I－A 141 Stress Distribution near the Penny shaped crack at the interface of an elastic half-space and Rigid Foundation

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1. Introduction

Considerable attention has been recently directed to the problems determining the distribution of stress and displacement fields in the neighbourhood of a crack between bonded dissimilar materials. The dissimilar material system is required to act as a single unit in that the loads are transmitted from one material to the next through the interfaces. The presence of flaws or cracks in one of the materials or at the interface could cause high elevation of local stresses and lead to failure if the crack reaches a critical size. Hence it is important to know the stress state associated with these cracks in the dissimilar material system. The analytical stress solutions exhibits a peculiar behavior near the tip of an interface crack where the stresses will have rapid oscillatory character [1, 2]. The present investigation formulates and solves a problem of a penny shaped crack at the interface of an elastic half-space bonded to a rigid foundation as shown in Fig.1.

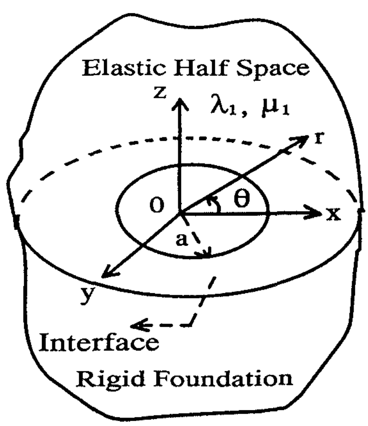

Fig.1 Interface Penny Shaped Crack

The crack surfaces are subjected to normal as well as shear tractions. The elastic material is isotropic with Lame's constants λ_1, μ_1. In terms of cylindrical polar coordinates (r, θ, z), Penny shaped crack is defined as $0 \le r \le a, z = 0\mp$ and the upper half-space ($z > 0$) is elastic while the lower half-space ($z < 0$) is rigid. The materials outside the crack region $(r > a, z = 0)$ are assumed to have perfect bonding. The non-zero stress and displacement components are denoted by $\sigma_{zz}(r,z)$, $\sigma_{rz}(r,z)$, $\sigma_{rr}(r,z)$, $\sigma_{\theta\theta}(r,z)$, $u_z(r,z)$, $u_r(r,z)$ and their limiting values as approaching to the interface plane $(z \to 0+)$ be denoted by $\sigma_{zz}^{(1)}(r,0)$, $\sigma_{rz}^{(1)}(r,0)$, $\sigma_{rr}^{(1)}(r,0)$, $\sigma_{\theta\theta}^{(1)}(r,0)$, $u_z^{(1)}(r,0)$, $u_r^{(1)}(r,0)$. In this formulation stress and displacement fields are represented in terms of one biharmonic function. The basic equations of linear theory of elasticity have been solved using Hankel transforms and Abel operators of the second kind[3]. An analytical solution is obtained for the crack problem reducing to a Riemann-Hilbert problem. Explicit expressions are given for the Stress Intensity Factors K_I and K_{II}.

2. Solution of the Interface Penny-Shaped Crack Problem

The stress and displacement fields in a semi-infinite elastic solid ($z > 0$) which is bonded to a rigid foundation at the plane $z = 0$ are determined in terms of two functions A and B defined by

$$\frac{\partial}{\partial\rho}\int_\rho^\infty \frac{\rho u_r^{(1)}(r,0)}{\sqrt{r^2-\rho^2}}dr = A(\rho),\ \rho \ge 0 \qquad (1)$$

$$\frac{\partial}{\partial\rho}\int_\rho^\infty \frac{r u_z^{(1)}(r,0)}{\sqrt{r^2-\rho^2}}dr = B(\rho),\ \rho \ge 0 \qquad (2)$$

Some similarity can be seen in the results given in [3] and those given in the present paper. The limiting values of stress and displacement components as $(z \to 0+)$ have been used to solve the interface penny-shaped crack problem. Let the penny shaped crack be situated at the interface plane z=0 of elastic half-space and rigid foundation. Crack faces are subjected to general surface loadings, but axisymmetric. The boundary conditions are given by

$$u_z^{(1)}(r,0) = 0,\ u_r^{(1)}(r,0) = 0,\ r > a \tag{3}$$

$$\sigma_{zz}^{(1)}(r,0) = H^*(r),\ 0 < r < a \tag{4}$$

$$\sigma_{rz}^{(1)}(r,0) = G^*(r),\ 0 < r < a \tag{5}$$

and at the tip of the crack continuity of the displacement has been assumed. The boundary conditions of the crack problem and the limiting values of stress and displacement components in terms of A and B lead to

