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EXPLICIT SOLUTION OF THE DOUBLE INTEGRAL IN TIME DOMAIN BEM

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INTRODUCTION

Comparing to the frequency domain BEM, researches directly on the time domain BEM are still new and fewer, although it has long been known that the time domain BEM may be more atractive, especially to nonlinear problems. Recently, among others, **Niwa** *et al*, **Cole** *et al*, **Antes** *et al*, **Fukui**, **Rice** *et al* and **Mansur** *et al* have investigated the dynamic responses of 2-D structures . In all these works, the convolution integrals with respect to time are evaluated analytically and the spacial integrations are carried out numerically at any time step. In this paper, explicit solutions of the double integrals are presented, for a straight boundary element and a time step with the interpolation functions of order 0,1,2 for both space and time. These solutions provide not only an accurate and efficient calculation but also a reliable base for examining the stability and accuracy of the time stepping algorithm, which is not yet well understood at present. The numerical solutions are computed and compared to the analytical ones, for a strip-line load with constant and impulsive time dependence on a half-space.

NUMERICAL IMPLEMENTATION FOR TIME-STEPPING BEM

Consider the two dimensional transient scalar wave problems in an isotropic, homogeneous, linearly elastic exterior domain D with the boundary B . For zero initial conditions and zero body forces, the boundary integral equation is

$$c(\mathbf{x})\mathbf{u}(\mathbf{x},\tau) = \int_B \mathbf{u}^*(\mathbf{y},\tau;\mathbf{x})*\mathbf{t}(\mathbf{y},\tau)\,dB(\mathbf{y}) - \text{p.v.}\int_B \mathbf{t}^*(\mathbf{y},\tau;\mathbf{x})*\mathbf{u}(\mathbf{y},\tau)\,dB(\mathbf{y}) \qquad \mathbf{x}\in B \tag{1}$$

By the discretization procedure, the boundary values are represented by distinct nodal values at each time step.

$$\{\ \mathbf{u}(\mathbf{y},\tau),\ \mathbf{t}(\mathbf{y},\tau)\ \} = \sum_{m=1}^{M}\sum_{n}\{\ \phi_u^m(\mathbf{y})\ \psi_u(\tau-n\Delta\tau)\ \mathbf{u}_n^m,\quad \phi_t^m(\mathbf{y})\ \psi_t(\tau-n\Delta\tau)\ \mathbf{t}_n^m\ \} \tag{2}$$

ϕ, ψ are the interpolation functions for space and time; $\mathbf{u}_n^m$, $\mathbf{t}_n^m$ are the displacement on m*th* node at n*th* step. When the time interpolation functions have at most $\pm\Delta\tau$ influence spane, substituting Eq.(2) to Eq.(1) leads

$$c(\mathbf{x}^k)\mathbf{u}_N^k = \sum_{m=1}^{M}\sum_{n=0}^{N-1}\int_B\int_{-\Delta\tau}^{\Delta\tau}\Big\{\mathbf{u}^*(\mathbf{y},n\Delta\tau-\tau;\mathbf{x}^k)\phi_t^m(\mathbf{y})\psi_t(\tau)\mathbf{t}_{N-n}^m-\mathbf{t}^*(\mathbf{y},n\Delta\tau-\tau;\mathbf{x}^k)\phi_u^m(\mathbf{y})\psi_u(\tau)\mathbf{u}_{N-n}^m\Big\}\,d\tau\, dB(\mathbf{y}) \tag{3}$$

or in its matix form:

$$(\ \mathbf{C}+\mathbf{H}_0\ )\ \mathbf{U}_N - \mathbf{G}_0\,\mathbf{T}_N = -\sum_{n=1}^{N-1}(\ \mathbf{H}_n\,\mathbf{U}_{N-n} - \mathbf{G}_n\,\mathbf{T}_{N-n}\ ) \tag{4}$$

We can see that the inversion of a $M\times M$ matrix needs to be carried out only once throughout the calculation.

