I-41　　三角形膜要素の有限変形解析

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本文は，Scheibe の有限変形解析を試みたものであり，筆者等が，骨組構造物の有限変形解析に就いて用いた手法を，FEM に適用する事を目的としている

[II] 滑節三角形膜要素

1. 要素辺長増分，要素辺方向余弦増分

$$x_j \triangleq [x_j,\ y_j]^*,\quad d_{ji} \triangleq x_j - x_i,\quad d_j \triangleq [d_{ji},\ d_{jk}]^*,\quad l_{ji} \triangleq \sqrt{d_{ji}^* \cdot d_{ji}},\quad \alpha_{ji} \triangleq [\alpha_{ji},\ \beta_{ji}]^* \triangleq \frac{d_{ji}}{l_{ji}} \quad \text{---}(1)$$

$$\alpha_j \triangleq [\alpha_{ji},\ \alpha_{jk}]^*,\quad N_j \triangleq [N_{ji},\ N_{jk}]^*\quad D_j = [X_j,\ Y_j]^*\qquad \mu = \left[\frac{N_{ji}+\Delta N_{ji}}{l_{ji}+\Delta l_{ji}},\ \frac{N_{jk}+\Delta N_{jk}}{l_{jk}+\Delta l_{jk}}\right]^* \quad \text{---}(2)$$

$$x \triangleq [x_i,\ x_j,\ x_k]^*\quad d \triangleq [d_i,\ d_j,\ d_k]^*\quad N \triangleq [N_i,\ N_j,\ N_k]^* \quad \text{---}(3)$$

と，おく。今，

$$(2l_{ji}+\Delta l_{ji})\Delta l_{ji} = (l_{ji}+\Delta l_{ji})^2 - l_{ji}^2 = (d_{ji}+\Delta d_{ji})^*(d_{ji}+\Delta d_{ji}) - d_{ji}^* \cdot d_{ji} \quad \text{---}(4)$$

から，

$$\omega_{ji} = \frac{2}{l_{ji}^2}\left(d_{ji}+\frac{1}{2}\Delta d_{ji}\right)^*\Delta d_{ji},\quad \chi_{ji} = 1 - \frac{1}{4}\omega_{ji} + \frac{1}{8}\omega_{ji}^2 - \frac{5}{64}\omega_{ji}^3 + \frac{7}{128}\omega_{ji}^4 - \frac{21}{512}\omega_{ji}^5 + \cdots \quad \text{---}(5)$$

と，おけば，

$$\Delta l_{ji} = \frac{1}{2}\chi_{ji}\cdot\omega_{ji}\cdot l_{ji} = \frac{\chi_{ji}}{l_{ji}}\left(d_{ji}+\frac{1}{2}\Delta d_{ji}\right)^*\Delta d_{ji},\quad \Delta\alpha_{ji} = \frac{\Delta d_{ji} - \alpha_{ji}\cdot\Delta l_{ji}}{l_{ji}+\Delta l_{ji}} = \frac{1}{l_{ji}}\left[e - \frac{\chi_{ji}}{l_{ji}}\alpha_{ji}\left(d_{ji}+\frac{1}{2}\Delta d_{ji}\right)^*\right]\Delta d_{ji} \quad \text{---}(6)$$

又，$\Delta u_j = [\Delta u_{ji},\ \Delta u_{jk}]^*$ とおけば

$$\Delta u \triangleq \begin{bmatrix}\Delta u_i\\ \Delta u_j\\ \Delta u_k\end{bmatrix} = \begin{bmatrix}\alpha_i^* & & \\ & \alpha_j^* & \\ & & \alpha_k^*\end{bmatrix}\begin{bmatrix}\Delta x_i\\ \Delta x_j\\ \Delta x_k\end{bmatrix} \triangleq G_u\cdot\Delta x \quad \text{---}(7)$$

$$G_{lji} = \frac{\chi_{ji}}{l_{ji}}\left(d_{ji}+\frac{1}{2}\Delta d_{ji}\right)^*\Delta d_{ji},\quad \Delta l \triangleq \begin{bmatrix}\Delta l_{ik}\\ \Delta l_{ji}\\ \Delta l_{kj}\end{bmatrix} = \begin{bmatrix}G_{lik} & & \\ & G_{lji} & \\ & & G_{lkj}\end{bmatrix}\begin{bmatrix}\Delta d_{ik}\\ \Delta d_{ji}\\ \Delta d_{kj}\end{bmatrix} = \begin{bmatrix}G_{lik} & & -G_{lik}\\ -G_{lji} & G_{lji} & \\ & -G_{lkj} & G_{lkj}\end{bmatrix}\begin{bmatrix}\Delta x_i\\ \Delta x_j\\ \Delta x_k\end{bmatrix} \triangleq G_l\cdot\Delta x \quad \text{---}(8)$$

