AR and MA Models of Nonstationary Ground Motions

Musashi Inst. of Tech. Toshiaki Chiba
" Masaru Hoshiya
" Zeki Hasgür

1. Introduction A multi-dimensional stochastic process model with nonstationary characteristics of amplitude and frequency contents has already proposed, where the identification and simulation of a system is represented with a frequency domain model. On the other hand, Akaike, Hino, Box and Jenkins and Hussain and Rao discussed the same topics with time domain models, namely, autoregressive (AR) model, moving average (MA) model and the combined (AR-MA) model. These models, however, have not been fully developed for the stochastic processes with the nonstationarity in the amplitude and frequency domains, especially for the multi-dimensional cases. This paper proposed time domain models for the multi-dimensional stochastic processes with the nonstationary characteristics of the amplitude and frequency contents.

2. Autoregressive (AR) Model of Multi-dimensional Nonstationary Stochastic Processes

A discrete autoregressive (AR) model for the multi-dimensional nonstationary stochastic processes $x_i(t);i=1,2,\ldots,m$ with zero mean is given by

$$x_i(j)=\sum_{p=1}^{i}\sum_{k=1}^{M} b_{ip}(k,j)\,x_p(j-k)+\varepsilon_i(j) \quad ; \; i=1,2,\cdots,m \tag{1}$$

where, discrete time $t=j\,t;j=1,2,\ldots,N$. The value of $x_i(j)$ at time $t=j\,t$ is represented as the linear summation of M values of $x_p(j-1),x_p(j-2),\ldots,x_p(j-M)$ with the unknown coefficient $b_{ip}(k,j)$. Eq.(1) has also nonstationary characteristics of the amplitude and frequency contents, since the unknown coefficient $b_{ip}(k,j)$ is the function of time j.

The coefficients $b_{ip}(k,j)$ are chosen in such a way that the mean square error $\sum_{i=1}^{m}E[\varepsilon_i^2(j)]$ should be minimum. Under this condition, the following relation may be led;

$$\begin{bmatrix} B_{n1}(j)\\ B_{n2}(j)\\ \vdots\\ B_{nn}(j)\end{bmatrix}=\begin{bmatrix} X_{11}(j), X_{21}(j), \cdots, X_{n1}(j)\\ X_{12}(j), X_{22}(j), \cdots, X_{n2}(j)\\ \vdots \quad\quad \vdots \quad\quad\quad \vdots\\ X_{1n}(j), X_{2n}(j), \cdots, X_{nn}(j)\end{bmatrix}^{-1}\begin{bmatrix} F_{n1}(j)\\ F_{n2}(j)\\ \vdots\\ F_{nn}(j)\end{bmatrix} \tag{2}$$

where,

$$B_{np}(j)=\begin{bmatrix} b_{np}(1,j)\\ b_{np}(2,j)\\ \vdots\\ b_{np}(M,j)\end{bmatrix}, \qquad F_{nq}(j)=\sum_{s=j-N'}^{j+N'}\begin{bmatrix} x_n(s)x_q(s-1)\\ x_n(s)x_q(s-2)\\ \vdots\\ x_n(s)x_q(s-M)\end{bmatrix}$$

$$X_{pq}(j)=\sum_{s=j-N'}^{j+N'}\begin{bmatrix} x_p(s-1)x_q(s-1), x_p(s-2)x_q(s-1), \cdots, x_p(s-M)x_q(s-1)\\ x_p(s-1)x_q(s-2), x_p(s-2)x_q(s-2), \cdots, x_p(s-M)x_q(s-2)\\ \vdots \quad\quad\quad \vdots \quad\quad\quad \vdots\\ x_p(s-1)x_q(s-M), x_p(s-2)x_q(s-M), \cdots, x_p(s-M)x_q(s-M)\end{bmatrix}$$

$$n=1,2,\cdots,m \qquad p=1,2,\cdots,n \qquad q=1,2,\cdots,n$$

After identifing $b_{ip}(k,j)$, the error term $\varepsilon_i(j)$ in Eq.(1) is determined, according to Akaike's procedure. The covariance matrix of $\varepsilon_i(j)$ is decomposed to the lower and upper triangular matrices.

