I-415 CLOSED-FORM COMPLEX MODES OF 2-DOF SYSTEM BY PERTURBATION TECHNIQUE

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SUMMARY Explicit formulas for modal frequencies, damping ratios, and complex-valued mode shapes of a two-degree-of-freedom(2-DOF) system are obtained through a new general perturbation technique, which assumes only that the damping nonproportionality is moderate or less. Subsequent parametric study of conditions rather common among composite systems of single-degree-of-freedom(SDOF) structure plus tuned mass damper(TMD), shows that moderate nonproportionality in damping has negligible effect on modal frequencies and damping ratio, but significantly affects the mode shapes.

PERTURBATION OF EIGENPROPERTIES For multi-degree-of-freedom(MDOF) systems described by $\mathbf{M\ddot{x}+C\dot{x}+Kx=f}$ (where: $\mathbf{x}$ ≡ vector of displacements in n degrees of freedom; $\mathbf{f}$ ≡ load vector; $\mathbf{M,C,K}$ ≡ mass, damping, stiffness matrices which are real-valued and symmetric; $\mathbf{M,K}$ are positive definite; $\mathbf{C}$ is positive semi-definite), the natural mode shapes, $\mathbf{y}_j$, and corresponding frequencies, λ_j, are defined such that each $\mathbf{x}_j \equiv \mathbf{y}_j \exp(\lambda_j t)$ is a solution of the homogeneous equation $\mathbf{M\ddot{x}+C\dot{x}+Kx=0}$. The resulting characteristic equation is:

$$(\lambda_j^2\mathbf{M}+\lambda_j\mathbf{C}+\mathbf{K})\mathbf{y}_j = \mathbf{0} \quad , \; j = 1,\cdots,r,\cdots,n,n+1,\cdots,n+r,\cdots,2n \tag{1}$$

where the (n+r)-th solution set is complex conjugate of the r-th set. $\mathbf{y}_j$ is complex-valued when $\mathbf{CM^{-1}K} \neq \mathbf{KM^{-1}C}$. The authors [Ref.1] developed a second order perturbation technique to obtain λ_j and $\mathbf{y}_j$ from λ_{oj} and $\mathbf{y}_{oj}$, the latter being solutions of another characteristic equation :

$$(\lambda_{oj}^2\mathbf{M}+\lambda_{oj}\mathbf{C_P}+\mathbf{K})\mathbf{y}_{oj} = \mathbf{0} \quad , \; \mathbf{y}_{oj} = \text{real-valued vector} \tag{2}$$

where $\mathbf{C_P} \equiv \mathbf{C}-\mathbf{C_N}$; $\mathbf{C_P M^{-1}K} = \mathbf{KM^{-1}C_P}$; and the elements of $\mathbf{C_N}$ are assumed to be one order of magnitude smaller than those of $\mathbf{C_P}$.(This perturbation is expected to be good approximation even for system with high overall damping, provided the nonproportionality of damping is indeed weak or moderate.) Modal damping ratio, ξ, and "undamped" modal circular frequency, ω, are defined from λ as follows:

$$\lambda_j = -\xi_j\,\omega_j \pm i\sqrt{1-\xi_j^2}\,\omega_j \quad , \; j = 1,\cdots,n \tag{3}$$

$$\lambda_{oj} = -\xi_{oj}\,\omega_{oj} \pm i\sqrt{1-\xi_{oj}^2}\,\omega_{oj} \quad , \; j = 1,\cdots,n \tag{4}$$

where $i \equiv \sqrt{-1}$. Note that ω_j is also referred to as "pseudo-undamped" frequency, because it is actually affected by level of damping. In contrast, ω_{oj} in Eq.4 which applies to proportionally damped system, is not affected by ξ_{oj}.

APPLICATION TO 2-DOF SYSTEM The above-described approximation of eigenproperties ω_j, ξ_j and $\mathbf{y}_j$ of nonproportionally damped system, leads to important physical insights when applied to a 2-DOF system as in Fig.1. For generality of discussion, the following nondimensional parameters are used: mass ratio, $\mu \equiv M_T/M_S$; tuning parameter, $\tau \equiv \omega_T/\omega_S$; damping ratio of TMD subsystem, $\xi_T \equiv C_T/2M_T\omega_T$;and damping proportionality, $\delta \equiv \xi_S/\xi_T$ for $\xi_T \neq 0$.