$$\therefore\ \Delta l = G_{lx}\cdot\Delta x = G_{lx}\cdot G_u^{-1}\cdot\Delta u \triangleq G_{lu}\cdot\Delta x \quad \text{---}(9)$$

2. 要素座標系格点力増分

要素内で，応力度を一定とし，これを

$$\sigma_0 = [\sigma_{x0},\ \sigma_{xy0},\ \sigma_{y0}]^* \quad \text{---}(10)$$

と，おけば，例えば，要素辺 $j-i$ の接線方向の応力は，

$$\sigma_{sji} = [\sigma_{x0},\ \sigma_{xy0},\ \sigma_{y0}][\alpha\cdot\beta,\ \alpha^2-\beta^2,\ -\alpha\cdot\beta]_{ji}^* \triangleq \sigma_0^*\cdot c_{ji} \quad \text{---}(11)$$

従って，$j-i$ に沿う周辺力の為す仮想仕事は，板厚を t_m とすれば，

$$\delta\Lambda_{sji} = t_m\cdot\delta\sigma_{sji}\cdot\Delta l_{ji} = t_m\cdot\delta\sigma_0^*\cdot c_{ji}\cdot\Delta l_{ji} \quad \text{---}(12)$$

全周辺について，

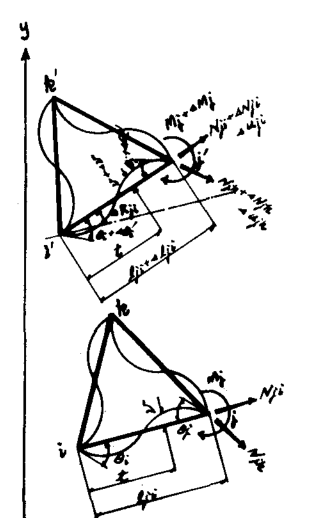
$$\delta\Lambda_s = t_m\cdot\delta\sigma_0^*[c_{ik},\ c_{ji},\ c_{kj}][\Delta l_{ik},\ \Delta l_{ji},\ \Delta l_{kj}]^* \triangleq t_m\cdot\delta\sigma_0^*\cdot C\cdot G_{lu}\cdot\Delta u \quad \text{---}(13)$$

内部応力の為す，仮想仕事は

$$D_s = \frac{1}{E}\begin{bmatrix}1 & -\nu & \\ -\nu & 1 & \\ & & 2(1+\nu)\end{bmatrix}\qquad A_m = \frac{1}{2}\begin{Vmatrix}1 & x_i & y_i\\ 1 & x_j & y_j\\ 1 & x_k & y_k\end{Vmatrix} \quad \text{---}(14)$$

と，おけば，

$$\delta\Lambda_I = t_m\int\delta\sigma_0^*\cdot D_s\cdot\sigma_0\cdot dA = t_m\cdot A_m\cdot\delta\sigma_0^*\cdot D_s\cdot\sigma_0 \quad \text{---}(15)$$



節点力の為す，仮想仕事は，

$$\delta\Lambda_N = \delta(\Delta\mathcal{N}^*)\Delta\mathcal{U} = \delta(\Delta\mathcal{U}^*)\Delta\mathcal{N} \quad \cdots \quad (16)$$

$$\because\ \delta\Lambda_\beta = \delta\Lambda_I \quad , t_m\cdot A_m\cdot\delta\sigma_0^*\cdot D_s\cdot\sigma_0 = t_m\cdot\delta\sigma_0^*\cdot\mathbb{C}\cdot G_{lu}\cdot\Delta\mathcal{U}\ , \quad \therefore\ \sigma_0 = \frac{D_s^{-1}\cdot\mathbb{C}\cdot G_{lu}}{A_m}\Delta\mathcal{U} \quad \cdots \quad (17)$$

