II－92　DEVELOPMENT OF WASTE DISPOSAL MANAGEMENT MODEL USING FINITE ELEMENT TECHNIQUE AND LINEAR PROGRAMMING OPTIMIZATION

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1. Introduction

Mathematical models have been successfuly applied in water quality management. These models normally provide some *necessary information for* decision makers who are in charge of water pollution control. In this study, a mathematical model is developed to compute the maximum loadihg of a selected substance which can be discharged into a number of river segments and still maintains the substance concentrations at some identified locations in the river within the specified limits. The finite element technique is used in transforming the substance balance equation which is in the form of partial differential equation to a set of algebraic equations. From the obtained finite element equations a set of constraint equations are formulated using matrix algebra. These constraint equations together with the objective function which is to maximize the total substance loading will form a linear programming model. This model can be solved by the Simplex method to obtain the optimal loading in each river segment.

2. Governing Equation

A vertical average two-dimensional substance balance equation (Pritchard, 1971) is used to describe dispersion of substance in a river. This equation is written as

$$\frac{\partial b}{\partial t} + u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} - \frac{1}{h}\left(\frac{\partial}{\partial x}(hD_x\frac{\partial b}{\partial x}) + \frac{\partial}{\partial y}(hD_y\frac{\partial b}{\partial y})\right) + k.b - R_b = 0 \qquad (1)$$

in which b is substance concentration, u and v are flow velocity in the x- and y-direction, respectively, h is water depth, D_x and D_y are dispersion coefficient, k is decaying rate, R_b is substance loading per unit volume.

In model formulation, two types of boundary are classified; namely So-boundary where substance concentration is specified and Sc-boundary where discharge flux is specified.

3. *Formulation of Finite Element Model*

The Galerkin weighted residual method is used in the formulation of substance dispersion finite element equations. The exacted solution is approximated by a trial function and the residual is forced to zero in an average sense, i.e. a weighting function is introduced and the inner product of the residual and the weighting function is set to zero (Zienkiewicz, 1977). In the Galerkin method the same interpolation function is used for the trial function and the weighting function. The weighted residual equation for substance dispersion is written as

$$\iint_\Omega W_b\left[\frac{\partial \hat{b}}{\partial t} + u\frac{\partial \hat{b}}{\partial x} + v\frac{\partial \hat{b}}{\partial y} - \frac{1}{h}\left(\frac{\partial}{\partial x}(hD_x\frac{\partial \hat{b}}{\partial x}) + \frac{\partial}{\partial y}(hD_y\frac{\partial \hat{b}}{\partial y})\right) + k\,\hat{b} - R_b\right]dA = 0 \qquad (2)$$

where $\hat{b}$ is approximated value of b; W_b is the weighting function.

By applying Gauss-Green theorem, Eq.(2) is transformed to

$$\iint_\Omega W_b\left[\frac{\partial \hat{b}}{\partial t} + u\frac{\partial \hat{b}}{\partial x} + v\frac{\partial \hat{b}}{\partial y} - \frac{D_x}{h}\frac{\partial h}{\partial x}\frac{\partial \hat{b}}{\partial x} - \frac{D_y}{h}\frac{\partial h}{\partial y}\frac{\partial \hat{b}}{\partial y} + k_1\hat{b} - R_b\right]dA$$
$$+ \iint_\Omega \left[D_x\frac{\partial W_b}{\partial x}\frac{\partial \hat{b}}{\partial x} + D_y\frac{\partial W_b}{\partial y}\frac{\partial \hat{b}}{\partial y}\right]dA - \int_s W_b\left[D_x\frac{\partial \hat{b}}{\partial x}dy - D_y\frac{\partial \hat{b}}{\partial y}dx\right] = 0 \qquad (3)$$

