III-107 SLIP-LINE ANALYSIS OF GRANULAR MATERIALS USING OBLIQUE COORDINATES SYSTEM

Tohoku University,Student member,A.H.Meiz.
Tohoku University,member,M.Satake.

ABSTRACT

An alternative derivation for the kinematic equations govering a certain class of flows (double shearing flows) in the plastic deformation of dilatant granular materials that given by Mehrabadi and Cowin (1978) using oblique coordinates system is presented. Also,an expression is derived which enables the rate of energy dissipation to be calculated for any pair of stress and velocity fields in a plane-strain which satisfies the stress equilibrium equations and Coulomb yield criterion.

1. INTRODUCTION

A kinematic theory for the initial planar deformation of dilatant granular materials based on a kinematic proposal of Butter field and Harkness (1972) was developed by Mehrabadi and Cowin (1978) using orthogonal coordinates system.As it is well known that the angle between the slip-lines is not equal to ($\pi/2$),so it may be reasonable to use a coordinate system in which the axes are inclined by an angle equal to the angle between the slip-lines. Considering a plane-strain deformation of a granular material,the components of the stress tensor σ referred to an oblique cartesian coordinates system (ξ , η) satisfy Coulomb yield criterion as shown in Figure (1) are given by

$$\begin{aligned}\sigma_\xi &= \frac{1}{\cos\phi}\left[-P+q\cos 2\psi\right]\\ \sigma_\eta &= \frac{1}{\cos\phi}\left[-P+q\cos 2(\psi-\alpha)\right]\\ \tau_{\xi\eta} &= \frac{1}{\cos\phi}\left[P\sin\phi+q\cos(2\psi-\alpha)\right]\end{aligned} \tag{1}$$

where c and ϕ are the cohesion and the angle of internal friction of the material,and α is the angle between (ξ , η)-axes. Assuming the weight of the material is negligible,the equilibriuum equations will be

$$\begin{aligned}\frac{\partial\sigma_\xi}{\partial\xi}+\frac{\partial\tau_{\xi\eta}}{\partial\eta}&=0\\ \frac{\partial\tau_{\xi\eta}}{\partial\xi}+\frac{\partial\sigma_\eta}{\partial\xi}&=0\end{aligned} \tag{2}$$

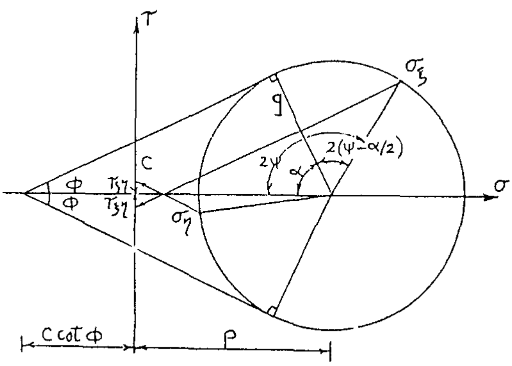

Fig.1. Presentation of the stresses in oblique coordinates on Mohr-circle.

2. DERIVATION OF VELOCITY EQUATIONS

In the slip-line analysis,there are two slip lines (α-,β-lines),the angle between them is always constant and equal the angle between the proposed oblique axes(ξ ,η) which is ($\pi/2$)+(ϕ). The slip lines passing through a typical point Q are illustrated in Figure (2). If the stress characteristics are obtained from the equations of equilibrium and Coulomb condition,then ψ ,represents the angle of inclination of the direction of algebrically-greater principal stress to the ξ axis.

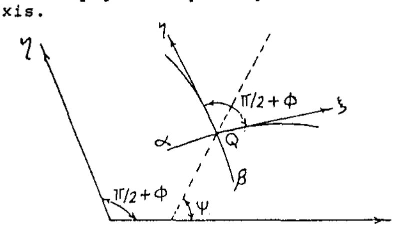

Fig.2. The two slip-lines passing through a typical point Q(t).

At a typical point Q,we introduce an oblique coordinate system (ξ , η) which in motion relative to the reference (ξ , η)-system. The positions of the coordinate system at two consecutive instants of time are shown in Figure (3).

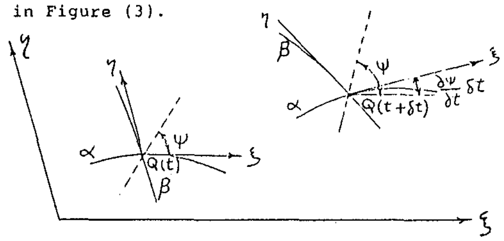

Fig.3. An illustration of the positions of the (ξ,η)-coordinates relative to the (ξ,η)-coordinates

We introduce a velocity vector $\underline{v}$ to denote the velocity of the particles relative to the (ξ , η)-coordinate system. The components of $\underline{v}$ are v_ξ and v_η .The kinematic proposal of Butter-field and Harkness is then that, the relative velocity in the (ξ , η)-frame of two successive points near Q along the α-line (β-line) is in the B-direction (A-direction). The A-,B-directions are illustrated in Figure (4),where ν is called the angle of dilatancy.