$$\because\ \delta\Lambda_N = \delta\Lambda_S \quad , \delta(\Delta\mathcal{U}^*)\Delta\mathcal{N} = t_m\cdot\delta\sigma_0^*\cdot\mathbb{C}\cdot G_{lu}\cdot\Delta\mathcal{U} = t_m\cdot\delta(\Delta\mathcal{U}^*)\cdot(\mathbb{C}\cdot G_{lu})^*\cdot\sigma_0 = t_m\cdot\delta(\Delta\mathcal{U}^*) \quad \cdots \quad (18)$$

$$= t_m\cdot\delta(\Delta\mathcal{U}^*)\ \frac{(\mathbb{C}\cdot G_{lu})^*\cdot D_s^{-1}\cdot(\mathbb{C}\cdot G_{lu})}{A_m}\Delta\mathcal{U} \quad \cdots \quad (19)$$

$$\therefore\ \mathbb{K}_S = \frac{t_m}{A_m}(\mathbb{C}\cdot G_{lu})^*\cdot D_s^{-1}\cdot(\mathbb{C}\cdot G_{lu}) \quad \cdots \quad (20)$$

と，置けば，要素座標系格点力増分と，絶対座標系変位増分の関係は．

$$\Delta\mathcal{N} = \mathbb{K}_S\cdot\Delta\mathcal{U} = \mathbb{K}_S\cdot G_u\cdot\Delta\mathcal{X} \triangleq \mathbb{K}_S'\cdot\Delta\mathcal{X} \quad \cdots \quad (21)$$

3. 絶対座標系格点力増分

格点 j に付いて

$$\Delta\mathcal{D}_j = (\alpha_j+\Delta\alpha_j)^*(\mathcal{N}_j+\Delta\mathcal{N}_j) - \alpha_j^*\mathcal{N}_j = \alpha_j^*\Delta\mathcal{N}_j + \Delta\alpha_j^*(\mathcal{N}_j+\Delta\mathcal{N}_j) \quad \cdots \quad (22)$$

ここで，

$$\Delta\alpha_j^*(\mathcal{N}_j+\Delta\mathcal{N}_j) = [\Delta\alpha_{ji}\ ,\ \Delta\alpha_{jk}]\begin{bmatrix} N_{ji} \\ N_{jk}\end{bmatrix} = \left[\frac{N_{ji}+\Delta N_{ji}}{l_{ji}+\Delta l_{ji}}\ ,\ \frac{N_{jk}+\Delta N_{jk}}{l_{jk}+\Delta l_{jk}}\right]\left[\begin{array}{c|c} e - \frac{x_{ji}}{l_{ji}}\alpha_{ji}(\alpha_{ji}+\frac{1}{2}\Delta\alpha_{ji}) & 0 \\ \hline 0 & e - \frac{x_{jk}}{l_{jk}}\alpha_{jk}(\alpha_{jk}+\frac{1}{2}\Delta\alpha_{jk})\end{array}\right]\begin{bmatrix}\Delta\alpha_{ji} \\ \Delta\alpha_{jk}\end{bmatrix}$$

$$\triangleq [\mu_{ji},\ \mu_{jk}]\begin{bmatrix} G_{ji} & \\ & G_{jk}\end{bmatrix}\begin{bmatrix}\Delta\alpha_{ji} \\ \Delta\alpha_{jk}\end{bmatrix} = \mu_{ji}\cdot G_{ji}\cdot\Delta\alpha_{ji} + \mu_{jk}\cdot G_{jk}\cdot\Delta\alpha_{jk} = [-\mu_{ji}\cdot G_{ji}\ ,\ \mu_{ji}\cdot G_{ji}+\mu_{jk}\cdot G_{jk}\ ,\ -\mu_{jk}\cdot G_{jk}][\Delta\mathcal{X}_i\ ,\ \Delta\mathcal{X}_j\ ,\ \Delta\mathcal{X}_k]^* \quad \cdots \quad (23)$$

