論 説 報 告

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THE DERIVATION OF INFLUENCE EQUATIONS OF STATICALLY INDETERMINATE STRUCTURES WITH PARTIAL FIXITY AT SUPPORTS.

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Synopsis.

This paper presents the solution of indeterminate structures with partial fixity at supports and is the continuation of the thesis on the "Derivation of Influence Equations of Statically Indeterminate Structures" which was published in the September 1932 number of this Journal.

INTRODUCTION.

The solution of indeterminate structures have been carried out with the assumption that they are perfectly fixed or perfectly hinged at supports.

But practically the conditions of supports are neither perfectly fixed nor perfectly hinged. This actual condition of the supports makes a difference in the distribution and character of stresses in the structure and therefore a difference in the results which would be obtained under the assumption that the supports are perfectly fixed or perfectly hinged.

The assumption of the conditions of the supports is important for the analysis of such structures in which the character of stresses are carefully noted, for example, reinforced concrete structure.

It is more logical to analyze the structure with an assumption that the supports of the structure have a certain "percentage of fixity."

The solution of a structure with supports having partial fixity is made as follows.

STRUCTURE WITH PARTIAL FIXITY AT SUPPORTS.

As an example let us take a structure having the left support partially fixed and the right one perfectly fixed as shown in **Fig. 1.**

If the left support is also perfectly fixed then from the angular relation the deflection angle $m_1 = \varphi_1$.

If the left support is perfetly hinged then we have from the deflection moment equations, $m_1 = -\frac{m_2}{2} - \frac{\Delta\alpha}{2}$.

It is imaginable that deflection angle at a, m_1, might change from $\left(-\frac{m_2}{2}-\frac{A\alpha}{2}\right)$ to φ_1 as the fixity increases from zero to 1.

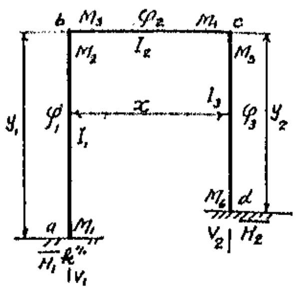

Fig. 1.

The author defines these conditions as follow: When the end is perfectly hinged the percentage of fixity is zero and when the end is perfectly fixed then the percentage of fixity is 100 %.

Now then if the support has k % fixity then

$$m_1=k\varphi_1-\left(\frac{m_2}{2}+\frac{A\alpha}{2}\right)(1-k)$$

Substituting this relation into the deflection moment equations,

$$AM_1^k=-(2\varphi_1k+km_2)-A\alpha k$$

$$AM_2=-\left[\frac{m_2}{2}(3+k)+k\varphi_1\right]+A\beta+A\frac{\alpha}{2}(1-k)$$

These are the general deflection moment equations when one end of a member has a fixity of k %.

Writing the deflection moment equations for the structure shown on **Fig. 1**:

$$AM_1^k=-(2\varphi_1k+km_2)-A\alpha_{ab}k \qquad BM_4=-(2m_4+m_3)+B\beta_{bc}$$

$$AM_2=-\left[\frac{m_2}{2}(3+k)+k\varphi_1\right]+A\beta_{ab}+A_2\frac{\alpha_{ab}}{2}(1-k) \qquad CM_5=-(2m_5+m_6)-C\alpha_{cd}$$

$$BM_3=-(2m_3+m_4)-B\alpha_{bc} \qquad CM_6=-(2m_6+m_5)+C\beta_{cd}$$

From the angular relations:

$$m_3=m_2-\varphi_1+\varphi_2, \qquad m_5=m_4-\varphi_2+\varphi_3, \qquad m_6=\varphi_3.$$

Substituting in above equations:

$$AM_1^k=-(2\varphi_1k+km_2)-A\alpha_{ab}k \qquad BM_4=-(2m_4+m_2-\varphi_1+\varphi_2)+B\beta_{bc}$$

$$AM_2=-\left[\frac{m_2}{2}(3+k)+k\varphi_1\right]+A\beta_{ab}+A\frac{\alpha_{ab}}{2}(1-k) \qquad CM_5=-(2m_4-2\varphi_2+3\varphi_3)-C\alpha_{cd}$$

$$BM_3=-(2m_2-2\varphi_1+2\varphi_2+m_4)-B\alpha_{bc} \qquad CM_6=-(m_4-\varphi_2+3\varphi_3)+C\beta_{cd}$$