The term $\int_s [D_x\frac{\partial \hat{b}}{\partial x}dy - D_y\frac{\partial \hat{b}}{\partial x}dx]$ represents substance discharge flux normal to the boundary and can be written as $\int_s Q_b\, dL$. Moreover, since the discharge flux along the Sc-boundary is not specified, the weighting function W_b is selected such that its value along the Sc-boundary equals zero. So, the term $\int_s W_b[D_x\frac{\partial \hat{b}}{\partial x}dy - D_y\frac{\partial \hat{b}}{\partial y}dx]$ can be written as $\int_{sc} W_b Q_b^* dL$, where Q_b^* is the specified discharge flux. The approximated solution $\hat{b}$ and the weighting W_b are expressed in terms of nodal values as

$$\hat{b} = \mathbf{N}^T\mathbf{B} \qquad W_b = \mathbf{N}^T\delta\mathbf{B} \qquad (4)$$

where $\mathbf{N}$ is an interpolation function, $\mathbf{B}$ and $\delta\mathbf{B}$ are matrices of the nodal values of b and W_b, respectively.

Substitute these expressions into Eq.(4) and with some arrangement, we obtain

$$\delta\mathbf{B}^T\Big[\iint_\Omega \mathbf{N}\mathbf{N}^T dA\,\frac{\partial \mathbf{B}}{\partial t} + \iint_\Omega \Big\{u\mathbf{N}\frac{\partial \mathbf{N}^T}{\partial x} + v\mathbf{N}\frac{\partial \mathbf{N}^T}{\partial y} - \frac{D_x}{h}\frac{\partial h}{\partial x}\mathbf{N}\frac{\partial \mathbf{N}^T}{\partial x} - \frac{D_y}{h}\frac{\partial h}{\partial y}\mathbf{N}\frac{\partial \mathbf{N}^T}{\partial y} + k_1\mathbf{N}\mathbf{N}^T$$
$$+ D_x\frac{\partial \mathbf{N}}{\partial x}\frac{\partial \mathbf{N}^T}{\partial x} + D_y\frac{\partial \mathbf{N}}{\partial y}\frac{\partial \mathbf{N}^T}{\partial y}\Big\}dA\,\mathbf{B} - \iint_\Omega R_b\mathbf{N}\,dA - \int_{s_c} Q_b^*\mathbf{N}\,dL\Big] = 0 \qquad (5)$$

Since the weighted parameter $\delta\mathbf{B}$ *can be arbitrarily* selected, Eq.(5) will be valid only when the expression in the brackets vanishes. In the finite element method, the study domain is divided into a number of elements. The domain integral can be written as a sum of element integrals as follows.

$$\sum_{e=1}^{L}\Big[\iint_{A^e} \mathbf{N}^e\mathbf{N}^{eT} dA\,\frac{\partial \mathbf{B}^e}{\partial t} + \iint_{A^e}\Big\{u^e\mathbf{N}^e\frac{\partial \mathbf{N}^{eT}}{\partial x} + v^e\mathbf{N}^e\frac{\partial \mathbf{N}^{eT}}{\partial y} - \frac{D_x^e}{h^e}\frac{\partial h^e}{\partial x}\mathbf{N}^e\frac{\partial \mathbf{N}^{eT}}{\partial x} - \frac{D_y^e}{h^e}\frac{\partial h^e}{\partial y}\mathbf{N}^e\frac{\partial \mathbf{N}^{eT}}{\partial y}$$
$$+ k_1^e\mathbf{N}^e\mathbf{N}^{eT} + D_x^e\frac{\partial \mathbf{N}^e}{\partial x}\frac{\partial \mathbf{N}^{eT}}{\partial x} + D_y^e\frac{\partial \mathbf{N}^e}{\partial y}\frac{\partial \mathbf{N}^{eT}}{\partial y}\Big\}dA\,\mathbf{B}^e - \iint_{A^e}\mathbf{N}^e dA\,R_b^e - \int_{s_c}\mathbf{N}^e\,dL\,Q_b^*\Big]$$
$$= 0 \qquad (6)$$

or in more compact form as

$$\sum_{e=1}^{l}\left[M_o^e \frac{\partial B^e}{\partial t} + F_b^e B^e - M^e R_b^e - M_{qb}^e\right] = 0 \quad (7)$$