同様にして，

$$\Delta\mathcal{D} \triangleq \begin{bmatrix}\Delta\mathcal{D}_i \\ \Delta\mathcal{D}_j \\ \Delta\mathcal{D}_k\end{bmatrix} = \begin{bmatrix}\alpha_i^* & & \\ & \alpha_j^* & \\ & & \alpha_k^*\end{bmatrix}\begin{bmatrix}\Delta\mathcal{N}_i \\ \Delta\mathcal{N}_j \\ \Delta\mathcal{N}_k\end{bmatrix} + \begin{bmatrix}\mu_{ij}\cdot G_{ij}+\mu_{ik}\cdot G_{ik} & -\mu_{ij}\cdot G_{ij} & -\mu_{ik}\cdot G_{ik} \\ -\mu_{ji}\cdot G_{ji} & \mu_{ji}\cdot G_{ji}+\mu_{jk}\cdot G_{jk} & -\mu_{jk}\cdot G_{jk} \\ -\mu_{ki}\cdot G_{ki} & -\mu_{kj}\cdot G_{kj} & \mu_{ki}\cdot G_{ki}+\mu_{kj}\cdot G_{kj}\end{bmatrix}\begin{bmatrix}\Delta\mathcal{X}_i \\ \Delta\mathcal{X}_j \\ \Delta\mathcal{X}_k\end{bmatrix} \triangleq \alpha^*\Delta\mathcal{N} + \mathcal{S}\cdot\Delta\mathcal{X} = (\alpha^*\cdot\mathbb{K}_S\cdot G_u + \mathcal{S})\Delta\mathcal{X} \quad \cdots \quad (24)$$

[III] 剛節三角形膜要素

1. 曲げモーメントに因る，要素辺長変形量

要素辺 j-i に付いて，要素座標系タワミ角が，

$$[\theta_i,\ \theta_j] \longrightarrow [\theta_i+\Delta\theta_i',\ \theta_j+\Delta\theta_j'] \quad \cdots \quad (25)$$

に変化した時の，要素辺長縮量は，適宜，suffix を略し

$$\varepsilon = \int_0^l\left[\sqrt{1+\left(\frac{dv}{dt}+\frac{d\Delta v}{dt}\right)^2} - \sqrt{1+\left(\frac{dv}{dt}\right)^2}\right]dt = \frac{1}{2}\int_0^l\sqrt{\left(\frac{dv}{dt}\right)^2 + 2\frac{dv}{dt}\cdot\frac{d\Delta v}{dt} + \left(\frac{d\Delta v}{dt}\right)^2}\,dt \quad \cdots \quad (26)$$

今，trial-function を

$$\begin{bmatrix} v \\ \Delta v\end{bmatrix} = \begin{bmatrix}\theta_i\ , & \theta_j \\ \Delta\theta_i'\ , & \Delta\theta_j'\end{bmatrix}\begin{bmatrix} 1, & -\frac{2}{l}\ , & \frac{1}{l^2} \\ & -\frac{1}{l}\ , & \frac{1}{l^2}\end{bmatrix}\begin{bmatrix} t \\ t^2 \\ t^3\end{bmatrix} \quad \cdots \quad (27)$$

とおけば

$$\varepsilon = \frac{l}{60}[2\theta_i+\Delta\theta_i',\ 2\theta_j+\Delta\theta_j']\begin{bmatrix} 4 & -1 \\ -1 & 4\end{bmatrix}\begin{bmatrix}\Delta\theta_i' \\ \Delta\theta_j'\end{bmatrix} \triangleq [r_i,\ r_j]\begin{bmatrix}\Delta\theta_i' \\ \Delta\theta_j'\end{bmatrix} = [r_i,\ r_j]\begin{bmatrix}\Delta\theta_i \\ \Delta\theta_j\end{bmatrix} - (r_i+r_j)\Delta R_{ji} \quad \cdots \quad (28)$$

$$\mathcal{E} = \begin{bmatrix}\varepsilon_{ik} \\ \varepsilon_{ji} \\ \varepsilon_{kj}\end{bmatrix} = \begin{bmatrix} r_{ik\cdot i} & & r_{ik\cdot k} \\ r_{ji\cdot i}\ , & r_{ji\cdot j} & \\ & r_{kj\cdot j} & ,\ r_{kj\cdot k}\end{bmatrix}\begin{bmatrix}\Delta\theta_i \\ \Delta\theta_j \\ \Delta\theta_k\end{bmatrix} - \begin{bmatrix} r_{ik\cdot i}+r_{ik\cdot k} & & \\ & r_{ji\cdot i}+r_{ji\cdot j} & \\ & & r_{kj\cdot j}+r_{kj\cdot k}\end{bmatrix}\begin{bmatrix}\Delta R_{ik} \\ \Delta R_{ji} \\ \Delta R_{kj}\end{bmatrix} \triangleq \gamma\cdot\Delta\Theta - \mathcal{g}\cdot\Delta R \quad \cdots \quad (29)$$