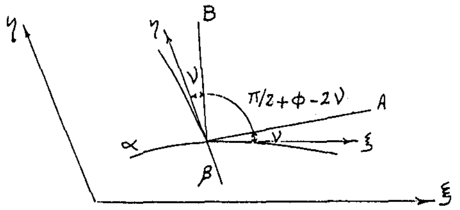

Fig.4. An illustration of the A- and B-directions associated with the α- and β- lines,respectively.

If s_α, s_β and $\grave{v}_\alpha, \grave{v}_\beta$ are length parameters and the components of $\grave{v}$ along the α-and β-lines respectively,then

$$\grave{v}_\xi = \grave{v}_\alpha \quad , \quad \grave{v}_\eta = \grave{v}_\beta \tag{3}$$

$$\frac{\partial \grave{v}_\eta}{\partial s_\beta} \Big/ \frac{\partial \grave{v}_\xi}{\partial s_\beta} = \frac{\sin\nu}{\cos(\phi-\nu)}$$
$$\frac{\partial \grave{v}_\eta}{\partial s_\alpha} \Big/ \frac{\partial \grave{v}_\xi}{\partial s_\alpha} = \frac{\cos(\phi-\nu)}{\sin\nu} \tag{4}$$

If (3) is substituted into (4),then

$$\frac{\partial \grave{v}_\beta}{\partial s_\beta}\cos(\phi-\nu) - \frac{\partial \grave{v}_\alpha}{\partial s_\beta}\sin\nu = 0 \tag{5}$$
$$\frac{\partial \grave{v}_\alpha}{\partial s_\alpha}\cos(\phi-\nu) - \frac{\partial \grave{v}_\alpha}{\partial s_\alpha}\sin\nu = 0$$

Equation (5) containing the kinematic hypothesis applied for the special chosen (ξ, η) coordinate system,and we wish to transform them into the general spatial frame of reference (ξ, η). If v denotes the velocity of a typical particle relative to the (ξ,η)-system,then let v_ξ and v_η be the components of v relative to the (ξ,η)-system and v_α and v_β the components of v in the α-and β-directions.The values of v_ξ, v_η, v_α,and v_β at the particle Q will be denoted by V_ξ, V_η, V_α,and V_β,respectively. $\dot\psi$ will denote the material derivative relative to (ξ,η)-system and is given by

$$\dot\psi = \frac{\partial\psi}{\partial t} + V_\xi\frac{\partial\psi}{\partial\xi} + V_\eta\frac{\partial\psi}{\partial\eta} = \frac{\partial\psi}{\partial t} + V_\alpha\frac{\partial\psi}{\partial s_\alpha} + V_\beta\frac{\partial\psi}{\partial s_\beta} \tag{6}$$

The three velocities v, $\grave{v}$ and V are related through the following equations

$$\grave{v}_\xi = v_\xi - V_\xi + \dot\psi\eta \tag{7}$$
$$\grave{v}_\eta = v_\eta - V_\eta - \dot\psi\xi$$

$$V_\xi\cos\phi = V_\alpha\cos(\psi-\pi/4-3\phi/2) + V_\beta\cos(\psi+\pi/4-\phi/2) \tag{8}$$
$$V_\eta\cos\phi = V_\alpha\sin(\psi-\pi/4-\phi/2) + V_\beta\sin(\psi+\pi/4+\phi/2)$$

Equations (5) containing the kinematic hypothesis can now changed from a condition on the velocity $\grave{v}$ to a condition on the velocity v using Eqs. (4),(7),and (8).After evaluating the result at ψ equals $(\pi/4)+(\phi/2)$,we get the desired velocity equations :

$$\cos(\phi-\nu)\left[\frac{\partial V_\alpha}{\partial s_\alpha}\cos\phi - (V_\beta - V_\alpha\sin\phi)\frac{\partial\psi}{\partial s_\alpha} - \dot\psi\sin\phi\right] - \sin\nu\left[\frac{\partial V_\beta}{\partial s_\alpha}\cos\phi + (V_\alpha - V_\beta\sin\phi)\frac{\partial\psi}{\partial s_\alpha} - \dot\psi\right] = 0 \tag{9}$$

$$\cos(\phi-\nu)\left[\frac{\partial V_\beta}{\partial s_\beta}\cos\phi + (V_\alpha - V_\beta\sin\phi)\frac{\partial\psi}{\partial s_\beta} + \dot\psi\sin\phi\right] - \sin\nu\left[\frac{\partial V_\alpha}{\partial s_\beta}\cos\phi - (V_\beta - V_\alpha\sin\phi)\frac{\partial\psi}{\partial s_\beta} + \dot\psi\right] = 0 \tag{10}$$

3. RATE OF ENERGY DISSIPATION

Cosider a plane-strain deformation of a granular material and suppose that the stress field σ is known together with the velocity field V corresponding to σ.

