_{-a}^{a} \frac{\Lambda(s)}{t - s} ds = g_1(t) \, , -a < t < a \quad (14)$$

The loading term $g_1(t)$ and material constant δ are given by

$$g_1(t) = \begin{cases} g(t) & 0 < t < a \\ g(-t) & 0 < t < a \end{cases} ; \delta = \frac{\beta_2 - \beta_1}{\alpha_2 + \alpha_2} \quad (15)$$

$$g(t) = \frac{\delta_3}{2(\alpha_1 + \alpha_2)} \int_0^t \frac{[tG^*(r) + irH^*(r)]dr}{\sqrt{t^2 - r^2}} + \frac{\delta_1}{2(\alpha_1 + \alpha_2)} [D(t) - iC(t)]$$
$$+ \frac{\delta_2}{2\pi i(\alpha_1 + \alpha_2)} \int_{-a}^{a} \frac{[D(s) - iC(s)]}{t - s} ds + C_3 \, , \; 0 < t < a \quad (16)$$

where C_3 is constant of integration and it can be settled using the continuity of the displacement at the rim of the crack (r = a), that is, $\int_0^a A(t)dt = 0$ and δ_1, δ_2, δ_3 are in term of material constants and are given by

$$\delta_1 = (\alpha_2 + \alpha_1)(\beta_2 + \beta_1) - (\alpha_2 - \alpha_1)(\beta_2 - \beta_1) \quad (17)$$

$$\delta_2 = (\alpha_2 + \alpha_1)(\alpha_2 - \alpha_1) - (\beta_2 + \beta_1)(\beta_2 - \beta_1) \quad (18)$$

$$\delta_3 = (\alpha_2 + \alpha_1)^2 - (\beta_1 - \beta_2)^2 \quad (19)$$

the constants α_k, β_k k=1, 2 are given by

$$\alpha_k = \frac{\lambda_k + 2\mu_k}{2\mu_k(\lambda_k + \mu_k)} ; \beta_k = \frac{1}{2(\lambda_k + \mu_k)} \, , k=1,2 \quad (20)$$

Using the plemelj formulae (Muskhesishvili [2])

$$\Lambda(x) = \Phi^+(x) - \Phi^-(x) \quad (21)$$

$$\frac{1}{\pi i} \int_{-a}^{a} \frac{\Lambda(s)}{s - x} ds = \Phi^+(x) + \Phi^-(x) \quad (22)$$

the singular integral equation (14) can be reduced to Riemann-Hilbert problem and is given by

$$\Phi^+(x) - \frac{(1+\delta)}{(1-\delta)} \Phi^-(x) = \frac{1}{(1-\delta)} g_1(x) \, , -a < x < a \quad (23)$$

Following the procedure of the Lowengrub and Sneddon [1] we can write the solution of the integral equation (23) as

$$\chi(\zeta) = (\zeta + 1)^{\frac{i\gamma}{2}} (\zeta - 1)^{-\frac{i\gamma}{2}} \, , \gamma = \frac{1}{2\pi} \log\left(\frac{1+\delta}{1-\delta}\right) \quad (24)$$

$$\Phi(\zeta) = \frac{\chi(\zeta)}{2\pi i(1-\delta)} \int_{-a}^{a} \frac{g_1(t)}{\chi^+(t)(t - \zeta)} dt \quad (25)$$

where ζ is complex variable $(\zeta = x + iz)$, $\Phi^+(x)$ denotes the value of $\Phi(\zeta)$ as ζ tends to the real value x in the interval (-a, a) through positive values of z and $\Phi^-(x)$ denote the corresponding value when ζ tend to x through negative values of z and $\chi(\zeta)$ satisfies the relation

$$\chi^+(x) - \frac{(1+\delta)}{(1-\delta)} \chi^-(x) = 0 \, , \; -a < x < a \quad (26)$$

We note that in deriving the solution (25) we used the fact that $\Phi(\zeta) \to 0$ as $\zeta \to \infty$ which settles the arbitrary constant in the general solution of the Riemann-Hilbert problem (23). The stress and displacement components are given in terms of A, B, C, D. These functions are derived in terms of prescribed loading conditions and are given by equations (13), (20), (24)-(25), (11). Hence in principle this completes the solution of the problem. In order to discuss the stress singularity at the rim of the crack the stress components on the crack plane are written as

$$\sigma_{zz}^{(1)}(r,0) + \sigma_{zz}^{(2)}(r,0) = -\frac{2(\alpha_1 + \alpha_2)}{\pi\delta_3} \frac{B(a)}{\sqrt{r^2 - a^2}}$$
$$+ \frac{1}{\pi\delta_3} \int_0^a \frac{[2(\alpha_1 + \alpha_2)B'(t) + \delta_1 C'(t)]}{\sqrt{r^2 - t^2}} dt \, , \; r > a \quad (27)$$

$$\sigma_{rz}^{(1)}(r,0) + \sigma_{rz}^{(2)}(r,0) = -\frac{2(\alpha_1 + \alpha_2)a}{\pi\delta_3 r} \frac{A(a)}{\sqrt{r^2 - a^2}}$$
$$+ \frac{1}{\pi\delta_3 r} \int_0^a \frac{[2(\alpha_1 + \alpha_2)A'(t) - \delta_1 D'(t)]}{\sqrt{r^2 - t^2}} t dt \, , r > a \quad (28)$$

Prime over the functions A, B, C, D denote the derivative with respect to the variable.

4. CONCLUDING REMARKS

A method is developed to solve a class of problems in which axisymmetric boundary conditions are prescribed on the interface plane. As an example the problem of Penny-Shaped crack in the interface under general surface loadings is solved.

By reducing the interface crack problem to that of solving the Riemann-Hilbert problem, a closed form solution is obtained. Explicit expressions are written for stress components on the crack plane.

For the cases of homogeneous material, and symmetric loading, results are deduced from the general case and compared with those available in the literature.

Stress components posses square root singularity. Oscillatory nature comes from plemelj functions. Oscillatory nature can be taken care when Λ is computed in the small neighbourhood of the rim of the crack.

5. REFERENCES

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