2. 部材角

$$\Delta R_{ji} = \frac{1}{l_{ji}}[-\beta_{ji},\ \alpha_{ji}]\begin{bmatrix}\Delta x_j - \Delta x_i \\ \Delta y_j - \Delta y_i\end{bmatrix} = \frac{1}{l_{ji}}[-\beta_{ji},\ \alpha_{ji}]\begin{bmatrix}\alpha_{ik}\cdot\Delta u_{ik} - \alpha_{jk}\cdot\Delta u_{jk} \\ \beta_{ik}\cdot\Delta u_{ik} - \beta_{jk}\cdot\Delta u_{jk}\end{bmatrix} = -\frac{\Delta u_{ik}}{l_{ji}}\begin{Vmatrix}\alpha_{ij} & \alpha_{ik} \\ \beta_{ij} & \beta_{ik}\end{Vmatrix} + \frac{\Delta u_{jk}}{l_{ji}}\begin{Vmatrix}\alpha_{ji} & \alpha_{jk} \\ \beta_{ji} & \beta_{jk}\end{Vmatrix} \triangleq -\frac{1}{l_{ji}}[\bar{\beta}_i\ ,\ \bar{\beta}_j]\begin{bmatrix}\Delta u_{ik} \\ \Delta u_{jk}\end{bmatrix}$$

$$= -\left[\frac{\bar{\beta}_i}{l_{ji}},\ 0 \;\middle|\; 0,\ \frac{\bar{\beta}_j}{l_{ji}} \;\middle|\; 0\ \ 0\right][\Delta u_{ik},\ \Delta u_{ij} \mid \Delta u_{ji}\ \ \Delta u_{jk} \mid \Delta u_{kj}\ \ \Delta u_{ki}]^* \triangleq G_{Rji}\cdot\Delta\mathcal{U} \quad \cdots \quad (30)$$

$$\therefore\ \Delta R \triangleq [\Delta R_{ik},\ \Delta R_{ji},\ \Delta R_{kj}]^* = [G_{Rik},\ G_{Rji},\ G_{Rkj}]\Delta\mathcal{U} \triangleq G_R\cdot\Delta\mathcal{U} \quad \cdots \quad (31)$$

3. 部材長

$$\Delta l_{ji} = \Delta u_{ji} + \Delta u_{ij} = [0\ \ 1 \mid 1\ \ 0 \mid 0\ \ 0][\Delta u_{ik},\ \Delta u_{ij} \mid \Delta u_{ji}\ \ \Delta u_{jk} \mid \Delta u_{kj}\ \ \Delta u_{ki}]^* \triangleq G_{lji}\cdot\Delta\mathcal{U} \quad \cdots \quad (32)$$

同様に

$$\Delta l = [\Delta l_{ik},\ \Delta l_{ji},\ \Delta l_{kj}]^* = [G_{lik},\ G_{lji},\ G_{lkj}]^*\Delta\mathcal{U} \triangleq G_l\cdot\Delta\mathcal{U} \quad \cdots \quad (33)$$

4 要素座標系格点力増分

$$[\sigma_x,\ \sigma_{xy},\ \sigma_y] = [\sigma_1+\sigma_2 y\ ,\ \sigma_3\ ,\ \sigma_4+\sigma_5 x]\ ,\quad [x\cdot y] = [x_i,\ y_i] + t[\alpha\cdot\beta] \quad \cdots \quad (34)$$

$$\begin{bmatrix} \mathbb{C}_{ji} \\ \mathbb{B}_{ji} \end{bmatrix} = \begin{bmatrix} \alpha^2 & -2\alpha\cdot\beta & \beta^2 \\ \alpha\cdot\beta & \alpha^2-\beta^2 & -\alpha\cdot\beta \end{bmatrix} \quad \text{---} \quad (35)$$

とおけば，要素辺 $j-i$ の接線方向，及び，鉛直方向の周辺力の為す，仮想仕事は，

$$\delta\Lambda_{sji} = t_m\cdot\mathbb{C}_{ji}\left[\int\sigma_x\cdot dt,\ \int\sigma_{xy}\cdot dt,\ \int\sigma_y\cdot dt\right]\frac{\Delta l_{ji}-\varepsilon_{ji}}{l_{ji}} + t_m\cdot\mathbb{B}_{ji}\left[\int\sigma_x\cdot\Delta v\cdot dt,\ \int\sigma_{xy}\cdot\Delta v\cdot dt,\ \int\sigma_y\cdot\Delta v\cdot dt\right] \quad \text{---} \quad (36)$$