$$\boldsymbol{\sigma}^2(j)=\begin{bmatrix}\sigma_{11}^2(j),\sigma_{12}^2(j),\cdots,\sigma_{1m}^2(j)\\ \sigma_{21}^2(j),\sigma_{22}^2(j),\cdots,\sigma_{2m}^2(j)\\ \vdots\quad\quad\vdots\quad\quad\quad\vdots\\ \sigma_{m1}^2(j),\sigma_{m2}^2(j),\cdots,\sigma_{mm}^2(j)\end{bmatrix}=\begin{bmatrix}C_{11}(j) & & & 0\\ C_{21}(j),C_{22}(j) & & & \\ \vdots\quad\quad\vdots & \ddots & & \\ C_{m1}(j),C_{m2}(j),\cdots,C_{mm}(j) & & & \end{bmatrix}\cdot\begin{bmatrix}C_{11}(j),C_{21}(j),\cdots,C_{m1}(j)\\ C_{22}(j),\cdots,C_{m2}(j)\\ \ddots\quad\vdots\\ 0\quad\quad C_{mm}(j)\end{bmatrix}=\mathbb{C}\,\mathbb{C}^T \tag{3}$$

where, elements of covariance matrix are given by

$$\sigma_{ij}^2(j)=E[\varepsilon_i(j)\,\varepsilon_g(j)]\cong\frac{1}{2N'}\sum_{s=j-N'}^{j+N'}\varepsilon_i(s)\,\varepsilon_g(s)$$
$$=\frac{1}{2N'}\sum_{s=j-N'}^{j+N'}\left\{x_i(s)-\sum_{p=1}^{i}\sum_{k=1}^{M}b_{ip}(k,s)x_p(s-k)\right\}\left\{x_g(s)-\sum_{p=1}^{g}\sum_{k=1}^{M}b_{gp}(k,s)x_p(s-k)\right\} \tag{4}$$

Then, $\varepsilon_i(j)$ is given by

$$\varepsilon_i(j)=\begin{bmatrix}\varepsilon_1(j)\\ \varepsilon_2(j)\\ \vdots\\ \varepsilon_m(j)\end{bmatrix}=\begin{bmatrix}C_{11}(j) & & 0\\ C_{21}(j),C_{22}(j) & & \\ \vdots\quad\quad\vdots & \ddots & \\ C_{m1}(j),C_{m2}(j),\cdots,C_{mm}(j) & & \end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_m\end{bmatrix} \tag{5}$$

where, ξ_i ; $i = 1,2,\ldots,m$ are mutually independent random variables with zero mean and its variances equal to unity.

3. Moving Average (MA) Model of the Multi-dimensional Nonstationary Stochastic Processes

A nonstationary discrete MA model is given by

$$x_i(j)=\sum_{p=1}^{i}\sum_{k=1}^{M}h_{ip}(k,j)a_p(j-k)+\varepsilon_i(j)\quad ;\ i=1,2,\cdots,m \tag{6}$$

where, $a_p(j)$ are m mutually independent random variables (white noise) with zero mean and its variances are equal to σ^2. In Eq.(6), if the white noise $a_p(j)$ are considered as the input to the filter $h_{ip}(k,j)$, $x_i(j)$ is the output of this system. Since the filter $h_{ip}(k,j)$ is the function of time j, Eq.(6) also represents the nonstationarity.

Letting the mean square error $\sum_{i=1}^{m}E[\varepsilon_i^2(j)]$ be minimum, $h_{ip}(k,j)$ are identified by the following equation.

$$\sum_{s=j-N'}^{j+N'}x_n(s)a_g(s-k)=\sum_{k=1}^{M}h_{ng}(k,j)\cdot 2N'\sigma^2 \tag{7}$$
$$j=1,2,\cdots,N\qquad n=1,2,\cdots,m\qquad g=1,2,\cdots,n\qquad l=1,2,\cdots,M$$

Eq.(7) can be represented as the following matrix form,

$$H_{ng}(j)=A_{gg}^{-1}F_{ng}(j)\ ;\ \begin{cases}n=1,2,\cdots,m\\ g=1,2,\cdots,n\end{cases} \tag{8}$$

where,

$$H_{ng}(j)=\begin{bmatrix}h_{ng}(1,j)\\ h_{ng}(2,j)\\ \vdots\\ h_{ng}(M,j)\end{bmatrix},\quad A_{gg}(j)=2N'\sigma^2\,\mathbb{I},\quad F_{ng}(j)=\sum_{s=j-N'}^{j+N'}\begin{bmatrix}x_n(s)a_g(s-1)\\ x_n(s)a_g(s-2)\\ \vdots\\ x_n(s)a_g(s-M)\end{bmatrix}$$

After getting $h_{ip}(k,j)$, error term $\varepsilon_i(j)$ can be given by the same procedure as in AR model.

4. References

(1) Hoshiya, Proc. of JSCE,Sept.1976, (2) Akaike and Nakagawa, "Stat. Anal. & Control" Science Book Co. 1972, (3) Hino, "Spec. Analysis" Asakura Book Co. 1977, (4) Box & Jenkins, "Time Series Analysis" Holden Day, 1970, (5) Hussain & Rao, Int. Jour. of Control, Vol.23, No.5, 1976, (6) Hoshiya, Proc. of JSCE, No.270, Feb., 1978