Eq.	M_1	M_2	M_4	M_6	H_1	V_1	M_o b	M_o c	M_o d	α_{ab}	β_{ab}	α_{bc}	β_{bc}	α_{cd}	β_{cd}	ΣAx	ΣAy
1	$-Ak$	$A+B$	$B+C$	C						$-A$	$-A$	$-B$	$-B$	$-C$	$-C$		
2	$-Ay_1$	$(2A+3B)y_1$	$3(B+C)y_1-Cy_2$	$3Cy_1-2Cy_2$						$-Ay_1$	$-2Ay_1$	$-3By_1$	$-3By_1$	$-3Cy_1+Cy_2$	$-3Cy_1+2Cy_2$		-3
3	$-3Ak$	$3A+B$	$-(B+3C)$	$-3C$						$-3A$	$-3A$	$-B$	B	$3C$	$3C$	[illegible]	
4	1	1			y_1		1										
5	1		1		y_1	x		1									
6	1			1	y_1-y_2	x			1								

Table. 1.

By the usual procedure eliminate "m" and "φ" from the equations combine with $\Sigma\Delta H=0$ and $\Sigma\Delta V=0$ and establish the Influence Equations as shown on **Table 1**. In the equations ξ is equal to $\left(\frac{1+k}{2k}\right)$.

NUMERICAL EXAMPLE.

Take the same structure used in the numerical example which appeared in the previous Journal, Sept. 1932, except that the fixity at the left support is $k\,\%$ as shown on **Fig. 2**.

By substituting the numerical values into Influence Equations given on **Table 1**, and solving we have:

$$M_1=\frac{38.889}{11.5\left(\frac{1+k}{2k}\right)-1.1}P\ ,\quad M_6=-\frac{27.778\left(\frac{1+k}{2k}\right)+20}{11.5\left(\frac{1+k}{2k}\right)-1.1}P$$

If $k\,\%$ approaches zero then,

$$M_1=\frac{38.889}{\infty}P=0,\quad M_6=-\frac{27.778(\infty)+20}{1.15(\infty)-1.1}P\doteqdot\frac{27.778(\infty)}{11.5(\infty)}P$$
$$\doteqdot-2.415\,P$$

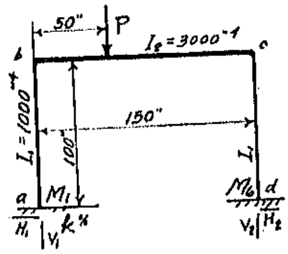

Fig. 2.

If $k\,\%$ become 100 % then,

$$M_1=\frac{38.889}{11.5(1)-1.1}P=3.739\,P,\qquad M_6=-\frac{27.778(1)+20}{11.5(1)-1.1}P=-4.594\,P$$

These results check with the results obtained by the Least Work principle.

The variation in values of M_1 and M_6 due to the change of $k\,\%$ are shown in the **Table 2** and curves (**Fig. 3**).

k%	M_1	M_6	k%	M_1	M_6
0	0	-2.415P			
10	0.626P	-2.780P	60	2.732P	-4.001P
20	1.164P	-3.094P	70	3.023P	-4.177P
30	1.633P	-3.367P	80	3.285P	-4.329P
40	2.044P	-3.607P	90	3.523P	-4.468P
50	2.408P	-3.818P	100	3.739P	-4.594P

Table 2.

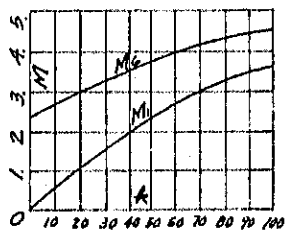

Fig 3.

AN ARCH WITH DIFFERENT PERCENTAGE OF FIXITY AT SUPPORTS.

Take the same arch used in the explanation of the fixed arch which appeared in the previous Journal with the exception that the percentage of fixity at supports a and h is different.

