I-5 A GENERAL EXPANSION FOR LINEAR CONTINUUM MECHANICS SUBJECT TO KINEMATIC FIELDS

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Introduction.- When a continuum body has a tendency in fundamental deformations, we usually deal with its mechanics, mathematically, within a frame of the displacement patterns. So far, there are many well-known governing equations obtained through that procedure: various beam and plate(or shell) theories and others. However, in the past, we find no intention to disclose a common feature of such formulations----this paper intends to do it within small displacements.

Kinematic Field.- Let $\{\xi^1,\xi^2,\xi^3\}$ be a set of curvilinear coordinates in a continuum body, where small displacements are decomposed into $\{\xi^i\}$-directions. When variables of two sets, $\{\varsigma^1,\cdots,\varsigma^\Lambda\}$ and $\{s^{1'},\cdots,s^{\Sigma'}\}$, are in a known one-to-one correspondence to $\{\xi^i\}$ $(\Lambda+\Sigma'=3)$, we can regard the set $\{(\varsigma^\lambda),(s^{\sigma'})\}$ as another Lagrangean coordinates: we put $\{\xi^i\}=\{(\varsigma^\lambda),(s^{\sigma'})\}$ without violating any generality. Now, we consider a kinematic field of the body in which the displacements $\{u^i\}$ are dependent on a finite number of unknown functions $\{\psi_1,\cdots,\psi_N\}$ of only $\{\varsigma^\lambda\}$ in the form:

$$\{u^i(\xi^j)\}=[\Phi^{im}(\varsigma^\lambda,s^{\sigma'})][D^{u}{}^{n}_{m}(\varsigma^\lambda)]\{\psi_n(\varsigma^\lambda)\} \qquad \text{.........(1)}$$

where $[\Phi^{im}]$ is matrix of known functions of both $\{\varsigma^\lambda\}$ and $\{s^{\sigma'}\}$; and $[D^{u}{}^{n}_{m}]$ is matrix of differential operators on $\{\varsigma^\lambda\}$-field. Let the domains of $\{\xi^i\}$ and $\{\varsigma^\lambda\}$ in the body be denoted by V and L, respectively, and the domain of $\{s^{\sigma'}\}$ at each $\{\varsigma^\lambda\}$ be $C(\varsigma^\lambda)$ and called a cross-section of the kinematic field.

In any case where displacements are represented in a linear form of (1), we can rearrange it into another linear form such that the column vectors of $[\Phi^{im}]$ are independent as functions of $\{s^{\sigma'}\}$ with each fixed $\{\varsigma^\lambda\}$, and that the row vectors of $[D^{u}{}^{n}_{m}]$ are also independent as differential operators. Thus, we consider Eq.(1) itself so arranged. With the above preliminaries, we now call the quantities defined by

$$\{v_m(\varsigma^\lambda)\}=[D^{u}{}^{n}_{m}(\varsigma^\lambda)]\{\psi_n(\varsigma^\lambda)\} \qquad \text{.........(2)}$$

a set of displacement parameters, which determine $\{u^i\}$ of all the material points on each $C(\varsigma^\lambda)$, completely. That is, in kinematic field (1), the displacements on $C(\varsigma^\lambda)$ are in a one-to-one to $\{v_m\}$. Let the highest order of the differentials in $[D^{u}{}^{n}_{m}]$ be Γ. It is to be noted that while $\{v_m\}$ can take any values at a cross-section, they can not, in general, over the entire field L ----indeed, they are restrained derivable through (2) from any differentiable $\{\psi_n\}$.

Strain Distribution.- To displacements $\{u^i\}$, the linear strain components are related as

$$e_{ij}=\frac{1}{2}\{g_{ik}u^k{}_{,j}+g_{jk}u^k{}_{,i}+(\Gamma_{kij}+\Gamma_{kji})u^k\} \qquad \text{.........(3)}$$

where subscript i after comma denotes the differentiation with respect to ξ^i ; and g_{ij} and Γ_{kij} are metric tensor components and Christoffel symbols, respectively. By substituting (1) and (2) into (3), we obtain

$$e_{ij}=\frac{1}{2}(f_{ijm}+h_{im}\partial_j+h_{jm}\partial_i)v_m \qquad \text{.........(4)}$$

