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STATIC ANALYSIS OF AN EMBEDDED SYMMETRICAL TWO-HINGED CIRCULAR ARCH UNDER VERTICAL LOADING CONDITION

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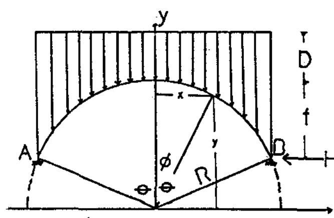
1. Introduction

As a result of soaring land prices and due to the scarcity of land, particularly in overpopulated cities, construction of human habitation under ground or under water has become the recent focus of research. Having a vast open space, a dome has been thought of serving as an enclosure to such type of future dwellings. This dome can be composed of intersecting circular arches as the supporting structural members. As such, in this paper, being the most fundamental part of the research, formulation of a general equation for the redundant horizontal reaction, H, of an embedded two-hinged circular arch under vertical loading using the classical Castigliano's theorem is presented.

2. Idealization

In Fig. A, consider the embedded circular arch represented by the parametric equations $x = R\sin\phi$ and $y = R\cos\phi$, where ϕ = angle that a tangent at an arbitrary point on the arch axis makes with respect to the horizontal ($-\theta \le \phi \le \theta$), and θ is the corresponding angle at the support ($-\pi/2 \le \theta \le \pi/2$). Likewise, from the figure, D = depth of embedment, f = rise = $R(1-\cos\theta)$, $2(l)$ = horizontal span and R = radius of curvature.



The Idealized Embedded Arch
Figure A

The horizontal reaction H at support B is considered to be the redundant force. Using the classical Castigliano's theorem, the horizontal deflection at B must vanish, i.e.,

$$\frac{\partial U}{\partial H} = 0 \; ; \qquad U = U_M + U_N + U_V \, ,$$

$$U_M = \int_0^s \frac{M^2}{2EI}\,ds \; , \quad U_N = \int_0^s \frac{N^2}{2AE}\,ds \quad \text{and} \quad U_V = \int_0^s \frac{V^2}{2GFA}\,ds \tag{1}$$

where U = total strain energy, M = bending moment, N = axial force, V = shear force, E = modulus of elasticity, I = moment of inertia, A = cross-sectional area, G= shear modulus of elasticity, F= shape factor, s= arc length and $ds = Rd\phi$.

3. Basic Derivation

Let N', M' and V' be the axial force, bending moment and shear force, respectively, of a simply-supported beam having the same span as the arch considered.

From (a), (b) and (c) of Figure B, the following equations can be obtained:

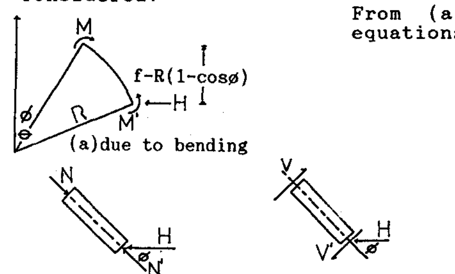


Figure B. Basic Derivation Diagram

$$M + H[f - R(1-\cos\phi)] = M' \qquad \frac{\partial M}{\partial H} = -[f - R(1-\cos\phi)]$$

$$N = N' + H\cos\phi \qquad \frac{\partial N}{\partial H} = \cos\phi$$

$$V = V' + H\sin\phi \qquad \frac{\partial V}{\partial H} = \sin\phi \tag{2}$$

Manipulating equations (1) and (2) and solving for H,

$$H = -\left[\frac{\frac{R}{EI}\int_{-\theta}^{\theta} M'(-f + R - R\cos\phi)d\phi + \frac{R}{AE}\int_{-\theta}^{\theta} N'\cos\phi d\phi + \frac{R}{GFA}\int_{-\theta}^{\theta}(V'\sin\phi)d\phi}{\frac{R}{AE}\int_{-\theta}^{\theta}\cos^2\phi d\phi + \frac{R}{EI}\int_{-\theta}^{\theta}(f^2 - 2fR + R^2 + 2fR\cos\phi - 2R^2\cos\phi + R^2\cos^2\phi)d\phi + \frac{R}{GFA}\int_{-\theta}^{\theta}\sin^2\phi d\phi}\right] \tag{3}$$

