CS 4

COMPARISION OF NUMERICAL SCHEMES FOR DAM-BREAK PROBLEM

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1. INTRODUCTION: Two recently developed first order accurate implicit schemes[1,2] based on split flux are compared with MacCormack scheme[3] with their application to the dam-break problem. MacCormack scheme is conservative but employs indiscriminate space differencing while the other two schemes use semi-conservative and fully conservative flux splitting technique.

2. GOVERNING EQUATIONS: The equations governing one-dimensional unsteady open channel flow are expressed in conservation law form as

$$\mathbf{U}_t + \mathbf{E}_x + \mathbf{S} = 0 \quad \ldots\ldots (1)$$

where, subscripts t and x denote partial derivative with respect to time and space respectively. Vectors **U**, **E** and **S** are

$$\mathbf{U} = \begin{bmatrix} h \\ uh \end{bmatrix};\ \mathbf{E} = \begin{bmatrix} uh \\ u^2.h + 0.5.gh^2 \end{bmatrix}, \text{and } \mathbf{S} = \begin{bmatrix} 0 \\ -gh(S_o - S_f) \end{bmatrix} \quad ..(2)$$

wherein, h = flow depth, u = velocity , g = acceleration due to gravity; S_o, S_f= bed slope and friction slope respectively. S_f is computed by Manning's formula. The non-conservation form of the governing Eq.(1) can be written by expressing **E** in terms of its Jacobian **A** with respect to **U** as

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x + \mathbf{S} = 0 \quad \ldots\ldots (3)$$

where, the Jacobian **A** in its diagonalized form is as follows

$$\mathbf{A} = \frac{1}{2c}\begin{bmatrix} 1 & -1 \\ u+c & -(u-c) \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} -(u-c) & 1 \\ -(u+c) & 1 \end{bmatrix} \quad \ldots\ldots (4)$$

where, $c = (gh)^{1/2}$ and λ_i's are eigenvalues of **A** giving the characteristic directions and are expressed as $\lambda_1 = u+c$ and $\lambda_2 = u-c$. Now, matrix **A** can be split into two components, positive and negative, by testing sign of the eigenvalues. Thus, the governing equation in split flux form can be written as

$$\mathbf{U}_t + \mathbf{A}^+\mathbf{U}_x + \mathbf{A}^-\mathbf{U}_x + \mathbf{S} = 0 \quad \ldots\ldots (5)$$

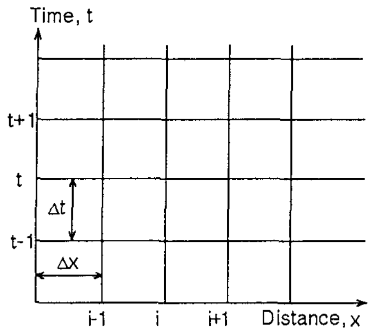

Fig.1 Finite Difference Grid

3. FINITE DIFFERENCE SCHEMES: The time derivative for both the models are approximated by forward time difference. The space derivatives are evaluated by either a backward or forward space difference depending on whether it is associated with positive or negative component of **A** respectively. Following difference operators are defined for writing finite difference equations

$$\Delta_x f_i = f_{i+1} - f_i\ ,\ \nabla_x f_i = f_i - f_{i-1} \quad \text{and} \quad \nabla_t f_i = f_i^{t+1} - f_i^t \quad \ldots.(6)$$

where, superscripts and subscripts in Eq.(6) stand for location in time and space respectively (*Fig.1*).

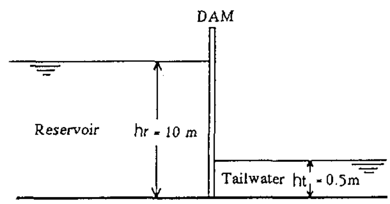

Fig.2 Dam-Break Problem

Model I: The continuity equation in Eq.(5) is evaluated conservatively although matrix **A** appears outside of the space derivative. However, the momentum equation is evaluated non-conservatively. The complete implicit finite difference equation for continuity and momentum equation can be written as[1]

$$\nabla_t h_i + \alpha\left\{\nabla_x\left[(A_{11}^+)^t h^{t+1}\right]_i + \Delta_x\left[(A_{11}^-)^t h^{t+1}\right]_i + \nabla_x\left[(A_{12}^+)^t (uh)^{t+1}\right]_i + \Delta_x\left[(A_{12}^-)^t (uh)^{t+1}\right]_i\right\} = 0 \quad \ldots.(7)$$

$$\nabla_t (uh)_i + \alpha\left\{(A_{21}^+)_i^t \nabla_x h_i^{t+1} + (A_{21}^-)_i^t \Delta_x h_i^{t+1} + (A_{22}^+)_i^t \nabla_x (uh)_i^{t+1} + (A_{22}^-)_i^t \Delta_x (uh)_i^{t+1} - gh_i^{t+1} S_o\right\} + \Delta t.\, g(uh)_i^{t+1}\left[\frac{n^2|u|}{h^{4/3}}\right]_i^t = 0 \quad \ldots(8)$$

where, $\alpha = \Delta x/\Delta t$, Δx is grid interval and Δt is the time step.