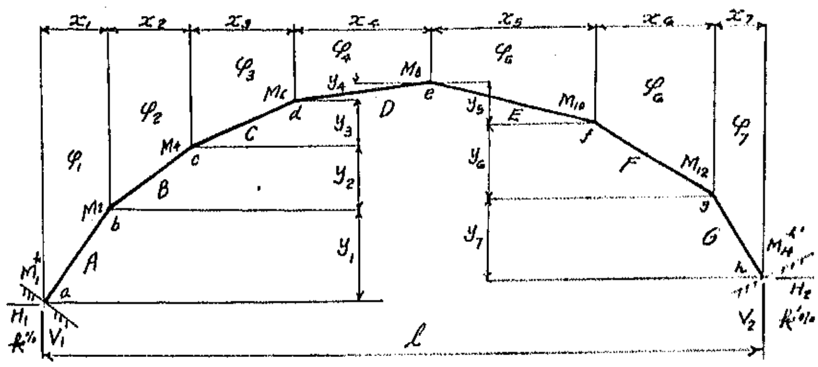

Fig. 4.

From the deflection moment equations:

$$AM_1^k = -(2\varphi_1 k + k m_2) \qquad DM_8 = -(2m_8 + m_7)$$

$$AM_2 = -\left[\frac{m_2}{2}(3+k) + k\varphi_1\right] \qquad EM_9 = -(2m_9 + m_{10})$$

$$BM_3 = -(2m_3 + m_4) \qquad EM_{10} = -(2m_{10} + m_9)$$

$$BM_4 = -(2m_4 + m_3) \qquad FM_{11} = -(2m_{11} + m_{12})$$

$$CM_5 = -(2m_5 + m_6) \qquad FM_{12} = -(2m_{12} + m_{11})$$

$$CM_6 = -(2m_6 + m_5) \qquad GM_{13} = -\left[\frac{m_{13}}{2}(3+k') + k'\varphi_7\right]$$

$$DM_7 = -(2m_7 + m_8) \qquad GM_{14}^{k'} = -(2\varphi_7 k' + k' m_{13})$$

From the angular relations:

$$m_3 = m_2 - \varphi_1 + \varphi_2, \qquad m_5 = m_4 - \varphi_2 + \varphi_3, \qquad m_7 = m_6 - \varphi_3 + \varphi_4$$

$$m_9 = m_8 - \varphi_4 + \varphi_5 \qquad m_{11} = m_{10} - \varphi_5 + \varphi_6 \qquad m_{13} = m_{12} - \varphi_6 + \varphi_7$$

Substituting these relations into the moment equations and eliminating "m" and "φ" combining with the conditions of $\Sigma \Delta H = 0$ and $\Sigma \Delta V = 0$ we have three Relational Moment Equations which are very similar to that of the fixed arch.

Eq.	M_1^k	M_2	M_4	M_6	M_8	M_{10}	M_{12}	$M_{14}^{k'}$	H_1	V_1	Temp. Ch.	R.S. Ch.	R.M. Ch.	(M_0) at b	c	d	e	f	g	h
1	$-A\left(\frac{4+k}{2k}\right)$	$(A+B)$	$(B+C)$	$(C+D)$	$(D+E)$	$(E+F)$	$(F+G)$	$G\left(\frac{4+k'}{2k'}\right)$												
2	$-A\phi_1$	$A\phi_2 + B\phi_3$	$B\phi_4 + C\phi_5$	$C\phi_6 + D\phi_7$	$D\phi_8 + E\phi_9$	$E\phi_{10} + F\phi_{11}$	$F\phi_{12} + G\phi_{13}$	$G\left\{\Sigma_1^6 y + \left(\frac{3+k'}{2k'}\right)\Sigma_1^7 y\right\}$			$\tau\Sigma_1^7 x$	$+\Delta_H$	$\tau\Delta S_x$							
3	$-A\theta_1$	$A\theta_2 + B\theta_3$	$B\theta_4 + C\theta_5$	$C\theta_6 + D\theta_7$	$D\theta_8 + E\theta_9$	$E\theta_{10} + F\theta_{11}$	$F\theta_{12} + G\theta_{13}$	$G\left\{\Sigma_1^6 x + \left(\frac{3+k'}{2k'}\right)\Sigma_1^7 x\right\}$			$\tau\Sigma_1^7 y$	$+\Delta_V$	$\tau\Delta S_y$							
4	1	1							y_1	x_1				1						
5		−1	1						y_2	x_2				−1	1					
6			−1	1					y_3	x_3					−1	1				
7				−1	1				y_4	x_4						−1	1			
8					−1	1			$-y_5$	x_5							−1	1		
9						−1	1		$-y_6$	x_6								−1	1	
10							−1	1	$-y_7$	x_7									−1	1