Simplifying the denominator and calling it K,

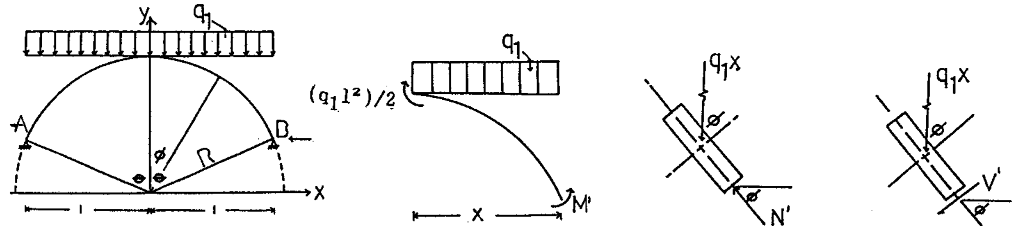
$$K = \left(\frac{R}{AE} + \frac{R^3}{EI}\right)(\theta + sc) + \frac{(f-R)}{EI}\,[2\theta R(f - R) + 4R^2 s] + \frac{R}{GFA}(\theta - sc) \tag{4}$$

where $s = \sin\theta$ and $c = \cos\theta$

4. Theoretical Analysis and Results

The analysis is divided into two cases. The first case, as shown in (a) of Fig. C, considers the uniform horizontal loading $q_1 = \gamma Db$, where γ = unit weight of the surrounding medium, b = cross-sectional width of the arch(equals diameter, if the cross section is a solid circle). The second case, as shown in (a) of Fig. D considers the loading $q_2 = \gamma bR(1-\cos\phi)$.

4.1 Case 1



(a)the idealized embedded arch (b)due to bending (c)due to axial force (d)due to shear

Figure C. Case 1

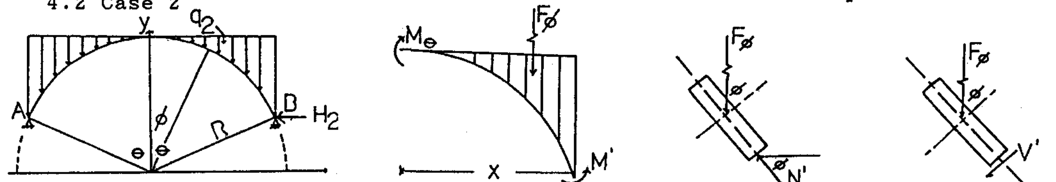
From figures (b), (c) and (d) above,

$$M' = \frac{q_1}{2}[l^2 - R^2\sin^2\phi], \qquad N' = q_1 R\sin^2\phi \quad \text{and} \quad V' = -q_1 R\sin\phi\cos\phi \tag{5}$$

Manipulating equations (3), (4) and (5) and solving for H_1,

$$H_1 = -\left[\frac{\frac{2}{3}q_1 R^2 S^3\left(\frac{1}{AE} - \frac{1}{GFA}\right) - \frac{q_1 R^4}{2EI}\left(\frac{4}{3}s^3 - 2\theta s^2 c + \theta c - sc^2\right)}{K}\right] \tag{6}$$

4.2 Case 2



(a)the idealized embedded arch (b)due to bending (c)due to axial force (d)due to shear

Figure D. Case 2

From figures (b), (c) and (d) above,

$$M' = M_\theta - \frac{\gamma bR^3}{2}\left[\frac{5}{3} - \cos^2\phi - \phi\sin\phi - \cos\phi + \frac{\cos^3\phi}{3}\right]$$

where

$$M_\theta = \frac{\gamma bR^3}{2}\left[\frac{5}{3} - \cos^2\theta - \theta\sin\theta - \cos\theta + \frac{\cos^3\theta}{3}\right] \tag{7}$$

also,

$$V' = -\gamma bR^2\left[\sin\phi\cos\phi - \frac{\phi\cos\phi}{2} - \frac{\sin\phi\cos^2\phi}{2}\right] \quad \text{and} \quad N' = \gamma bR^2\left[\sin^2\phi - \frac{\phi\sin\phi}{2} - \frac{\sin^2\phi\cos\phi}{2}\right]$$

Manipulating equations (3), (4) and (7) and solving for H_2,

$$H_2 = -\left\{\gamma bR^3\left(\frac{1}{AE} - \frac{1}{GAF}\right)\left[\frac{2}{3}s - sc\left(\frac{2}{3}c + \frac{3}{8} - \frac{c^2}{4}\right) + \frac{\theta}{2}\left(c^2 - \frac{3}{4}\right)\right] + \frac{\gamma bR^5}{72EI}\left[3\theta(9 - 64c^2 - 12c + 24s^2c - 8s^2c^2) + sc(155 + 84c - 26c^2 - 72\theta^2) - 48s\right]\right\} / K \tag{8}$$

Thus, by superimposing cases 1 and 2, $H = H_1 + H_2$. From this result, other reactions can be obtained using the three equations of statics.

5. Conclusions

Formulation of the general expression for the redundant horizontal reaction at the right support of an embedded symmetrical two-hinged circular arch under vertical loading condition using Castigliano's theorem has been thoroughly presented, serving as the most fundamental analysis. As such, further analysis considering horizontal forces, together with the dynamic effect of water waves or ground motions, depending upon whether the structure is to be embedded under water or under ground, has to be taken into account for future studies. Finally, integration of intersecting arches into the final embedded structural dome is another point for future consideration.

6. References

Blake, Alexander, ed., Handbook of Mechanics, Materials and Structures, John Wiley and Sons, New York, 1985.

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