where
$$f_{ijm}(=f_{jim})=g_{ik}\Phi^k_{m,j}+g_{jk}\Phi^k_{m,i}+(\Gamma_{kji}+\Gamma_{kij})\Phi^k_m$$
$$h_{im}=g_{ik}\Phi^k_m \qquad \text{.........(5.a,b)}$$

and $\partial_i=,_i$ ($\partial_{\sigma'}v_m=0$, since v_m are functions of only $\{\varsigma^\lambda\}$). By introducing (2), Eqs.(4) are transformed into the following matrix form:

$$\{e_{ij}\}=[\Psi_{(ij)}{}^{k}(\varsigma^\lambda,s^{\sigma'})][D^{\epsilon}{}^{n}_{k}(\varsigma^\lambda)]\{\psi_n(\varsigma^\lambda)\} \qquad \text{.........(6)}$$

where $[\Psi_{(ij)}{}^{k}]$ is matrix of known functions of $\{\varsigma^\lambda\}$ and $\{s^{\sigma'}\}$; and $[D^{\epsilon}{}^{n}_{k}]$ is matrix of differential operators on $\{\varsigma^\lambda\}$-field. Again, we can arrange their elements such that the column vectors of $[\Psi_{(ij)}{}^{k}]$ and the row vectors of $[D^{\epsilon}{}^{n}_{k}]$ are independent, respectively. And, based on that, on each cross-section, strain distributions are uniquely determined by the quantities:

$$\{\epsilon_k(\varsigma^\lambda)\}=[D^{\epsilon}{}^{n}_{k}(\varsigma^\lambda)]\{\psi_n(\varsigma^\lambda)\} \qquad \text{.........(7)}$$

we call the $\{\epsilon_k\}$ a set of <u>strain parameters</u> in kinematic field (1), and Eq.(7), a <u>generalized strain-displacement relation</u>. By the same reason to $\{v_m\}$ in (2), $\{\epsilon_k\}$ can take any values at a point of $\{\varsigma^\lambda\}$, but they can not, over the $\{\varsigma^\lambda\}$-field.

From relations (4), we can see that the differentials in $[D^{\epsilon n}_{k}]$ are, at most, once higher than those in $[D^{un}_{m}]$. Then, operator matrix $[D^{\epsilon n}_{k}]$ may be represented as

$$[D^{\epsilon n}_{k}] = [D^{\Delta}{}^{k}_{m}][D^{u}{}^{n}_{m}] \quad \text{..........(8)}$$

where $[D^{\Delta}{}^{k}_{m}]$ is matrix of differential operators of first order, with the rows being independent.

<u>Mechanical Relations from Virtual-Work Principle</u>.- Let the boundaries of L in R^{λ} be B . We denotes the body forces, per unit region of $\{\xi^i\}$ and decomposed into $\{\xi^i\}$-directions, by $\{\bar{p}_i(\xi^j)\}$; and the surface forces acting on the cross-sections at B , per unit region of $\{s^{\alpha'}\}$, by $\{\bar{q}_i(\xi^j)\}$. With σ^{ij} being the stress components associated to e_{ij} , the equation of virtual work is written down as

$$\delta W = \int_V [\sigma^{ij}\delta e_{ij} - \bar{p}_i \delta u^i]\, dV - \int_{B_1}\int_C \bar{q}_i \delta u^i \, dC\, dB = 0 \quad \text{........(9)}$$

where B_1 is the subregion of B subject to mechanical conditions. Into Eq.(9) substituting (1),(2),(6) and (7), we define <u>stress resultants</u>, $\{M^k(\varsigma^\lambda)\}$, and <u>body</u>- and <u>surface-force resultants</u>, $\{\bar{P}^m(\varsigma^\lambda)\}$ and $\{\bar{Q}^m(\varsigma^\lambda)\}$, as follows:

$$\{M^k(\varsigma^\lambda)\} = \int_C [\psi^{k}_{(ij)}]^T\{\sigma^{ij}\}\, dC$$

$$\{\bar{P}^m(\varsigma^\lambda)\} = \int_C [\Phi^{im}]^T\{\bar{p}_i\}\, dC, \quad \{\bar{Q}^m(\varsigma^\lambda)\} = \int_C [\Phi^{im}]^T\{\bar{q}_i\}\, dC \quad \text{.......(10.a-c)}$$

and make the equation of virtual work transformed into

$$\delta W = \int_L [\{M^k\}^T[D^{\epsilon n}_{k}]\delta\{\chi_n\} - \{\bar{P}^m\}^T[D^{un}_{m}]\delta\{\chi_n\}]\, dL - \int_{B_1}\{\bar{Q}^m\}^T[D^{un}_{m}]\delta\{\chi_n\}\, dB = 0$$

