A Simplified Method of Analysis of a Plate Problem

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In general, an "exact" solution of a plate problem is either hard to be obtained or difficult to be evaluated even if it can be obtained. Therefore, it is desirable to seek an easier method to obtain a reasonably approximate solution even if the solution obtained in this way may have less degree of accuracy compared with the "exact result.

A simple method which falls into such a category is developed in this paper. This method is illustrated by solving for the slopes at an end of a simply supported rectangular plate under uniform load. As an example of application of this method as well as the slopes obtained in this way, an approximate expression of torsional stresses in an edge beam supporting a rectangular plate is developed. When the edge beam is rigid enough so that the rotation of this beam caused by the twisting moment may be considered as small, the plate can be assumed as fixed at the supports. In this case, the torsional moment at the edge beam is directly proportional to the variation of the slopes on the corresponding boundary of the simply supported plate.

Consider a simply supported rectangular plate under uniformly distributed load, q, which satisfies the usual assumptions of small deformation theory of plate. Let this plate be divided by several pairs of mutually orthogonal beam strips which are parallel to each of the corresponding boundaries. Each pair is assumed to be entirely independent of other pairs and each beam strip of this pair is supposed to support a part of the concentrated load at the junction of two beams by satisfing the deflection compatibility condition only. When this procedure is carried out over the entire surface of the plate, a result similar to the influence surface can be obtained and the problem of a plate under uniform load can be handled.

Let a concentrated load, P, be applied at the junction of such a pair of beams. When the coordinates of these two beams are expressed as $x = x$ and $y = y$, respectively, this junction is at (x, y). Defining P_x and P_y as the distributed loads on the beams in the x - and y - directions, respectively,

$$P = P_x + P_y . \tag{1}$$

The deflection of the beam in the x - direction is

$$\delta_x = \frac{P_x X (a-x)}{6 E_x I_x a} (2 ax - 2 x^2). \tag{2}$$

A similar expression can be obtained for that in the y - direction, δ_y. Assuming that this plate is homogeneous and isotropic, the following result can be obtained from the condition $\delta_x = \delta_y$:

$$P_x = \frac{\frac{P y (Na-y)}{Na} (2 Nay - 2 y^2)}{\frac{x (a-x)}{a} (2 ax - 2 x^2) + \frac{y (Na-y)}{Na} (2 Nay - 2 y^2)} \tag{3}$$

in which $N = \frac{b}{a}$

where

a : length of the plate in the x - direction

b : length of the plate in the y - direction.

This procedure is carried out with different values of x, within the region (o, a), while the value of y is fixed. Then, the load on the beam strip at $y = y$ is a continuous function in P_x and the variation of this load is shown in Figure 1. Because of the symmetry, the slope in the x - direction on the boundary $x = o$ can be shown as :

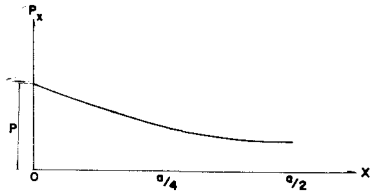

Figure 1, Typical Distributed Load Pattern on a Beam in the x - Direction.

$$\left.\frac{dw_1}{dx}\right|_{x=o} = \frac{1}{EI}\int_o^{a/2}\left\{ x\int_o^{a/2} P_x dx - \int_o^{x} P_x\,(x-\xi)\, d\xi \right\} dx \qquad (4)$$

Since the integrand of this equation is too complicated, one may readily agree that it is very difficult to integrate this equation by an ordinary method. In this paper, polynomial interpolation scheme is used to transform such a function, which is difficult to integrate by the ordinary method, to polynomial functions which can be easily operated on.

Assuming that P_x can be expressed as

$$P_x = a_o + a_1 x + \ldots + a_{n-1} x^{n-1} \qquad (5)$$

Let P_{x_i} be the value of P_x evaluated at x_i . If P_x is evaluated at n different x, the following matrix equation can be obtained.

$$C \cdot A = P_x \qquad (6)$$

where

$$C = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

$$A = \begin{bmatrix} a_o \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \end{bmatrix} \qquad P = \begin{bmatrix} P_{x_1} \\ P_{x_2} \\ \cdot \\ \cdot \\ \cdot \\ P_{x_n} \end{bmatrix}$$

then

$$A = C^{-1} \cdot P_x$$

and

$$P_x = D \cdot A \qquad (7)$$

where

$$D = (1, x, \ldots, x^{n-1})$$

Equation (7) is evaluated for the cases of $n = 3, 4,$ and 5, within the region ($o, \frac{a}{2}$). These results are compared with that of the original equation and it is concluded that the $n = 3$ case is accurate enough for engineering purposes (Figure 2). Equation (4) is integrated after taking three point polynomial interpolation on P_x and the results are compared with those of "exact" solution of Navier, assuming that the intensity of q on unit area is equal to P (Figure 3).

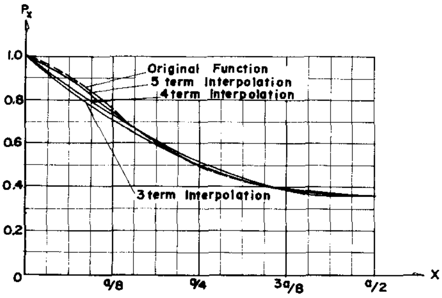

Figure 2, Results of Polynomial Interpolations.