積分を実行すれば

$$\left[\int\sigma_x\cdot dt,\ \int\sigma_{xy}\cdot dt,\ \int\sigma_y\cdot dt\right] = l\left[\sigma_1+\frac{l}{2}\beta\cdot\sigma_2,\ \sigma_3,\ \sigma_4+\frac{l}{2}\alpha\cdot\sigma_5\right] \quad \text{---} \quad (37)$$

$$\begin{bmatrix} \int\sigma_x\cdot\Delta v\cdot dt \\ \int\sigma_{xy}\cdot\Delta v\cdot dt \\ \int\sigma_y\cdot\Delta v\cdot dt \end{bmatrix} = \frac{l^2}{60}\begin{bmatrix} 5\sigma_1+2l\cdot\beta\cdot\sigma_2 \\ 5\sigma_3 \\ 5\sigma_4+2l\cdot\alpha\cdot\sigma_5 \end{bmatrix}\Delta\theta_i' - \frac{l^2}{60}\begin{bmatrix} 5\sigma_1+3l\cdot\beta\cdot\sigma_2 \\ 5\sigma_3 \\ 5\sigma_4+3l\cdot\alpha\cdot\sigma_5 \end{bmatrix}\Delta\theta_j' \quad \text{---} \quad (38)$$

ここで，

$$a = \frac{l^2}{60}\begin{bmatrix} 5,\ 2l\cdot\beta & & \\ & 5 & \\ & & 5,\ 2l\cdot\beta \end{bmatrix},\quad b = \frac{l^2}{60}\begin{bmatrix} 5,\ 3l\cdot\beta & & \\ & 5 & \\ & & 5,\ 3l\cdot\beta \end{bmatrix} \quad \text{---} \quad (39)$$

$$\bar{\mathbb{C}}_{ji} = \left[\alpha\cdot\beta,\ \left(\frac{l}{2}\beta\right)\alpha\cdot\beta,\ \alpha^2-\beta^2,\ -\alpha\beta,\ -\left(\frac{l}{2}\alpha\right)\alpha\cdot\beta\right]_{ji},\quad \sigma_0 = [\sigma_1\ \ \sigma_2\ \ \sigma_3\ \ \sigma_4\ \ \sigma_5]^* \quad \text{---} \quad (40)$$

とおけば

$$\begin{aligned}\delta\Lambda_{sji} &= t_m\cdot\bar{\mathbb{C}}_{ji}\cdot\delta\sigma_0(\Delta l_{ji}-\varepsilon_{ji}) + t_m\cdot\mathbb{B}_{ji}\{a_{ji}\cdot\Delta\theta_i - b_{ji}\cdot\Delta\theta_j - (a_{ji}-b_{ji})\Delta R_{ji}\}\delta\sigma_0 = t_m\cdot\delta\sigma_0^*\cdot\bar{\mathbb{C}}_{ji}^*(\Delta l_{ji}-\varepsilon_{ji}) \\ &\quad + t_m\cdot\delta\sigma_0^*\left[(\mathbb{B}_{ji}\cdot a_{ji})^*\Delta\theta_i - (\mathbb{B}_{ji}\cdot b_{ji})^*\Delta\theta_j - (\mathbb{B}_{ji}\cdot a_{ji}-\mathbb{B}_{ji}\cdot b_{ji})^*\Delta R_{ji}\right] = t_m\cdot\delta\sigma_0^*\cdot\bar{\mathbb{C}}_{ji}^*(\Delta l_{ji}-\varepsilon_{ji}) \\ &\quad + t_m\cdot\delta\sigma_0^*\left[(\mathbb{B}_{ji}\cdot a_{ji})^*,\ -(\mathbb{B}_{ji}\cdot b_{ji}),\ 0\right]\begin{bmatrix}\Delta\theta_i \\ \Delta\theta_j \\ \Delta\theta_k\end{bmatrix} - t_m\cdot\delta\sigma_0^*\left[0,\ \{\mathbb{B}_{ji}(a_{ji}-b_{ji})\}^*,\ 0\right]\begin{bmatrix}\Delta R_{ik} \\ \Delta R_{ji} \\ \Delta R_{kj}\end{bmatrix} \quad \text{---} \quad (41)\end{aligned}$$

