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IMPROVEMENT OF SHALLOW WATER FLOW MODELLING

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Introduction

Shallow water flows are found in a wide variety of natural water bodies such as shallow lakes and coastal zones. A common characteristic is that the water depth is much smaller than a typical length in the horizontal directions. These flows can be modelled using numerical models based on the shallow water equations (SWE).

In the Netherlands 2DH (two-dimensional horizontal) models were used to predict the water levels and flow velocities during the construction of the closure dams in the Rhine-Meuse delta. The similarity between the predicted and measured water levels and flow velocities was good in most parts of the estuaries. But, directly downstream from submerged dams the predicted water levels were too high.

These deviations might caused by errors in the SWE. These equations are derived by depth integrating the continuity equation and the Navier-Stokes equations in flow direction, assuming vertical profiles of pressure (hydrostatic) and velocity (logarithmic) as in uniform flow. In accelerating flow the velocity profiles become more nearly uniform and in a deceleration flow less uniform. For steep slopes, separation of the flow even occur.

The above described effects in the three-dimensional (3D) non-uniform flow already appear in two-dimensional flow in a vertical plane (2DV flow). Therefore this study is restricted to 2DV flow only.

This paper describes a set of more general 1DH steady SWE, valid for uniform flows as well as for non-uniform flows. Modifications of the conventional equations are expressed by coefficients in the convection, the pressure and the friction term. These coefficients are derived by analytically solving the two-dimensional steady Navier-Stokes equations for flow over a sill in a vertical plane.

Shallow water equations

Simplifications made in the conventional SWE are assumed to be responsible for the differences between the measured and predicted water levels and velocities directly downstream from submerged dams. Therefore a set of more general SWE is derived. The depth-integrated continuity equation for steady 2DV flow is kept unchanged (equation (1)), while the momentum equation of the SWE, the depth-integrated Navier-stokes equation in flow direction, slightly changes (equation (2)).

$$\frac{\mathrm{d}Uh}{\mathrm{d}x} = 0 \quad (1), \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha \rho U^2 h\right) + \zeta \rho g h \frac{\mathrm{d}\eta}{\mathrm{d}x} + \gamma\left(\lambda \rho U^2\right) = 0 \quad (2)$$

In these equations U is the depth-averaged velocity, h the water depth, α, ζ and γ the correction coefficients for momentum flux, pressure and friction, respectively, ρ is the density, η is the water level, g is the gravitational acceleration and λ is the friction parameter for uniform flow. The convection coefficient α describes the influence of the non-uniformity of the velocity profile on the depth-integrated momentum flux ($\alpha \geq 1$). The pressure coefficient ζ describes the influence of the deviation from the hydrostatic pressure. The friction coefficient γ describes the deviation of the bed shear stress from its value for uniform flow.

In uniform flow, $\zeta = 1$ and $\gamma = 1$ by definition, whereas $\alpha = 1 + \lambda/\kappa^2$ for a logarithmic velocity distribution. The logarithmic velocity profiles in turbulent free-surface flows deviate hardly from uniform velocity profiles and yield values for α that are only a little in excess of 1. In the conventional SWE all three coefficients are equal to 1.

To obtain the correction coefficients for the improvement of the SWE equations analytically, the non-linear second-order differential Navier-Stokes equations have to be solved. In general analytical solutions are not possible. Here the method of weighted residuals are used to obtain approximate solutions. This method is a generalisation of the method used by Madsen & Svendsen [3] to compute the velocity distribution in a hydraulic jump.

The method of weighted residuals

In the method of weighted residuals (MWR) similarity of velocity profiles is assumed. The choice of the shape of the similarity profiles is more or less free. Here the velocity profile is assumed to consist of a zeroth-order profile, u_0, and a first-order profile, u_1, multiplied with a weighting factor, Γ. The weighting factor varies in streamwise direction and it is one of the variables to be solved. The solutions found for the velocity distribution with this method are not exact, but they will be the best fit for the chosen similarity profile. The choice of the shape of the velocity profiles has an important influence on the elevation of the water level. Therefore the choice of the shape of the velocity profiles requires serious attention. In this study, zeroth- and first-order velocity profiles based on analytically solved velocity profiles by means of the method of asymptotic expansions (Blom [1]), are used. Two different first-order profiles are used, one in the acceleration zone and the other in the deceleration zone. Using one similarity profile for the entire domain was found to give inadequate results.