The pseudo-undamped modal frequency ω_j, modal damping ratio ξ_j, and complex-valued mode shape $\mathbf{y}_j$, for j=1,2, are approximated by the following explicit formulas.

$$\omega_j = \omega_{oj}\sqrt{(1+\alpha_{jk})} \quad , \; (j,k) = (1,2),(2,1) \tag{5}$$

$$\xi_j = \xi_{oj}\sqrt{(1+\beta_{jk})} \tag{6}$$

$$\mathbf{y}_j = (\mathbf{y}_{oj}+\zeta_{jk}\mathbf{y}_{ok}) + i(\eta_{jk}\mathbf{y}_{ok}) \tag{7}$$

where ω_{oj}, ξ_{oj} and $\mathbf{y}_{oj}$ are solutions of Eqs.2 and 4, namely:

$$\omega_{oj} = \omega_S\,\sigma_j \tag{8}$$

$$\xi_{oj} = \xi_T[\delta(\tau^2-\sigma_j^2)^2+\mu\tau\sigma_j^4]/[\sigma_j\{(\tau^2-\sigma_j^2)^2+\mu\tau^4\}] \tag{9}$$

$$\mathbf{y}_{OJ} = \begin{Bmatrix} 1 \\ \tau^2/(\tau^2-\sigma_J^2) \end{Bmatrix} 1/\sqrt{[1+\mu\{\tau^2/(\tau^2-\sigma_J^2)\}^2]M_S} \tag{10}$$

$$\sigma^2_{1,2} = [(1+\tau^2+\mu\tau^2) \mp \sqrt{(1+\tau^2+\mu\tau^2)^2-4\tau^2}\,]/2 \tag{11}$$

and α_{JK}, β_{JK}, ζ_{JK}, η_{JK} are real-valued functions of τ, μ, δ, ξ_T. These functions become zero when δ equals $1/\tau$, i.e. when the damping becomes proportional, or when $\delta \fallingdotseq 1$ and $\xi_T \fallingdotseq 0$. See also Fig.2.

$$\alpha_{JK} = \frac{\rho^2}{D_{JK}}\left[(\sigma_K^2-\sigma_J^2)\left\{1+\frac{\rho^2}{4D_{JK}}(\sigma_K^2-\sigma_J^2)\right\} + \frac{\rho^2}{4D_{JK}}\frac{1}{1-\xi_{OJ}^2}\left\{\xi_{OJ}(\sigma_K^2-\sigma_J^2)-2\sigma_K(\xi_{OJ}\sigma_K-\xi_{OK}\sigma_J)\right\}^2\right] \tag{12}$$

$$\beta_{JK} = \frac{1}{1+\alpha_{JK}}\left[\frac{\rho^2}{D_{JK}}\left\{\frac{\sigma_K}{\xi_{OJ}}(\xi_{OJ}\sigma_K-\xi_{OK}\sigma_J)\right\}\left\{2+\frac{\rho^2}{D_{JK}}\frac{\sigma_K}{\xi_{OJ}}(\xi_{OJ}\sigma_K-\xi_{OK}\sigma_J)\right\}-\alpha_{JK}\right] \tag{13}$$

$$\zeta_{JK} = [\sigma_J\xi_{OJ}(\sigma_J^2+\sigma_K^2)-2\xi_{OK}\sigma_J^2\sigma_K]\,\rho/D_{JK} \tag{14}$$

$$\eta_{JK} = [\sigma_J\sqrt{1-\xi_{OJ}^2}\,(\sigma_J^2-\sigma_K^2)]\,\rho/D_{JK} \tag{15}$$

$$D_{JK} = 2(\sigma_J^2+\sigma_K^2)(\xi_{OJ}\sigma_J-\xi_{OK}\sigma_K)^2 + (\sigma_J^2-\sigma_K^2)\{(\sigma_J^2-\sigma_K^2)-2(\xi_{OJ}^2\sigma_J^2-\xi_{OK}^2\sigma_K^2)\} \tag{16}$$

$$\rho = 2\xi_T(\delta-1/\tau)\sqrt{\mu\tau^4/[(1+\tau^2+\mu\tau^2)^2-4\tau^2]} \tag{17}$$