ここで，

$$\bar{\mathbb{B}}_{ji} \triangleq \left[(\mathbb{B}_{ji}\cdot a_{ji})^*,\ -(\mathbb{B}_{ji}\cdot b_{ji})^*,\ 0\right],\quad \hat{\mathbb{B}}_{ji} \triangleq \left[0,\ \{\mathbb{B}_{ji}(a_{ji}-b_{ji})^*\},\ 0\right] \quad \text{---} \quad (42)$$

とおけば

$$\delta\Lambda_{sji} = t_m\cdot\sigma_0^*\cdot\bar{\mathbb{C}}_{ji}(\Delta l_{ji}-\varepsilon_{ji}) + t_m\cdot\delta\sigma_0^*(\bar{\mathbb{B}}_{ji}\cdot\Delta\Theta - \hat{\mathbb{B}}_{ji}\cdot\Delta\mathbb{R}) \quad \text{---} \quad (43)$$

全周に付いて加算し

$$\begin{aligned}\delta\Lambda_s &= t_m\cdot\delta\sigma_0^*\left[\bar{\mathbb{C}}_{jk}^*,\ \bar{\mathbb{C}}_{ji}^*,\ \bar{\mathbb{C}}_{kj}^*\right]\left[\Delta l_{ik}-\varepsilon_{ik},\ \Delta l_{ji}-\varepsilon_{ji},\ \Delta l_{kj}-\varepsilon_{kj}\right]^* = t_m\cdot\delta\sigma_0^*\left[\bar{\mathbb{B}}_{ik}+\bar{\mathbb{B}}_{ji}+\bar{\mathbb{B}}_{kj}\right]\Delta\Theta - t_m\cdot\delta\sigma_0^*\left[\hat{\mathbb{B}}_{ik}+\hat{\mathbb{B}}_{ji}+\hat{\mathbb{B}}_{kj}\right]\Delta\mathbb{R} \\ &= t_m\cdot\delta\sigma_0^*\cdot\bar{\mathbb{C}}(\Delta\mathbb{l}-\mathcal{E}) - t_m\cdot\delta\sigma_0^*\cdot\bar{\mathbb{B}}\cdot\Delta\Theta - t_m\cdot\delta\sigma_0^*\cdot\hat{\mathbb{B}}\cdot\Delta\mathbb{R} = t_m\cdot\delta\sigma_0^*\left[\bar{\mathbb{C}}(\mathbb{G}_R+\mathbb{g}\cdot\mathbb{G}_R)-\hat{\mathbb{B}}\cdot\mathbb{G}_R,\ \hat{\mathbb{B}}-\bar{\mathbb{C}}\cdot\mathbb{r}\right]\begin{bmatrix}\Delta\mathbb{u} \\ \Delta\Theta\end{bmatrix} \triangleq t_m\cdot\delta\sigma_0^*\cdot\mathbb{T}\cdot\Delta\tilde{\mathbb{u}} \quad \text{---} \quad (44)\end{aligned}$$

内部応力の為す，仮想仕事は

$$\mathbb{I}_0 = \begin{bmatrix}1 & & \\ & 0\ 1 & \\ & & 1\ 0\end{bmatrix}\quad \mathbb{I}_x = \begin{bmatrix}0 & & \\ & 0\ 0\ 0 & \\ & & 1\end{bmatrix}\quad \mathbb{I}_y = \begin{bmatrix} & 1\ 0\ 0\ 0 \\ 0 & \\ 0 & \end{bmatrix}\quad \bar{\sigma} = \begin{bmatrix}\sigma_x \\ \sigma_{xy} \\ \sigma_y\end{bmatrix} = (\mathbb{I}_0+x\cdot\mathbb{I}_x+y\cdot\mathbb{I}_y)\sigma_0 \quad \text{---} \quad (45)$$