$$A(t) = 0,\ B(t) = 0,\ t > a \tag{6}$$

and for $0 < t < a$ reduce to a set pair of simultaneous integral equations. The pair of integral equations can be reduced to

$$\Phi^+(x) - \left(\frac{1+\delta}{1-\delta}\right)\Phi^-(x) = \frac{1}{(1-\delta)}g_1(x) \tag{7}$$

which is Riemann-Hilbert problem. The functions $\Phi^+(x)$ and $\Phi^-(x)$ can be written using Plemelj formulae

$$\Lambda(x) = \Phi^+(x) - \Phi^-(x) \tag{8}$$

$$\frac{1}{\pi i}\int_{-a}^{a}\frac{\Lambda(s)}{s-x}ds = \Phi^+(x) + \Phi^-(x) \tag{9}$$

and δ is in terms of material constants which can be written as

$$\delta = \frac{\mu_1}{(\lambda_1 + 2\mu_1)} \tag{10}$$

where x is a point in the interval (-a, a) and the integral (9) is interpreted as a principal value. In the present problem $\Lambda(t)$ is given by

$$\Lambda(t) = A(t) + iB(t) \tag{11}$$

and loading term $g_1(t)$ is

$$g_1(t) = \begin{cases} g(t), & 0 < t < a \\ \overline{g(-t)}, & -a < t < 0 \end{cases} \tag{12}$$

$$g(t) = \frac{(\lambda_1 + 3\mu_1)}{2\mu_1(\lambda_1 + 2\mu_1)}\int_0^t \frac{\left[tG^*(r) + irH^*(r)\right]}{\sqrt{t^2 - r^2}}dr \tag{13}$$

Closed form solution can be obtained for the integral equation (7) and it is given by

$$\Phi(\zeta) = \frac{\chi(\zeta)}{(1-\delta)2\pi i}\int_{-a}^{a}\frac{g_1(t)dt}{\chi^+(t)(t-\zeta)} + c_4\chi(\zeta) \tag{14}$$

$$\chi(\zeta) = (\zeta+1)^{\frac{i\gamma}{2}}(\zeta-1)^{\frac{-i\gamma}{2}} \tag{15}$$

where c_4 is an arbitrary constant and $\chi(\zeta)$ is the solution of the homogeneous part of the equation (7) and γ, ζ are given by

$$\gamma = \frac{1}{2\pi}\log\left[\frac{1+\delta}{1-\delta}\right];\quad \zeta = x + iz \tag{16}$$

where $\Phi^+(x)$ and $\Phi^-(x)$ are limiting values of $\Phi(\zeta)$ as approaching to crack plane $z = 0$ from positive and negative values of z respectively. The arbitrary constant c_4 can be settled using the continuity of the displacement at the tip of the crack, that is, $\int_{-a}^{a}\Lambda(t)dt = 0$. Since the stress and displacement components are in terms of A and B, therefore, they can be simplified substituting values of A and B from equations (8), (11), (14)-(16). In principle this completes the solution of the crack problem. For the sake of engineering applications, an explicit expressions are derived for mixed mode Stress Intensity Factors using the definitions

$$K_I = \lim_{r\to a+}\sqrt{[2(r-a)]}\sigma_{zz}^{(1)}(r,0) \tag{17}$$

$$K_{II} = \lim_{r\to a+}\sqrt{[2(r-a)]}\sigma_{rz}^{(1)}(r,0) \tag{18}$$

and are given by

$$K_{II} + iK_I = -\frac{4\mu_1(\lambda_1 + 2\mu_1)}{\pi(\lambda_1 + 3\mu_1)\sqrt{a}}\Lambda(a) \tag{19}$$

3. Concluding Remarks

If we take lower half-space is rigid in [2], the integral equations are same as those given in this paper. If both half spaces are elastic and crack faces are subjected to general surface loadings, similar way problem can be solved, but the general case is very complicated from which the present case can be obtained as a special case. Results will be reported in the separate communication. Mode-I and Mode-II Stress Intensity Factors are derived considering the square root singularity at the tip of the crack. The oscillatory nature will come from the term $\Lambda(a)$.This oscillatory nature is confined to a small neighbourhood of the rim of the crack. The integrals involved in the function $\Lambda(a)$ can be computed numerically but, oscillatory nature should be taken care. Both the Stress Intensity factors depend on the material constants.

4. References:

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[3] K. S. Parihar and J. V. S. Krishna Rao, Int. J. Engng. Sci., 31 (1993) 953-966