Table. 3.

Establish Influence Equations with Relational Moment Equations and moment equations about sections passed at each pannel point as shown for the fixed arch.

The values of ϕ and θ are the same as in the fixed arch.

NUMERICAL EXAMPLE.

Take an arch with dimensions as given in **Fig. 5** and having a constant arch rib.

Divide the arch into 16 segments and establish the Influence Equations by inspection as given on **Table 4.** Then substitute the numerical values in the equations and solve.

The arch was solved for various percentage of fixity, k and k', at the supports the results of which are given by the influence lines on **Fig. 6** and **7**.

From these curves various interpretations may be obtained for the arch problem.

Then the question of the actual values of k and k' arises. What should be their values for practical purposes; since they vary according to the condition of supports? For a two hinged arch, the reasonable values of k and k' can be obtained by considering the friction between the pin and shoe at springing of the arch.

Table 5 illustrates the solution of simultenious equations for the arch.

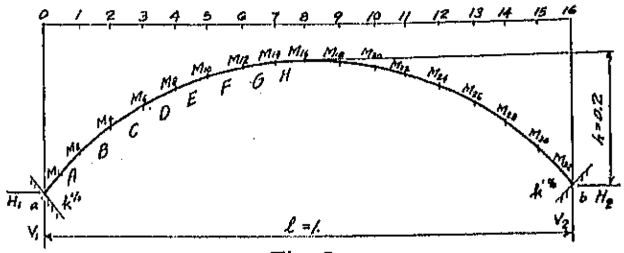

Fig. 5.

Pt.		x	y	ℓ	Member Ratio ℓ/I	
16	1	.0625	.0531	.0820	A	.0820
15	2	"	.0424	.0755	B	.0755
14	3	"	.0337	.0710	C	.0710
13	4	"	.0263	.0678	D	.0678
12	5	"	.0195	.0655	E	.0655
11	6	"	.0141	.0640	F	.0640
10	7	"	.0083	.0630	G	.0630
9	8	"	.0026	.0626	H	.0626

$$\phi_{(2n-1)}=3\sum_1^{(n-1)}y\pm y_n \qquad \phi_{2n}=3\sum_1^{(n-1)}y\pm 2y_n$$

$$\theta_{(2n-1)}=3\sum_1^{(n-1)}x+x_n \qquad \theta_{2n}=3\sum_1^{(n-1)}x+2x_n$$