From Figure 3, it can be concluded that the result obtained by this simple method has less degree of accuracy compared with that of the "exact" method but is far better than that of one - dimensional analysis case, i.e., $n = \infty$. However, a modified theory can be employed to minimize the discrepancies between the two methods as long as one is interested in calculating a traction which is obtainable by an existing "exact" theory, such as the slope on the boundary.

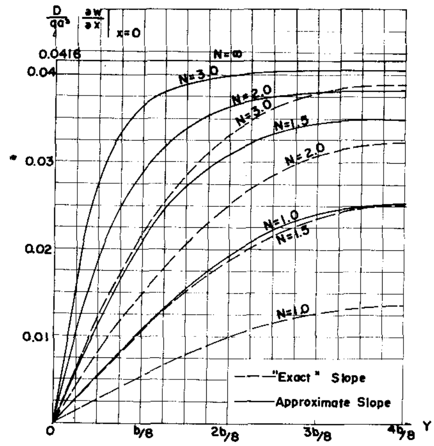

Figure 3, Comparison of the Approximate Result with the "Exact" One.

The slope at $x = o$ can be expressed by a series of polynomial terms in y. This slope is also a function of the plate shape factor, N. Let $F (y, N)$ be the slopes obtained by the simple method and be expressed as

$$F (y, N) = \frac{D}{qa^3} \frac{dw_1}{dx}\Big|_{x=o} = b_o + b_1 y + \ldots + b_4 y^4 \qquad (8)$$

where D is the flexural rigidity of the plate. Let $G (y, N)$ be defined such that

$$\frac{D}{qa^3} \frac{\partial w}{\partial x}\Big|_{x=o} = G (y, N) F (y, N) \qquad (9)$$

and

$$G (y, N) = C_o + \ldots + C_4 y^4$$

in which W is the "exact" deflection. Evaluating $\frac{D}{qa^3} \frac{\partial w}{\partial x}\Big|_{x=o}$ and $F (y, N)$ at $y_1, \ldots, y_5$, the following matrix equation can be obtained :

$$\left[\frac{\frac{D}{qa^3} \frac{\partial w}{\partial x}\Big|_{x=o}}{F (y, N)} \right] = \left[G (y, N) \right]$$

$$\begin{bmatrix} 1, & y_1, & \ldots, & y_1^4 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1, & y_5, & \ldots, & y_5^4 \end{bmatrix} \begin{bmatrix} C_0 \\ \cdot \\ \cdot \\ \cdot \\ C_4 \end{bmatrix} \qquad (10)$$

or

$$G = Y\ C$$

and

$$C = Y^{-1}\ G \qquad (11)$$

then

$$G(y, N) = \left[1, y, \ldots, y^4\right] C \qquad (12)$$

The coefficients b's and c's are given in Figures 4 to 5. The results obtained by this method is shown in Figure 6, and it is concluded that these results are accurate enough for engineering purposes.

As an example of application of this slope, consider a plate supported by stiff edge beams. The torsional moment at the edge beam is proportional to $\frac{\partial^2 w}{\partial x\,\partial y}$ and the following result can be obtained :

$$T = \frac{qa^2}{D\ N} K\ (F'\ G + G'\ F) \qquad (13)$$

where K is the factor dependent on the shape of the beam.

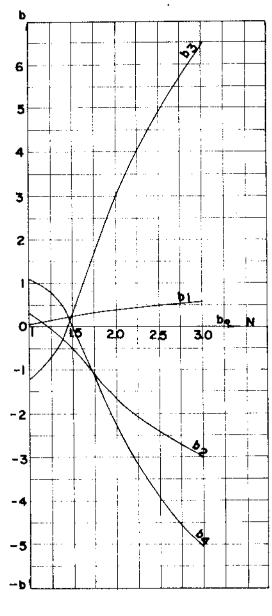

Figure 4, b

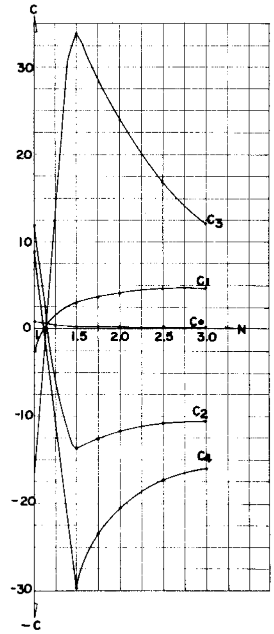

Figure 5, c

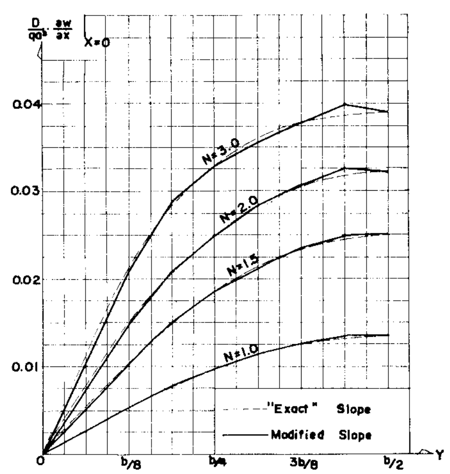

Figure 6, Comparison of the Modified Slope with "Exact" One.