Model II: This is one parameter scheme which can be from fully explicit to fully implicit. To evaluate the Jacobians in Eq.(5) conservatively, Roe's[4] technique of approximate Jacobian is used. Therefore, $\mathbf{A}^+$ is evaluated at (i-1/2) and $\mathbf{A}^-$ at (i+1/2).The velocities at half grid point (i+1/2) or (i-1/2) are computed as the square root averaging of velocities at node i and (i+1) or (i-1) respectively. The depths at half grid points are computed as the geometric mean of the depths at adjacent nodes. With these considerations the implicit finite difference scheme is written as[2]

$$\mathbf{U}_i^{t+1} + \alpha\theta\left\{\mathbf{A}_{i-1/2}^{+}\nabla_x\left[\mathbf{U}_i^{t+1}\right] + \mathbf{A}_{i+1/2}^{-}\Delta_x\left[\mathbf{U}_i^{t+1}\right]\right\} = \mathbf{U}_i^{t} + \alpha(1-\theta)\left\{\mathbf{A}_{i-1/2}^{+}\nabla_x\left[\mathbf{U}_i^{t}\right] + \mathbf{A}_{i+1/2}^{-}\Delta_x\left[\mathbf{U}_i^{t}\right]\right\} + \mathbf{S}_i^{t} \quad .....(9)$$

where, θ is time weighting factor.

4. RESULTS AND DISCUSSION: Model I, Model II and MacCormack scheme is applied to the one-dimensional flood wave propagation problem due to instantaneous collapse of dam (Fig.2). The horizontal and frictionaless channel is assumed to be rectangular in shape. The depth ratio, DR, defined as the ratio of tailwater depth, h_t to the reservoir depth, h_r, is specified as 0.05. At this depth ratio the flow downstream of the breach becomes supercritical while the flow upstream of the breach remains subcritical. Model I and Model II are run at a Courant number of unity and the time weighting factor, θ for Model II is taken as 0.6. Since MacCormack scheme is explicit it must satisfy the Courant stability criteria. Therefore, MacCormack scheme is run at a Courant number of 0.95. The results for all the models are obtained at time 30+Δt seconds. The depth profiles along the channel as computed by the three models are shown in Fig.3 along with the analytical solution by Stoker method.

It can be seen from Fig.3 that MacCormack scheme fails to simulate this problem. It exhibits a sudden drop in depth of flow near the breach which invalidates the solution. The reason for MacCormack scheme's failure is that it does not take into account the different direction of signal propagation in upstream and downstream of the breach as the flow changes from subcritical to supercritical. Model I and Model II, incorporating flux splitting technique, do not blow-up in this situation. However, Model I gives slower wave front celerity and higher height as seen from its comparison with analytical solution. This may be because the momentum equation is solved non-conservatively. When conservative flux splitting technique of Roe[4] is employed in Model II and both continuity and momentum equations are solved conservatively the results improve considerably. The wave front celerity and the front height is very well simulated by Model II.

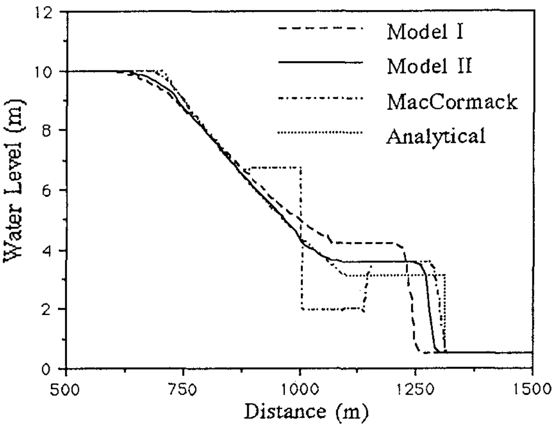

Fig.3 Water Surface Profile Along the Channel

5. CONCLUSION: It is important that the finite difference schemes take into account the directional property of signal propagation, specially for such problem as dam-break where both subcritical and supercritical flows may be present simultaneously. The difference on this account between MacCormack and two new schemes is obvious by the formers inability to handle the problem presented in this study. It is also important for dam-break problem that the conservative property is maintained while employing flux splitting technique. This is noted from the fact that the Model II employing conservative flux splitting technique gives the best result.

6. REFFERENCES:

1. Jha et al ,(1993), Proc. of Hyd. Engg., JSCE, Vol. 37, pp.745-750.
2. Jha et al, (under preparation)
3. MacCormack (1969), Amer. Inst. Aero. Astro. Paper 69-354, Cincinnati, Ohio.
4. Roe, P.L.(1981), Journal. of Comp. Phy., Vol. 43, pp.357-372.