Eq.	M_1	M_2	M_4	M_6	M_8	M_{10}	M_{12}	M_{14}	M_{16}	M_{18}	M_{20}	M_{22}	M_{24}	M_{26}	M_{28}	M_{30}	M_{32}	H_1	V_1	ΔφEI	ΔθEI	(M_0) Load at Pt. 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	A(ℓ₁k/2x)	A/B	B/C	C/D	D/E	E/F	F/G	G/H	2H	G/H	F/G	E/F	D/E	C/D	B/C	A/B	[illegible]																			
2	$-A\phi_1$	$A\phi_2$ $B\phi_3$	$B\phi_4$ $C\phi_5$	$C\phi_6$ $D\phi_7$	$D\phi_8$ $E\phi_9$	$E\phi_{10}$ $F\phi_{11}$	$F\phi_{12}$ $G\phi_{13}$	$G\phi_{14}$ $H\phi_{15}$	$2H\phi_{16}$	$G\phi_{18}$ $H\phi_{17}$	$F\phi_{20}$ $G\phi_{19}$	$E\phi_{22}$ $F\phi_{21}$	$D\phi_{24}$ $E\phi_{23}$	$C\phi_{26}$ $D\phi_{25}$	$B\phi_{28}$ $C\phi_{27}$	$A\phi_{30}$ $B\phi_{29}$	[illegible]	[illegible]	[illegible]																	
3	$-A\theta_1$	$A\theta_2$ $B\theta_3$	$B\theta_4$ $C\theta_5$	$C\theta_6$ $D\theta_7$	$D\theta_8$ $E\theta_9$	$E\theta_{10}$ $F\theta_{11}$	$F\theta_{12}$ $G\theta_{13}$	$G\theta_{14}$ $H\theta_{15}$	$H\theta_{16}$ $H\theta_{17}$	$G\theta_{18}$ $H\theta_{17}$	$F\theta_{20}$ $G\theta_{19}$	$E\theta_{22}$ $F\theta_{21}$	$D\theta_{24}$ $E\theta_{23}$	$C\theta_{26}$ $D\theta_{25}$	$B\theta_{28}$ $C\theta_{27}$	$A\theta_{30}$ $B\theta_{29}$	[illegible]																			
4	+1	1															= $A\phi_1$	y_1	x																	
5		-1	1															y_2	x			x														
6			-1	1													when $\Sigma_1^{16}y=0$	y_3	x			x	x													
7				-1	1													y_4	x			x	←→	x												
8					-1	1												y_5	x			x	←	→	x											
9						-1	1											y_6	x			x	←		→	x										
10							-1	1										y_7	x			x	←			→	x									
11								-1	1									y_8	x			x	←				→	x								
12									-1	1								$-y_8$	x			x	←					→	x							
13										-1	1							$-y_7$	x			x	←						→	x						
14											-1	1						$-y_6$	x			x	←							→	x					
15												-1	1					$-y_5$	x			x	←								→	x				
16													-1	1				$-y_4$	x			x	←									→	x			
17														-1	1			$-y_3$	x			x	←										→	x		
18															-1	1		$-y_2$	x			x	←											→	x	
19																-1	1	$-y_1$	x			x	←												→	x

Table. 4.

Eq.	M 1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	H_1	V_1	ΔφGEI	ΔS/GE	(M_0) Load at Pt. 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	$-K_1$	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}	K_{11}	K_{12}	K_{13}	K_{14}	K_{15}	K_{16}	K_{17}			$\frac{1}{2}$	Δ_H															
2	K_1	K_1																y_1K_1	xK_1																	
3		$-K_1^2$	K_1^2															$y_2K_1^2$	xK_1^2			xK_1^2														
4			$-K_1^3$	K_1^3														$y_3K_1^3$	xK_1^3			xK_1^3	xK_1^3													
5				$-K_1^4$	K_1^4													$y_4K_1^4$	xK_1^4			xK_1^4	←→	xK_1^4												
6					$-K_1^5$	K_1^5												$y_5K_1^5$	xK_1^5			xK_1^5	←	→	xK_1^5											
7						$-K_1^6$	K_1^6											$y_6K_1^6$	xK_1^6			xK_1^6	→		→	xK_1^6										
8							$-K_1^7$	K_1^7										$y_7K_1^7$	xK_1^7			xK_1^7	→			→	xK_1^7									
9								$-K_1^8$	K_1^8									$y_8K_1^8$	xK_1^8			xK_1^8	→				→	xK_1^8								
10									$-K_1^9$	K_1^9								$-y_8K_1^9$	xK_1^9			xK_1^9	→					→	xK_1^9							
11										$-K_1^{10}$	K_1^{10}							$-y_7K_1^{10}$	xK_1^{10}			xK_1^{10}	←						→	xK_1^{10}						
12											$-K_1^{11}$	K_1^{11}						$-y_6K_1^{11}$	xK_1^{11}			xK_1^{11}	→							→	xK_1^{11}					
13												$-K_1^{12}$	K_1^{12}					$-y_5K_1^{12}$	xK_1^{12}			xK_1^{12}	→								→	xK_1^{12}				
14													$-K_1^{13}$	K_1^{13}				$-y_4K_1^{13}$	xK_1^{13}			xK_1^{13}	→									→	xK_1^{13}			
15														$-K_1^{14}$	K_1^{14}			$-y_3K_1^{14}$	xK_1^{14}			xK_1^{14}	←										→	xK_1^{14}		
16															$-K_1^{15}$	K_1^{15}		$-y_2K_1^{15}$	xK_1^{15}			xK_1^{15}	→											→	xK_1^{15}	
17																$-K_1^{16}$	K_1^{16}	$-y_1K_1^{16}$	xK_1^{16}			xK_1^{16}	→												→	xK_1^{16}
18	$\frac{1}{2}(K_1^7)$ $-\frac{1}{2}(y_1K_1^{16}+y_2K_1^{15}+y_3K_1^{14}+y_4K_1^{13}+y_5K_1^{12}+y_6K_1^{11}+y_7K_1^{10}+y_8K_1^{9})$																		$\Sigma K_1^{1\text{-}16}$	$\frac{1}{2}$	$\frac{\Delta_H}{2}$	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	K_1^{16}