The profiles $u = u_0 + \Gamma u_1$ are substituted into the continuity equation and the Navier-Stokes equation in flow direction which are subsequently integrated with respect to the depth to obtain the modified SWE. For the pressure term a hydrostatic distribution is assumed. This assumption appeared to be correct even for flow over rather steep slopes (Blom [1]). These modified SWE contain three variables to be solved: the depth-averaged velocity U, the water level η and the weighting factor Γ. This means that in addition to the continuity equation (1) and the momentum equation (2) another equation is needed. The third equation used here is the depth-averaged energy equation. The total set of equations describing the non-uniform steady flow then reads:

$$\frac{\mathrm{d}Uh}{\mathrm{d}x} = 0 \qquad (1)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha \rho U^2 h\right) + \zeta \rho g h \frac{\mathrm{d}\eta}{\mathrm{d}x} + \gamma\left(\lambda \rho U^2\right) = 0 \qquad \text{with } \alpha = \int_0^1 \tilde{u}^2 \mathrm{d}\sigma = f(\Gamma), \quad \zeta = 1, \quad \gamma = \frac{\tau_{bx}}{\lambda \rho U^2} = g(\Gamma) \qquad (2)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\alpha_1 \rho U^3 h\right) + \zeta_1 \rho g h U \frac{\mathrm{d}\eta}{\mathrm{d}x} + \gamma_1\left(\lambda \rho U^3\right) = 0 \qquad \text{with } \alpha_1 = \int_0^1 \tilde{u}^3 \mathrm{d}\sigma = f_1(\Gamma), \ \zeta_1 = 1, \ \gamma_1 = \frac{\int_0^1 \tilde{u} \frac{\partial \tau_{\sigma x}}{\partial \sigma} \mathrm{d}\sigma}{\lambda \rho U^2} = g_1(\Gamma) \qquad (3)$$

in $\tilde{u}$ is the relative velocity, which τ_{bx} is the bottom shear stress and $\tau_{\sigma x}$ is the Reynolds shear stress. In the turbulence model used to obtain the Reynolds shear stress a parabolic eddy-viscosity profile is applied. In that case the correction coefficients depend on the velocity profile only through the weighting factor $\Gamma(x)$, in a known manner (analytical expressions are available in Blom [1]). The set of coupled first-order differential equations (1) through (3) is to be integrated to derive the water level η, the averaged velocity U and the weighting factor Γ.

Several computations were made for different bottom geometries. The results of only one computation are shown here, (Fig(1)). In this computation the initial water depth is 20m, the discharge per unit width is 13m^2/s, the length of the crown of the sill is 50m, the slope length of the sinusoidal sill is 75m, while its maximum slope is $^1/_5$. In this figure the form of the sill and the height of the water levels, velocity profiles, convection coefficient, α, and the friction coefficient, γ, calculated with a 2DV numerical computations (description is available in Blom [1]) and those calculated with the MWR, are compared.

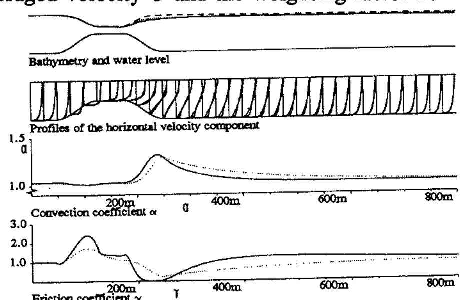

Figure 1. Coefficients and water levels, situation with maximum slope $^1/_5$.
—— computed using numerical model. ······ computed using MWR.
— — computed using eq. (1) and (2) with α=1, ζ=1 and γ=1.

Although the agreement between the water levels and coefficients derived with the 2DV model and the MWR is quite satisfactory, this formulation of the velocity profiles has some disadvantages. It is hardly possible to use two first-order velocity profiles for unsteady flow in which reversion of the mean flow can occur. The disadvantage to apply this method to 2DH SWE models is the dependency of the perturbation profiles on the depth-averaged velocity. In 2DH flow a secondary flow in transverse direction can occur without an depth-averaged component in that direction. To make the MWR also applicable for such flows is dealt with in the next section.

The improved method of weighted residuals.

It is necessary to formulate slightly equations (1) and (2) to make the MWR also applicable for unsteady flows or flows in which secondary flow occurs without a depth-averaged component in traverse direction. Again the flow velocities are expressed as a linear combination of a series of functions, the so-called shape functions. The procedure is almost the same as described before. The assumed velocities are again substituted into the basic equations and then integrated over the depth. The resulting residual functions are then minimised using the Galerkin technique. The starting set of equations reads:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \quad (4), \qquad u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z}=0 \quad (5), \qquad \frac{1}{\rho}\frac{\partial p}{\partial z}+g=0 \quad (6)$$