DISCUSSION Comparing ω_J, ξ_J, and a norm of $\mathbf{y}_J$ as obtained presently, with corresponding "exact" values from conventional matrix iteration procedure, indicated that the percentage difference, i.e. |[perturbed - exact]/exact| ×100%, is 1% or much less in all cases concerned in Fig.2. An advantage of the present perturbation technique is that very large magnitudes of the coefficients $\alpha_{JK}, \beta_{JK}, \zeta_{JK}, \eta_{JK}$ themselves would be the indication of large perturbation errors due to gross violation of initial assumption, i.e. moderate nonproportionality.

In the present examples, moderate nonproportionality of damping has little effect on modal frequencies and damping ratios. Eq.7 and ζ_{JK}, η_{JK} from Fig.2, however, show how classical real modes are "coupled" to form complex modes. Eqs. 5-7 shall be very useful in physically interpreting complex-modal superposition equations.

REFERENCE [1] B.M. Pacheco, H.Kim, and Y.Fujino : " Eigenvalue Analysis of Nonproportionally Damped Systems Using a Perturbation Technique and Its Application to Modal Analysis ", Symp. on Comp. Methods in Structural Eng. and Related Fields, July 18-20,1988 (In Japanese)

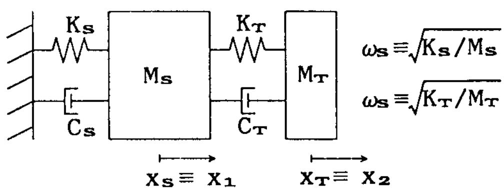
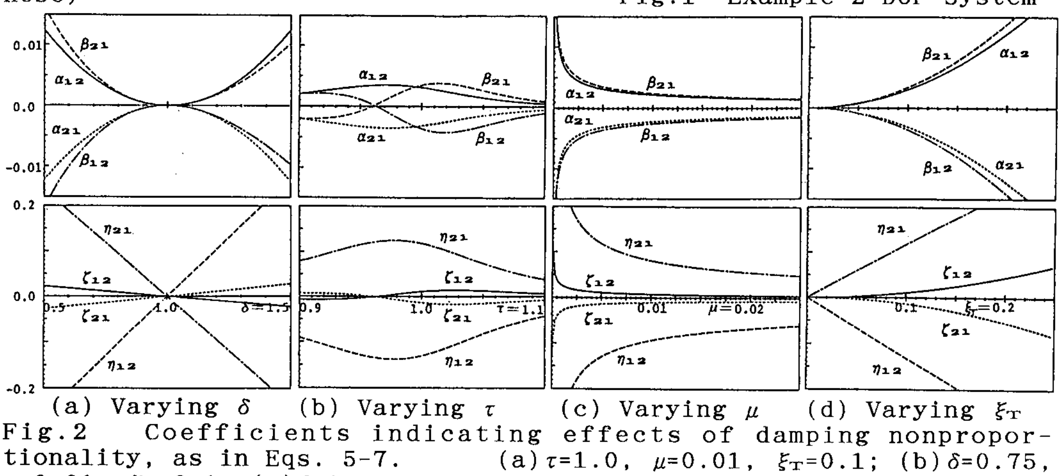


Fig.1 Example 2-DOF System



(a) Varying δ (b) Varying τ (c) Varying μ (d) Varying ξ_T

Fig.2 Coefficients indicating effects of damping nonproportionality, as in Eqs. 5-7. (a) τ=1.0, μ=0.01, ξ_T=0.1; (b) δ=0.75, μ=0.01, ξ_T=0.1; (c) δ=0.75, τ=1.0, ξ_T=0.1; (d) δ=0.75, τ=1.0, μ=0.01.