The element matrices are assembled to form system matrices. Finally, the substance dispersion finite element equations are

$$M_o \frac{\partial B}{\partial t} + F_b B - M R_b - M_{qb} = 0 \quad (8)$$

In the steady state case, the term with time derivative vanishes, so we obtain

$$F_b B - M R_b - M_{qb} = 0 \quad (9)$$

4. Formulation of Optimization Model

The obtained steady state finite element dispersion equations are in the form of linear algebraic equations. This enables formulation of linear constraint equations and thus linear programming optimization can be applied. The objective of this model is to determine the maximum loading of substance that can be discharged into a number of river segments and can still maintain substance concentrations at some identified locations within the specified limits. The objective function of this model can be written as

$$\text{Maximize} \quad z = \sum_{m} V^e R_{bc}^e = V^T R_{bc} \quad (10)$$

where R_{bc} is the controllable substance loading discharged into the e^{th} segment, V^e is volume of that river segment.

The constraint equations are

$$b_j \le b_j^* \quad \text{at any identified node } j. \quad (11)$$

The substance loading R_b can be classified as controllable and uncontrollable loadings and the matrix R_b is divided to R_{bc} and R_{bu}, respectively. Corresponding to R_{bc} and R_{bu}, the matrix M in Eq.(9) is divided into M_m and M_n, such that

$$M_m R_{bc} + M_n R_{bu} = M R_b \quad (12)$$

Then, Eq.(9) can be written as

$$F_b B - M_m R_{bc} - M_n R_{bu} - M_{qb} = 0 \quad (13)$$

The matrix B is also divided into 2 matrices, namely B_u and B_s, corresponding to nodal points with non-specified and specified substance concentrations, respectively. The matrix F_b is then divided into F_{bu} and F_{bs}, such that

$$F_{bu} B_u + F_{bs} B_s = F_b B \quad (14)$$

Substitute into Eq.(13), we obtain

$$F_{bu} B_u + F_{bs} B_s - M_m R_{bc} - M_n R_{bu} - M_{qb} = 0 \quad (15)$$

On the So-boundary, the substance concentration is specified and so the equations corresponding to those nodal points can be eliminated. This can be done by eliminating all corresponding rows in the matrices F_{bu}, F_{bs}, M_m, M_n and M_{qb}. Then, we obtain

$$F'_{bu} B_u + F'_{bs} B_s - M'_m R_{bc} - M'_n R_{bu} - M'_{qb} = 0 \quad (16)$$

The matrix B_u can be written as

$$B_u = F'^{-1}_{bu} \{ M'_m R_{bc} + M'_n R_{bu} + M'_{qb} - F'_{bs} B_s \} \quad (17)$$

or in more compact form

$$B_u = G_b R_{bc} + E_b \quad (18)$$

Then, the constraint equations become

$$B_{uj} = G_{bj} R_{bc} + E_{bj} \le B_j^* \quad (19)$$

where B_{uj} is the matrix of substance concentrations at some identified nodal points, B_j^* is matrix of the limiting concentrations. G_{bj} and E_{bj} are respectively obtained from the elements of G_b and E_b which are corresponded to the identified nodes.

In conclusion, the following optimization model is obtained

$$\text{Objective Function :} \quad \text{Maximize} \quad z = V^T R_{bc} \quad (20)$$

Subjected to :

$$G_{bj} R_{bc} + E_{bj} \le B_j^* \quad (21)$$

and

$$R_{bc} \ge 0 \quad (22)$$

The Simplex method is then applied to solve for the optimal loading R_{bc} which is allowed to discharge into each river segment and still satisfies the specified constraints.

5. References

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(3) B.G. Galerkin, Vestn. Inzh. Tech., Vol.19, pp.897-908, 1915.