The rate of dissipation $\dot W$ is everywhere positive.Let d denote the strain-rate tensor, then

$$\dot W = \sigma .. d \tag{11}$$

If we substitute Eq. (1) into (11) at which $\alpha = \pi/2+\phi$,we get

$$\dot W = \frac{-p}{\cos\phi}\left[\frac{\partial V_\xi}{\partial\xi} + \frac{\partial V_\eta}{\partial\eta} + \gamma_{\xi\eta}\sin\phi\right] + \frac{q}{\cos\phi}\left[\frac{\partial V_\xi}{\partial\xi}\cos 2\psi - \frac{\partial V_\eta}{\partial\eta}\cos(2\psi-2\phi) + \sin(2\psi-\phi)\gamma_{\xi\eta}\right] \tag{12}$$

After transforming Eq. (12) to stress characteristic coordinates and evaluating the result at $\psi = \pi/4+\phi/2$,we get

$$\dot W = \frac{-p}{\cos^3\phi}\left[\frac{\partial V_\alpha}{\partial s_\alpha} + \frac{\partial V_\beta}{\partial s_\beta} + \frac{1}{\cos\phi}\left[(V_\alpha - V_\beta\sin\phi)\frac{\partial\psi}{\partial s_\beta} - (V_\beta - V_\alpha\sin\phi)\frac{\partial\psi}{\partial s_\alpha}\right]\right] + \frac{q}{\cos^3\phi}\left[\frac{\partial V_\alpha}{\partial s_\beta} + \frac{\partial V_\beta}{\partial s_\alpha} + \frac{1}{\cos\phi}\left[(V_\alpha - V_\beta\sin\phi)\frac{\partial\psi}{\partial s_\alpha} - (V_\beta - V_\alpha\sin\phi)\frac{\partial\psi}{\partial s_\beta}\right]\right] \tag{13}$$

Equation (11) can written as

$$\dot W = -p\,\mathrm{tr}\,d + \mathrm{tr}(\grave\sigma . \grave d) \tag{14}$$

where $\grave\sigma, \grave d$ denote the deviatoric stress,strain-rate tensors,respectively.The condition

$$\dot W \geq 0 \tag{15}$$

must hold at each point in the region under cosideration.Suppose now that the granular material obeys the kinematic equations (9),(10). Adding Eqs. (9),(10)and using (13),gives

$$\mathrm{tr}\,d = \frac{\sin\nu}{\cos(\phi-\nu)}\,\frac{\mathrm{tr}(\grave\sigma . \grave d)}{q} \tag{16}$$

If we eliminate tr d between Eqs.(14),(16),we get

$$\dot W = \left(q - \frac{p\sin\nu}{\cos(\phi-\nu)}\right)\frac{\mathrm{tr}(\grave\sigma . \grave d)}{q} \tag{17}$$

In particular,if the material obeys Coulomb yield criterion

$$q = p\sin\phi + c\cos\phi \tag{18}$$

Then

$$\dot W = M\,\mathrm{tr}(\grave\sigma . \grave d) \tag{19}$$

where

$$M = \frac{\cos\phi\left[c + p\tan(\phi-\nu)\right]}{c\cos\phi + p\sin\phi}$$

Since $q \geq 0$, for $\phi \geq \nu \geq 0$, then M must be positive. Returning now to Eq.(19),we see that inequality (15) is satisfied provided that

$$\frac{\mathrm{tr}(\grave\sigma . \grave d)}{q} \geq 0 \tag{20}$$

or from Eq. (13)

$$\frac{\partial V_\alpha}{\partial s_\beta} + \frac{\partial V_\beta}{\partial s_\alpha} + \frac{1}{\cos\phi}\left[(V_\alpha - V_\beta\sin\phi)\frac{\partial\psi}{\partial s_\alpha} - (V_\beta - V_\alpha\sin\phi)\frac{\partial\psi}{\partial s_\beta}\right] \geq 0 \tag{21}$$

CONCLUSION

In section 3,expressions for the rate of energy dissipation $\dot W$ are given in Eqs.(12),(13) relative to oblique cartesian coordinates for a granular material satisfying the stress eq - ilibrium equations and Coulomb yield criterion.

In the case of Mehrabadi and Cowin model a kinematic inequality for the non-negative rate of energy dissipation $\dot W$ is presented in equation (21).

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