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RESPONSE OF A VISCOELASTICALLY DAMPED SYSTEM A FRACTIONAL DERIVATIVE APPROACH

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1. Introduction

To model the frequency dependent mechanical property of viscoelastic material, a fractional derivative model has a advantage, over the commonly used linear integer models. The attractive features of fractional derivative are as follows; Fractional derivative model has its foundation in accepted molecular theories governing the mechanical behavior of viscoelastic materials. The model satisfies the second law of thermodynamics and predict the stress-strain hysteresis loops for VE materials accurately. This viscoelastic model uses few parameters, thereby leading itself to straight forward and accurate least-square fits to measured mechanical properties Ref.[1]. The above mentioned features motivates the use of FDM for response analysis of viscoelastically damped structure. In this paper the salient features of fractional derivative approach in calculating the response of viscoelastically damped system has been discussed with the help of a numerical solution . The numerical analysis has been carried out in time domain as well as in frequency domain. The problems associated with numerical solution and their remedies will be discussed briefly.

2. Fractional Derivative Model for VE Material

Adopting the well known Kelvin's model, constitutive relationship can be expressed as:

$$\sigma(t) = GD^0\epsilon(t) + bD^\alpha(t) \quad \cdots\cdots (1)$$

where G,b,and α are the constitutive parameters and D^α is a generalized differential operator with the definition given by Liouville as: $D_t^\alpha[f(t)] = \frac{d^\alpha f(t)}{[d(t)]^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}[\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau], \quad n > \alpha \geq 0, n \ \ integer$. For elastic and viscous material, the uniaxial stress-strain relationship can be expressed as: $\sigma = k\epsilon(t) = kD^0\epsilon(t)$ and $\sigma = kd\epsilon(t)/dt = kD^1\epsilon(t)$ respectively. Therefore, a viscoelastic material which has a properties in between elastic and viscous material could be represented by D^α, where α lies between 0 and 1. Now similiar to constitutive Equ.(1) the force-displacement relationship VE device can be expressed as $f(t) = kx(t) + bD^\alpha x(t) = k^*x(t)$; which renders $k^* = k + bD^\alpha$. Let the equation of motion of SDOF system be $m\ddot{x} + k^*x(t) = F(t)$. This equation in derivative form becomes:

$$mDx^2 + bD^\alpha x + kx = F(t) \quad \cdots\cdots (2)$$

Taking the Laplace Transform of eq(2)

$$(ms^2 + bs^\alpha + k)X(s) = F(s) \quad \cdots\cdots (3)$$

Taking Laplace Transform for Eqn(1) it becomes $\sigma(s) = GD^0\epsilon(s) + bD^\alpha(s)$, substituting $s = i\omega$ and $i = cos(0.5\pi) + isin(0.5\pi)$ in $\sigma(s) = GD^0\epsilon(s) + bD^\alpha(s)$ then separating the real and imaginary parts, we get storage modulus $(G') = G + b\omega^\alpha cos(0.5\alpha\pi)$ and loss modulus$(G'') = b\omega^\alpha sin(0.5\alpha\pi)$.The constitutive parameters (G,b,α),can be estimated by a least square fit over the data of storage modulus Vs frequency.

3. Numerical Scheme

Time Domain

To facilitate the incorporation of nonlinearity a numerical step by step solution technique has to be developed. The typical term in equation (2) is $D^\alpha x$ which is evaluated by quadrature formula's by Oldham Ref.[3]. The $D^\alpha x$ can be expressed in quadrature form as; $D^\alpha x_n = \frac{1}{h^\alpha}\sum_{j=0}^n w_j x_j, \quad 0 \leq \alpha < 1$ where w_0, w_{n-j} and w_n are weights Ref.[3] for details). Where n = Total no. of time steps ; h = time step size. Acceleration is approximated by central difference. Substituting this value of acceleration in Eq(2), $\frac{m}{h^2}(x_{n+1} - 2x_n + x_{n-1}) + \frac{b}{h^\alpha}\sum_{j=0}^n w_j x_j + kx_n = f(nh)$ is obtained. This renders a multistep numerical scheme:$\bar{w}_{n+1}x_{n+1} = f(nh) - \sum_{j=0}^n \bar{w}_j x_j$ (Ref.[2] for details).