By substituting (8) and using an integration by parts on the first term of the integrand in L ,

$$\delta W = \int_L([D_{\Delta}{}^{m}_{k}]\{M^k\} - \{\bar{P}^m\})^T[D^{un}_{m}]\delta\{\chi_n\}\, dL + \int_B([F_{\Delta}{}^{m}_{k}]\{M^k\} - \{\bar{Q}^m\})^T\delta\{v_m\}\, dB = 0$$

where $[D_{\Delta}{}^{m}_{k}]$ is matrix of differential operators of first order; and $[F_{\Delta}{}^{m}_{k}]$ is matrix of functions of $\{\varsigma^\lambda\}$. Further, by repeating integrations by parts concerning $[D^{u}{}^{n}_{m}]$, as a result of mathematical expansions, we finally attain the form:

$$\begin{aligned}\delta W = & \int_L([D_u{}^{n}_{m}]([D_{\Delta}{}^{m}_{k}]\{M^k\} - \{\bar{P}^m\}))^T\delta\{\chi_n\}\, dL \\ & + \int_B[([D_v{}^{m}_{k}]\{M^k\} - [D'_v{}^{m}_{m}]\{\bar{P}^m\} - \{\bar{Q}^m\})^T\delta\{v_m\} \\ & \qquad + ([D'_u{}^{i}_{m}]([D_{\Delta}{}^{m}_{k}]\{M^k\} - \{\bar{P}^m\}))^T\delta\{(\partial\chi)_i\}]\, dB = 0 \quad \text{.........(11)}\end{aligned}$$

where $[D_u{}^{n}_{m}]$, $[D_v{}^{m}_{k}]$, $[D'_v{}^{m}_{m}]$ and $[D'_u{}^{i}_{m}]$ are matrices of differential operators, with the highest orders of the differentials being Γ , $\Gamma-1$, $\Gamma-1$ and less than $\Gamma-1$, respectively; and $\{(\partial\chi)_i\}$ are a set of all the lower derivatives of $\{\chi_n\}$ independent to $\{v_m\}$. $\delta\{\chi_n\}$ and $\delta\{v_m\}$ are arbitrary in L and B_1 , respectively. Since the displacements of a cross-section are entirely determined by parameters $\{v_m\}$ (the geometrical boundary conditions take the form: $v_m = \bar{v}_m$ on $B - B_1$), the derivatives $\{(\partial\chi)_i\}$ are free on any part of B . Thus, we obtain

<u>Equilibrium Equations</u>: $[D_u{}^{n}_{m}]([D_{\Delta}{}^{m}_{k}]\{M^k\} - \{\bar{P}^m\}) = \{0\}$ in L .....(12)

<u>Mechanical Boundary Conditions</u>:

$$[D_v{}^{m}_{k}]\{M^k\} - [D'_v{}^{m}_{m}]\{\bar{P}^m\} - \{\bar{Q}^m\} = \{0\} \text{ on } B_1 \quad \text{................(13)}$$

(and if exist such derivatives $(\partial\chi)_i$)

$$[D'_u{}^{i}_{m}]([D_{\Delta}{}^{m}_{k}]\{M^k\} - \{\bar{P}^m\}) = \{0\} \text{ on entire } B \quad \text{............(14)}$$

<u>Constitutive Relations</u>.- Provided that the original constitutive equations take the form: $\sigma^{ij} = E^{ijkl}e_{kl}$ (E^{ijkl}: elastic moduli), by the use of (6), those are generalized to

$$\{M^k(\varsigma^\lambda)\} = [C^{kl}(\varsigma^\lambda)]\{\epsilon_l\} \quad \text{..................................(15)}$$

$$[C^{kl}(\varsigma^\lambda)] = \int_C [\psi^{k}_{(ij)}]^T[E^{ijmn}][\Phi^{l}_{(mn)}]\, dC \quad \text{..........................(16)}$$