$\frac{1}{I}(\Delta\phi \ldots)$

By the same manner establish two more equations from R.M.E. (1) and (3) and solve these 3 equations and find the Influence Values of M_1, H_1 and V_1. The values of other moments shall be obtained by following eqs.

Eq.	M 1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	H_1	V_1			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
19	1																1	0	16x			15x	14x	13x	12x	11x	10x	9x	8x	7x	6x	5x	4x	3x	2x	x
20	1															1			$\Sigma_1^7 y$	14x			13x	12x	11x	10x	9x	8x	7x	6x	5x	4x	3x	2x	x		
21	1												1					$\Sigma_1^6 y$	12x			11x	10x	9x	8x	7x	6x	5x	4x	3x	2x	x				
22	1										1							$\Sigma_1^5 y$	10x			9x	8x	7x	6x	5x	4x	3x	2x	x						
23	1								1									$\Sigma_1^4 y$	8x			7x	6x	5x	4x	3x	2x	x								
24	1						1											$\Sigma_1^3 y$	6x			5x	4x	3x	2x	x										
25	1				1													$\Sigma_1^2 y$	4x			3x	2x	x												
26	1		1															$\Sigma_1^1 y$	2x			x														

Table. 5.

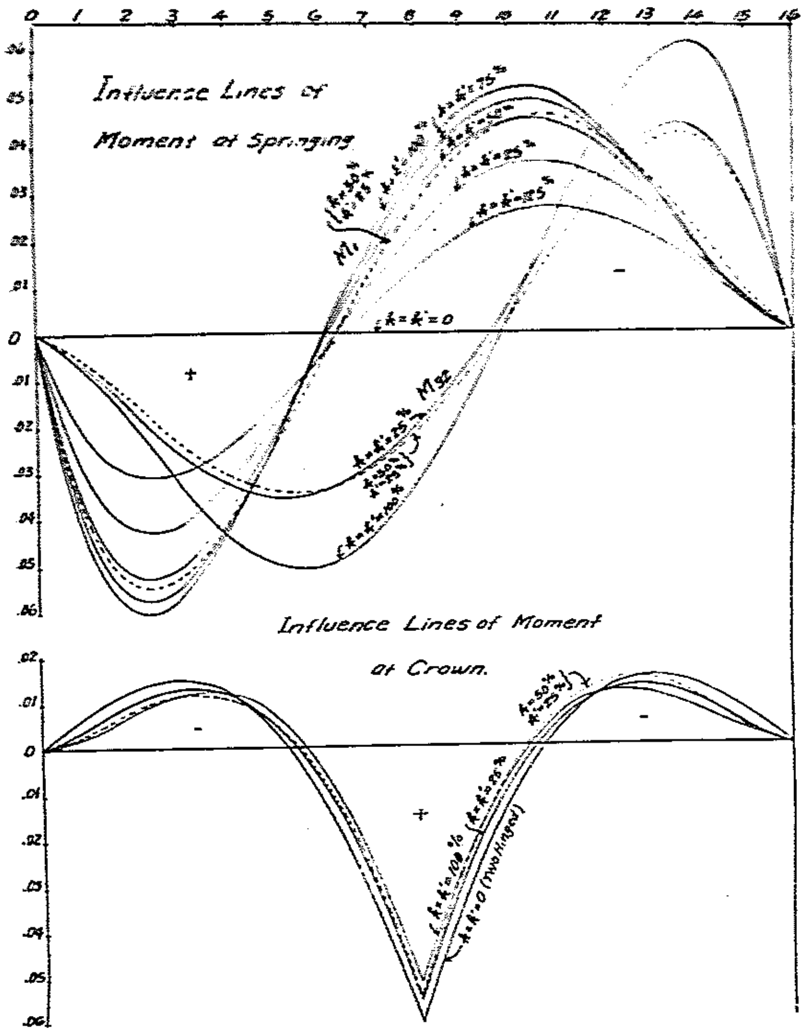
Influence Lines of
Moment at Springing
Influence Lines of Moment
at Crown.