Laplace Domain

From Eqn(3) the transfer function of the system is written as H(s)=$(ms^2+bs^\alpha+k)^{-1}$; As $X(s) = H(s)F(s)$, Taking taking inverse laplace transform of $X(s)$ will give time history of response($X(t)$).

4. Numerical Results and Disscussion

The numerical analysis are carried out on SDOF system defined by Eqn(2) for sinusoidal excitation($sin\Omega t$). Numerical values taken for analysis are: mass(m) = 1.0, k = 1.0, b = 0.1, α=0.5, h = 0.5 sec, Ω=1.0 rad/s.

Time Domain

The evaluation of quadrature weights at a particular step recalls all the previous step, which is commonly known as memory characteristic of VE material. Now if the time step size is small, the computational time becomes large. To overcome this problem a algorithm proposed by Koh & Kelly (Ref.[2]), which facilitate how many previous step are to be recalled as $D^\alpha x_n = \frac{1}{h^\alpha}\sum_{j=0}^{N} w_j x_{n-N+j}, \quad 0 \le \alpha < 1.$ where at n^{th} step; N = no. of previous steps to be recalled; w_0, w_{n-j} and w_n are weights (Ref.[3] for details). Now to estimate the number of previous step to be recalled different researchers have their own thumb rule. Ref.[3]. We have used the no. of step to be recalled(N) by assigning the relative error. So the no. of steps to be recalled(N) are calculated by using a formula: no.of step to be recalled(N)$\approx$ $\alpha^2/2$ relative error Ref.[4]. So for α=0.5 and 1% relative error, renders N =12. Results of this analysis has been shown in the figure below.

Laplace Domain

The laplace domain solution of the system requires solving the characteristics equation expressed as

$$s^2 + 0.1s^{0.5} + 1 = 0 \quad \cdots\cdots (4)$$

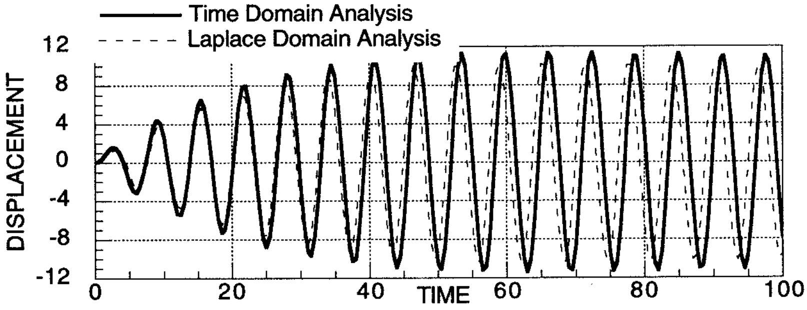
Since the Equ(4) involves fractional power to s, for mathematical simplicity, it is to be converted into an integer. Therefore Equ(5) is written as

$$s^4 + 2s^2 + 0.01s^{0.5} + 1 = 0 \quad \cdots\cdots (5)$$

However it should be notified that in writing Equ(5) the order of the system has been increased and thus the solution will involve some "false poles" related to the added higher order of s due the transformation from Eqn(4) to Equ(5). It is the designers skill to seperate the false poles and choose the correct one(Ref.[1] for details). The poles for the current problem is obtained as

$$s = -0.0353443 \pm 1.03537i \quad \cdots\cdots (6)$$

Now, the inverse laplace transform of Equ(3), with the poles given by Equ(6) gives the time history of response as $X(t) = 9.489e^{-0.0353t}cos(1.035t + 0.768) - 9.824cos(t + 0.8031252)$ and is plotted in figure along with the solution obtained by time domain analysis.



Comparison of Time and Frequency Domain Analysis

5. Conclusion

A comparison of response in time domain as well as frequency domain has been presented as shown is figure above. The solution from both the approaches are very close to each other. Using the FDM the equations of motion of a VE damped structure can be solved in a reasonably straight forward manner. But their is a need to develop a robust numerical scheme, which can address the problem associated with the convergence and memory parameter.

References

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