If arbitrary solutions of the velocities and pressure are substituted in the equations (4) through (6), the left hand-side of these equations is only 0 when these substituted solutions are the exact solutions. Otherwise a residual (R) appears at the right-hand side of the equations (4) through (6) and a residual (r) appears in the boundary conditions. For a chosen variation δu of a displacement field $\bar{u}$ the residuals R and r should become 0 for an exact solution.

$$\iiint_V R\,\delta u\,\mathrm{d}V-\iint_S r\,\delta u\,\mathrm{d}S=0 \qquad (7)$$

This condition can only satisfy every arbitrary variation δu when R=0 in V and r=0 on S. δu is written as a linear combination of a series of shape functions multiplied with a scalar: $\delta u(x,y,z)=\Sigma\,\delta u_i\,N_i(x,y,z)$. Applying this to the equations (4) and (5) these equations become:

$$\iiint_V\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)N_i\,\mathrm{d}V+\iint_S\left(u\frac{\mathrm{d}\eta}{\mathrm{d}x}-w\right)N_i+\left(u\frac{\mathrm{d}z_b}{\mathrm{d}x}-w\right)N_i\,\mathrm{d}S=0 \qquad (8)$$

$$\iiint_V\left(u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z}\right)N_i\,\mathrm{d}V+\iint_S\left(u\frac{\mathrm{d}\eta}{\mathrm{d}x}-w\right)N_i+\left(u\frac{\mathrm{d}z_b}{\mathrm{d}x}-w\right)N_i\,\mathrm{d}S=0 \quad (9), \qquad \iiint_V\left(\frac{1}{\rho}\frac{\partial p}{\partial z}+g\right)N_i\,\mathrm{d}V+\iint_S(P)N_i\,\mathrm{d}S=0 \quad (10)$$

in which P is the value of the pressure at the water level.

The next step is the choice of the velocity profiles, pressure distribution and shape functions. Again the water pressure is supposed to be hydrostatic. The horizontal profiles are based on the previous zeroth- and first-order velocity profiles, with some small modifications. The velocity profiles consists of a zeroth-, a first- and a second-order profile. The zeroth-order velocity profile is dependent on U with a logarithmic vertical distribution. The first-order and second-order profiles are dependent on a unknown depth-averaged velocity, but the vertical distribution is chosen to be just like in the previous part. Only, the first-order profile is the previous first-order profile for the acceleration zone and the second-order profile is the first-order profile in the deceleration zone. The vertical velocities are calculated from equation (4) after substituting these assumed horizontal velocities.

Substitution of the hydrostatic pressure distribution in equation (10) implies that this equation is satisfied for every arbitrary shape function. Substitution of the horizontal and vertical velocity profiles implies that equation (8) reduces to equation (1). Substitution of the velocity profiles in equation (9) gives:

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(h\int_0^1\rho N_i u^2\mathrm{d}\sigma\right)+\rho g h\frac{\mathrm{d}\eta}{\mathrm{d}x}\int_0^1 N_i\mathrm{d}\sigma-\tau_{\sigma x}N_i\Big|_{\sigma=0}-h\int_0^1\left(\rho u^2\frac{\mathrm{d}N_i}{\mathrm{d}x}+\frac{\rho u^2}{h}\left(\frac{\mathrm{d}z_b}{\mathrm{d}x}-\sigma\frac{\mathrm{d}h}{\mathrm{d}x}\right)\frac{\mathrm{d}N_i}{\mathrm{d}\sigma}+\rho w u\frac{\mathrm{d}N_i}{\mathrm{d}z}+\frac{\tau_{\sigma x}}{h}\frac{\mathrm{d}N_i}{\mathrm{d}x}\right)\mathrm{d}\sigma\right)\mathrm{d}y\,\mathrm{d}x=0$$

Integration over y can be eliminated in all the terms. Integration over x is the required change of the water level and velocity profiles. This should be done numerically. The only analytical integration to be done is the integration over the depth. But before integration, the shape functions have to be chosen. There are four variables to solve, U, h, u_1 and u_2 and there are $i+1$ equations available. Therefore three shape functions have to be chosen. A convenient choice for a shape-function is N_1=1. This yields the depth-averaged momentum equation (2). For the other two equations two of the three velocities u_0, u_1 and u_2 can be chosen.

Although the results of these calculations are not available on this moment it seems that this method is convenient to improve the 2DH SWE. Due to the inclusion of the vertical velocities, such a 2DH SWE model can be called a quasi-three-dimensional (Q3D) model. The above described formulation has the advantage that the influence of the convection terms on the water level is dealt with. This is not the case in ordinary Q3D models which make use of the Vertical-Horizontal-Splitting algorithm (Jin [2]).

References

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