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A COMBINED UPPER AND LOWER BOUND ANALYSIS

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1. INTRODUCTION

A method of limit analysis which combines the features of the lower and upper bound approaches is outlined. The method is illustrated with a simple example and its advantages over separate lower and upper bound analyses are discussed.

2. PROPOSED COMBINED UPPER AND LOWER BOUND ANALYSIS

In a lower bound analysis, the ultimate strength of a structural system is usually expressed as a function of static variables such as stresses or stress resultants and then maximised without violating the yield conditions so as to obtain the best lower bound solution. On the other hand, in an upper bound approach, the ultimate strength is expressed as a function of kinematic variables and is minimised, without violating the compatibility conditions, to give the best upper bound solution (1,2).

In the event that the best lower bound solution coincides with the best upper bound solution, the exact or true collapse load is said to have been obtained since the conditions of equilibrium, yield and compatibility are simultaneously satisfied for such a case. However, it is noted that both the upper and lower bound approaches involve the consideration of the equilibrium state of the structure. To eliminate such a repetition, a combined upper and lower bound analysis is proposed as below.

Consider a structure subjected to a system of external forces which are proportional to one another at any level of loading (Fig. 1). From the equilibrium conditions of the structure, the representative force, P, can be expressed as a function of a set of kinematic variables **k** and static variables **s**, that is:

$$\left.\begin{aligned} &P = P(\mathbf{k},\mathbf{s}) \\ &\text{where} \\ &\mathbf{k} = \{k_1,\ k_2,\ \ldots,\ k_m\} \\ &\mathbf{s} = \{s_1,\ s_2,\ \ldots,\ s_n\} \end{aligned}\right\} \quad (1)$$

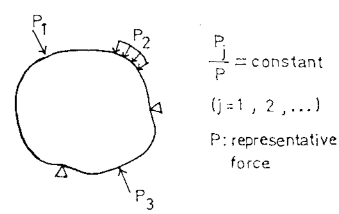

Fig. 1 Structure under a System of Forces

The kinematic variables are parameters which describe the kinematically admissible velocity field or collapse mechanism of the structure. On the other hand, the static variables are stresses or stress resultants which define the statically admissible stress field of the structure. It is obvious that these variables are subjected to constraints:

$$C_i(\mathbf{k},\mathbf{s}) \geqslant 0 \quad , \quad (i = 1,\ 2,\ \ldots) \qquad (2)$$

for the collapse mechanism to be valid and for the yield conditions to be satisfied.

By the upper bound and lower bound theorems of limit analysis, the ultimate strength or true collapse load, P_u, of the structure is then given by the value of P which is minimised with respect to the kinematic variables **k** and maximised with respect to the static variables **s**. For a structure with given geometrical and material properties and load distribution, the value of P_u is uniquely determined and can be expressed as:

$$\left.\begin{aligned} P_u &= \max_{\mathbf{s}} \left\{ \min_{\mathbf{k}} P(\mathbf{k},\mathbf{s}) \right\} \\ &= \min_{\mathbf{k}} \left\{ \max_{\mathbf{s}} P(\mathbf{k},\mathbf{s}) \right\} \end{aligned}\right\} \quad (3)$$

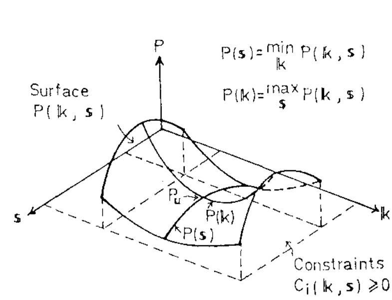

Fig. 2 Concept of Combined Upper and Lower Bound Analysis

where **k** and **s** are subjected to constraints at (2).

The concept of the proposed method of analysis is shown diagrammatically in Fig. 2.

The true collapse load, P_u, corresponds to the 'saddle point' of the surface represented by the function $P(k,s)$. It can be seen that the combined upper and lower bound analysis is essentially a constrained optimization problem.

3. A SIMPLE EXAMPLE

To illustrate the use of the proposed method of analysis, consider a two-span continuous steel beam subjected to a uniformly distributed load p, as shown in Fig. 3(a). The beam has a uniform cross-section and the only possible collapse mechanism is as shown in Fig. 3(b) where plastic hinges are formed at D, B and E. Denoting the bending moments at D and B by $\alpha_1 M_1$ and $\alpha_2 M_2$, the free body diagrams of members AD and DB can be shown as in Figs. 3(c) and (d). From the equilibrium conditions, the value of p can be written as:

$$p = p(\alpha_1, \alpha_2, x, M_1, M_2) = \frac{2\{\alpha_1 M_1 \ell + \alpha_2 M_2 x\}}{\ell x(\ell - x)} \tag{4}$$

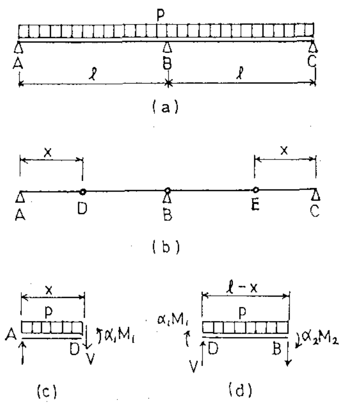

Fig. 3 Two-span Continuous Beam

To ensure the validity of the collapse mechanism shown in Fig. 3(b) and to satisfy yield conditions, the kinematic variables, α_1, α_2 and x, and the static variables, M_1 and M_2, are subjected to the following constraints:

$$\left.\begin{array}{l} \alpha_1 \geqslant 1 \quad , \quad \alpha_2 \geqslant 1 \quad , \quad 0 < x < \ell \\ 0 \leqslant M_1 \leqslant M_p \quad , \quad 0 \leqslant M_2 \leqslant M_p \end{array}\right\} \tag{5}$$

where M_p is the full plastic moment of the beam section. By the proposed combined upper and lower bound analysis, the true collapse load, p_u, is given by:

$$p_u = \max_{\{M_1, M_2\}} \left\{ \min_{\{\alpha_1, \alpha_2, x\}} p(\alpha_1, \alpha_2, x, M_1, M_2) \right\}$$
$$= p(\alpha_1 = \alpha_2 = 1,\ x = 0.414\ell,\ M_1 = M_2 = M_p)$$
$$= 11.66\ M_p/\ell^2 \tag{6}$$

4. DISCUSSIONS AND CONCLUSIONS

The attractiveness of the proposed method lies in the fact that the derivation of equilibrium equations need not be repeated as in separate upper and lower bound analyses. The method is more advantageous when the number of unknown variables increases. It is especially useful when the exact mode of the collapse mechanism (such as the position of the plastic hinge within the beam in the above example) is not exactly known in advance. In addition, since the values of the kinematic and static variables are determined together with the value of the true collapse load, the method not only allows the exact collapse mode to be determined but it also enables the various internal forces and hence their contributions to the ultimate strength to be calculated simultaneously.

A further advantage of the combined analysis is that the order of maximization and minimization is arbitrary. Hence, if there are multiple kinematic or static variables, the corresponding optimization processes can be performed in a manner such that the true collapse load can be computed easily. The maximization and minimization processes in the combined upper and lower bound analysis may be complicated for some cases. In such cases, it is usually true that the formulation is also complicated even for a separate upper bound or lower bound analyses. The problem then lies in the nature of the model used and a modification of the model may be needed instead. On the other hand, with the advent of computer softwares, it is believed that a combined upper and lower analysis may be more advantageous.

5. REFERENCES

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