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APPLICATION OF BOUNDARY INTEGRAL EQUATION METHOD TO CONTROL PROBLEMS FOR WAVE EQUATION

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Introduction

The time domain boundary integral equation method (BIEM) has so far been applied successfully to various problems governed by the wave equation. We now attempt to investigate further applications of this method in the so called boundary control problems, where BIEM is expected to be ideal as a tool of numerical analysis because of its 'boundary only' nature.

The control problem under consideration is the following: Suppose that a field u defined in $D \times (t > 0)$ satisfies the wave equation and initial conditions:

$$\Delta u = \ddot{u} \quad \text{in } D \times (t > 0), \qquad u|_{t=0} = u_0, \quad \dot{u}|_{t=0} = u_1 \quad \text{in } D, \tag{1 - 3}$$

where $\dot{} := \partial/\partial t$ and (u_0, u_1) are given functions, respectively. We are now interested in steering u to rest at a given time $T(> 0)$ by prescribing an appropriate Dirichlet boundary condition

$$u = f \quad \text{on } \partial D \times (0, T). \tag{4}$$

Control Strategies

Lions[1] showed that this problem can be solved in the following manner: Let (e_0, e_1) be a pair of functions defined in D such that $e_0 = 0$ holds on ∂D. One solves the wave equation for ϕ in $D \times (0, T)$ with the initial conditions $\phi|_{t=0} = e_0$, $\dot{\phi}|_{t=0} = e_1$ in D and the homogeneous Dirichlet boundary condition. Subsequently, one solves the wave equation for ψ backward from $t = T$ to $t = 0$ with the homogeneous initial conditions at $t = T$ and a Dirichlet boundary condition given by $\psi = \partial\phi/\partial n$ on $\partial D \times (0, T)$. These steps define a linear operator Λ given by $\Lambda(e_0, e_1) = \left(\dot{\psi}|_{t=0}, -\psi|_{t=0}\right)$. One then solves

$$\Lambda(e_0, e_1) = (u_1, -u_0) \tag{5}$$

for (e_0, e_1) and utilises the solution (e_0, e_1) thus obtained to determine the control f by $f = \partial\phi/\partial n$. This method is called the Hilbert Uniqueness Method (HUM), and has been tested numerically by Glowinski et al.[2] with the help of FEM.

As noticed in [1], HUM is not the only way to solve this control problem. In the cases of odd spatial dimensions, for example, one may have the following choice[1]:

$$f(\boldsymbol{x}, t) = \int_D G(\boldsymbol{x} - \boldsymbol{y}, t) u_1(\boldsymbol{y}) dV_y + \int_D \frac{\partial G(\boldsymbol{x} - \boldsymbol{y}, t)}{\partial t} u_0(\boldsymbol{y}) dV_y, \tag{6}$$

where G is the fundamental solution of the wave equation. This control is obtained assuming that the boundary ∂D is completely transmitting. In practice, however, one may possibly be satisfied by making u small, rather than driving u to an absolute rest. In this case the control in (6) may be used in 2D problems as well.

Numerical Analysis

We now use the time domain BIEM to solve the 2D control problem. We discretise (6) by

$$< \{u_1, -u_0\}, \hat{\boldsymbol{e}} > \; = \; < \Lambda \boldsymbol{e}, \hat{\boldsymbol{e}} > \; = \int_0^T \int_{\partial D} \frac{\partial \phi}{\partial n} \frac{\partial \hat{\phi}}{\partial n} dS dt,$$

where $<,>$ stands for (roughly speaking) the inner product, $\hat{e}$ is a base function for (e_0, e_1) and $\hat{\phi}$ indicates the ϕ corresponding to $\hat{e}$. This formula eliminates the need of the backward analysis in HUM.

We now compare the numerical performances of HUM and the control based on (6) by using simple numerical examples. Figure 1 shows an example of such comparison. In this example the domain D is assumed to be circular with the radius of $r_0 = 15$. The time domain collocation BIEM with spatially linear shape functions is used with all the necessary integrations carried out analytically. The time shape functions are taken to be piecewise linear (constant) in single (double) layer potentials. As the initial conditions we set $u_0 = 0$, $u_1 = \omega J_0(\omega|\boldsymbol{x}|)$, $\omega = j_{0,1}/r_0$, where J_0 is the Bessel function of the 0th order and $j_{0,1}$ is the smallest zero of J_0, respectively. Also, we put T to be equal to 200 arbitrarily. The computed values of u at the centre of D are plotted in this figure. The thin line indicates u with $f = 0$, the thick line the results of HUM and the intermediate line the consequence of the control given by (6). As this picture shows the control with HUM gives rise to an oscillatory behaviour of the solution. The control with (6), on the other hand, yields a quick decay initially, but becomes less effective as the time increases.

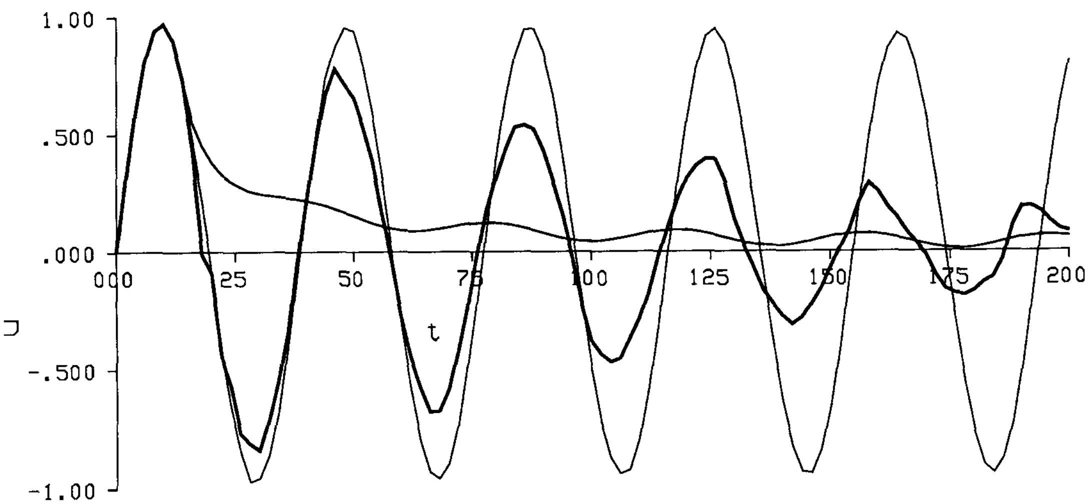

Figure 1. u with various boundary controls vs t.

References

[1] J. L. Lions, Exact controllability, stabilization and perturbations for distributed systems, SIAM Review, 30, 1–68, 1988.

[2] R. Glowinski, C.-H. Li and J.L. Lions, A numerical approach to the exact boundary controllability of the wave equation (I) Dirichlet controls: description of the numerical methods, Japan J. Appl. Math., 7, 1–76, 1990.