Fig. 6.

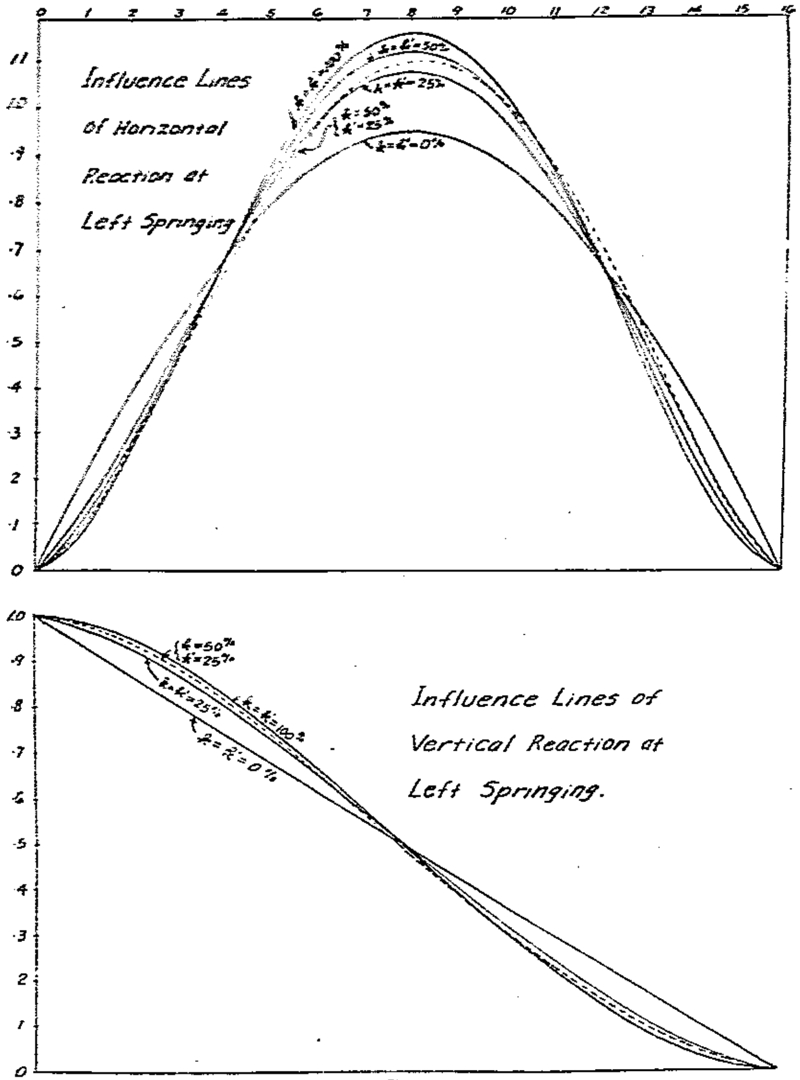
Influence Lines
of Horizontal
Reaction at
Left Springing
Influence Lines of
Vertical Reaction at
Left Springing.

Fig. 7.