ANALYTICAL EVALUATION OF THE DOUBLE INTEGRALS

It can be seen that the essential part of the time domain BEM is the evaluation of the double integral

$$\int_{-1}^{1}\int_{-\Delta\tau}^{\Delta\tau}\mathbf{g}(\ \mathbf{y}(\xi),t-\tau\,;\,\mathbf{x})\ \phi(\xi)\psi(\tau)\,d\tau\, d\xi \tag{5}$$

where $\mathbf{g}$ denotes the Green functions of displacement or traction. In the case of a straight line boundary element, the problem can be simplified if we manipulate the interpolation functions to a combination of the function $\xi^m\tau^n Sgn(\xi)H(\tau)$. For example, the interpolation function with order 0 in space and 1 in time can be written as

$$\big\{Sgn(\xi+1)-Sgn(\xi-1)\big\}\big\{(\tau+\Delta\tau)H(\tau+\Delta\tau)-2\tau H(\tau)+(\tau-\Delta\tau)H(\tau-\Delta\tau)\big\}/2\Delta\tau \tag{6}$$

Then, we only have to evaluate the integrals

$$\{\ P_m^n\ ,\ Q_m^n\ \} = \int_0^t\int_{-\infty}^{\infty}\Big\{\ \frac{1}{2\pi\sqrt{(t-\tau)^2-r^2}}\ ,\ \ \frac{\partial}{\partial z}\frac{-1}{2\pi\sqrt{(t-\tau)^2-r^2}}\ \Big\}\xi^m\tau^n Sgn(\xi)\,d\xi\, d\tau \tag{7}$$

in which x, z are the local coordinates of the source point; $r=\sqrt{(x-\xi)^2+z^2}$. The explicit solution of Eq.(7) can be obtained by a direct integration or integral transform method. The later one is recommended here. By use of the Cagniard-De Hoop method, the integrals can be reduced to a convolution integral with respect to time only. In this paper both methods are employed and justified by each other. The solutions are of the form

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$$P_m^n = Sgn(x^{m+1})\ \{A_0^{mn}H(t-z) + \frac{1}{\pi}[\ A_1^{mn}f_1 + A_2^{mn}f_2 + A_3^{mn}f_3 + A_4^{mn}f_4\ ]\ H(t-r)\} \tag{8}$$

$$Q_m^n = Sgn(x^{m+1}z)\ \{B_0^{mn}H(t-z) + \frac{1}{\pi}[\ B_1^{mn}f_1 + B_2^{mn}f_2 + B_3^{mn}f_3 + B_4^{mn}f_4\ ]\ H(t-r)\} \tag{9}$$

in which

$$f_1 = \tan^{-1}\frac{\sqrt{t^2-r^2}}{x}\ ,\quad f_2 = \tan^{-1}\frac{z\sqrt{t^2-r^2}}{xt}\ ,\quad f_3 = \cosh^{-1}\frac{t}{r}\ ,\quad f_4 = \sqrt{t^2-r^2} \tag{10}$$

The coefficients A and B are multinormal functions of x, z and t. The solutions of P_m^n and Q_m^n for m,n=0,1,2 are obtained. From them, the solutions of the double integrals for the interpolation functions of order 0,1,2 can be constructed. For example, for Eq.(6), the solution for the displacement Green function is given by

$$[P_0^1(x+1,t+\Delta\tau)-P_0^1(x-1,t+\Delta\tau)-2P_0^1(x+1,t)+2P_0^1(x-1,t)+P_0^1(x+1,t-\Delta\tau)-P_0^1(x-1,t-\Delta\tau)]/2\Delta\tau$$

NUMERICAL EXAMPLE

The analytical solutions are applied to calculate the displacements at the surface of a half-space to a strip-line load with constant and impulsive time dependence. In this example, a strip of 2 unit legnth is divided into 61 elements. 150 time steps with the increment $\Delta\tau$=0.15 are computed. The solutions are plotted in Fig.1 for the impulsive load. and in Fig.2 for the force of constant time dependence; In Fig.1 , Fig.2, the circles denote the numerical solutions and the solid lines the analytical ones.

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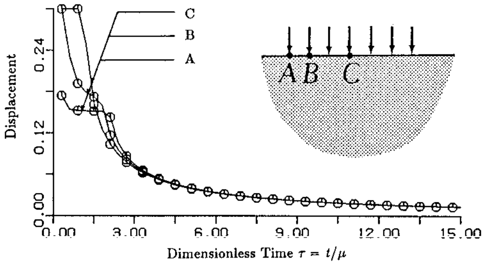

Fig.1 Displacement for Impulsive Loading

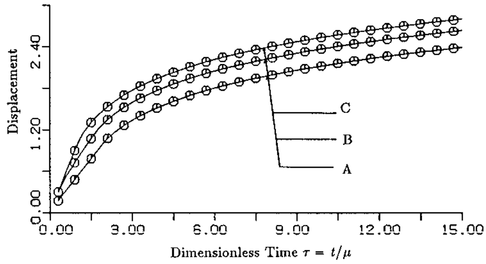

Fig.2 Displacement for Constant Loading