とおけば

$$\delta\Lambda_I = t_m\cdot\int\delta\bar{\sigma}^*\cdot D_s\cdot\bar{\sigma}\,dA \quad \text{---} \quad (46)$$

$$\begin{aligned}\delta\bar{\sigma}^*\cdot D_s\cdot\bar{\sigma} &= \delta\bar{\sigma}^*(\mathbb{I}_0+x\mathbb{I}_x+y\mathbb{I}_y)^*\cdot D_s\cdot(\mathbb{I}_0+x\mathbb{I}_x+y\mathbb{I}_y) = \delta\sigma_0^*(\mathbb{I}_0^*\cdot D_s\cdot\mathbb{I}_0 + 2x\cdot\mathbb{I}_x^*\cdot D_s\cdot\mathbb{I}_0 + 2y\cdot\mathbb{I}_y^*\cdot D_s\cdot\mathbb{I}_0 + 2xy\,\mathbb{I}_x^*\cdot D_s\cdot\mathbb{I}_y \\ &\quad + x^2\cdot\mathbb{I}_x^*\cdot D_s\cdot\mathbb{I}_x + y^2\cdot\mathbb{I}_y^*\cdot D_s\cdot\mathbb{I}_y) \quad \text{---} \quad (47)\end{aligned}$$

今，

$$G_x = \frac{2}{3}(x_i+x_j+x_k),\quad G_y = \frac{2}{3}(y_i+y_j+y_k),\quad I_{xx} = \frac{1}{3}\left[x_1^2+x_2^2+x_3^2+x_1x_2+x_2x_3+x_3x_1\right],\quad I_{yy} = \frac{1}{3}\left[y_1^2+y_2^2+y_3^2+y_1y_2+y_2y_3+y_3y_1\right] \quad \text{---} \quad (48)$$

$$I_{xy} = \frac{1}{6}\left[2(x_1y_1+x_2y_2+x_3y_3)+(x_1y_2+x_2y_1)+(x_1y_3+x_3y_2)+(x_2y_3+x_3y_2)+(x_3y_1+x_1y_3)\right] \quad \text{---} \quad (49)$$

$$\therefore\ \delta\Lambda_I = t_m\cdot A_m\cdot\delta\sigma_0^*\left[\mathbb{I}_0^*D_s\mathbb{I}_0 + \mathbb{I}_x^*D_s\cdot\mathbb{I}_0\cdot G_x + \mathbb{I}_y^*D_s\cdot\mathbb{I}_0\cdot G_y + \mathbb{I}_x^*\cdot D_s\cdot\mathbb{I}_y\cdot I_{xy} + \mathbb{I}_x^*\cdot D_s\cdot I_{xx} + \mathbb{I}_y^*\cdot D_s\cdot I_{yy}\right]\sigma_0 \triangleq A_m\cdot\delta\sigma_0^*\cdot H\cdot\sigma_0 \quad \text{---} \quad (50)$$

節点力の為す，仮想仕事は

$$\Delta f \triangleq [\Delta N,\ \Delta M]^* \qquad \therefore\ \delta\Lambda_N = \delta(\Delta f^*)\cdot\Delta\tilde{u} \quad \text{---} \quad (51)$$

$$\because\ \delta\Lambda_S = \delta\Lambda_I \qquad \therefore\ t_m\cdot A_m\cdot\delta\sigma_0^*\cdot H\cdot\sigma_0 = t_m\cdot\delta\sigma_0^*\cdot\mathbb{T}\cdot\Delta\tilde{u} \qquad \therefore\ \sigma_0 = \frac{H^{-1}\cdot\mathbb{T}}{A_m}\Delta\tilde{u} \quad \text{---} \quad (52)$$

$$\because\ \delta\Lambda_N = \delta\Lambda_S \qquad \therefore\ \delta(\Delta\tilde{u}^*)\cdot\Delta f = t_m\cdot\delta\sigma_0^*\cdot\mathbb{T}\cdot\Delta\tilde{u} = t_m\cdot\delta(\Delta\tilde{u}^*)\cdot\mathbb{T}^*\cdot\sigma_0 = t_m\cdot\delta(\Delta\tilde{u}^*)\cdot\mathbb{T}\cdot\frac{H^{-1}\cdot\mathbb{T}}{A_m}\Delta\tilde{u} \quad \text{---} \quad (53)$$

$$\therefore\ \Delta f = \frac{t_m}{A_m}\mathbb{T}^*\cdot H^{-1}\cdot\mathbb{T}\cdot\Delta\tilde{u} \triangleq \mathbb{K}\cdot\Delta\tilde{u} \quad \text{---} \quad (54)$$

[IV] 参考文献

1. 前田：「立体ラーメンの有限変形解析」（才24回 応用力学連合会）
2. 前田：「平面応力場の有限変形解析」（才1回 土木学会 関東支部会）