論說報告

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THE DERIVATION OF INFLUENCE EQUATIONS OF STATICALLY INDETERMINATE STRUCTURES.

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Synopsis.

The object of this paper is to present the derivation of Influence Equations and their applications to the solution of statically indeterminate bending stresses in structures.

The Influence Equations embody the bending, shear and thrust relations existing among the members of a structure when distorted by any cause, entirely separate from the forces producing the distortion. They contain only a series of known terms in question and enable taking full account of axial deformations of members.

The application of the principle is universal and practical for all statically indeterminate structures.

INTRODUCTION.

Heretofore the solution of statically indeterminate bending stresses has generally been accomplished by applying the principle of "Least Work" or "Castigliano" principle. More recently the principle of slope deflection has been developed. The latter is recognized to be much more simple than the former, particularly for complicated structures. This method was introduced by Manderla in 1878 and was somewhat developed by Mohr in 1892. In 1915 Merrs. W. M. Wilson and G. A. Maney carried its development much further and others* have employed it for certain classes of problems. In 1925 the writer, however, attacking this problem from a different angle developed a general method of Influence Equations.

The subject of this paper is divided into two parts: the first part is devoted to

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* W. Gehler: Rahmenberechnung mittels Drehwinkel. Otto Mohr zumachtigsten Geburtstage, Berlin. 1916.

F. P. Witmer: Indeterminate Structures, 1917.

W. M. Wilson, F. E. Richart and Camillo Weiss: Analysis of Statically Indeterminate Structures by the Slope Deflection Method, Illinois Bulletin No. 108, 1918.

Case: Strength of Materials, 1925.

Ostenfeld: Die Deformationsmethode, 1926.

鷹部屋福平: 架構新論. 1928.

A. W. Ross: Wind Moments of Office Building Frames, A.S.C.E. Proceedings 1928.

Kleinlogel: Rahmenformeln, 1929.

H. Cross: Analysis of Continuous Frames by Distributing Fixed-End Moments, A. S. C. E. Proceedings 1930.

Others.

the derivation of the Influence Equations; the second part deals with the application of these equations to practical problems.

INFLUENCE EQUATIONS

The influence equations consist of five fundamental elements as follow:

(1) Deflection moment equations.

(2) Angular relations among intersecting members.

(3) Summations of horizontal and vertical displacements of joints between supports.

(4) Summation of moments about a joint.

(5) Summation of moments about a section passed.

(1) Deflection Moment Equations.

Consider a member which is not acted upon by any intervening loads. This may readily be derived by the well known method, thus: In **Fig. 1** let M_1 and M_2 be the moments applied at the ends of a beam AB. Let V_1 and V_2 be the end shears necessary to produce equilibrium with M_1 and M_2.

Let m_1 and m_2 be the deflection angles at the ends measured from the straight line axis AB; that is, the values of $\frac{dy}{dx}$ which for small angles may be considered as equal to the angles themselves.

The straight line axis AB is the final axis of the beam after deformation. This final position of the beam is the datum axis to establish the Deflection Moment Equations which are simplified by taking the datum line in this manner. Moreover, the datum line can be taken in any inclination.

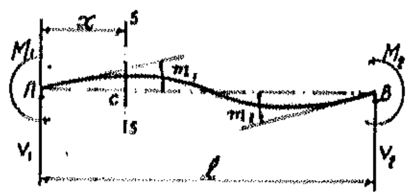


Fig 1.

The bending moment M_c at any section *ss* is equal to the algebraic sum of the moments of all the forces on either side of the section about *c*.

Assume that the directions of V_1 and V_2 are unknown, and consider the moment of V_1 as positive, for purposes of analysis.

Then
$$M_c = EI\frac{d^2y}{dx^2} = M_1 + V_1 x$$

Integrating, remembering that $\frac{dy}{dx} = m_1$ if $x = 0$, and $\frac{dy}{dx} = m_2$ if $x = l$

$$M_1 = -\frac{2EI}{l}(2m_1 + m_2) \quad \ldots\ldots (1)$$

$$M_2 = -\frac{2EI}{l}(2m_2 + m_1) \quad \ldots\ldots (2)$$

Since m_1 and m_2 are shown in **Fig. 1** as positive when measured from the straight line axis AB, and M_1 and M_2 are both shown in a counter-clockwise direction, it follows from (1) and (2) that positive moments should be considered as clockwise in order to be consistent with the usual convention for the measurement of angles: this assumption, therefore, will be made for all moments throughout the following analysis.

Consider the deflection moment equations for a beam similar to the foregoing method but with an intervening load:

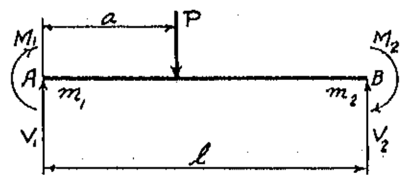


Fig. 2.

By a similar procedure we may write for the left hand side of P:

$$EI\frac{d^2y}{dx^2} = M_1 + V_1 x$$

On the right hand side of P:

$$EI\frac{d^2y}{dx^2} = M_1 + V_1 x - P(x-a)$$

Integrating and solving the above we obtain:

$$M_1 = -\frac{2EI}{l}(2m_1+m_2) - \frac{Pa}{l^2}(l-a)^2 \quad \dots\dots(3)$$

$$M_2 = -\frac{2EI}{l}(2m_2+m_1) + \frac{Pa^2}{l^2}(l-a) \quad \dots\dots(4)$$

In a similar manner the deflection moment equations can be written for all loadings.

After the examination of the deflection moment equations for any loading the first terms of the right hand side of the equations, i.e. $-\frac{2EI}{l}(2m_1+m_2)$ and $-\frac{2EI}{l}(2m_2+m_1)$ always remain the same and the second portions of the equations are variable according to the nature of the loading.

Generally they may be written:

$$AM_1 = -(2m_1+m_2) - A\alpha \quad \dots\dots(5)$$

$$AM_2 = -(2m_2+m_1) + A\beta \quad \dots\dots(6)$$

where $A = \frac{l}{2EI}$, $\alpha = \frac{Pa}{l^2}(l-a)^2$ and $\beta = \frac{Pa^2}{l^2}(l-a)$

Generally α and β are the second portions which will be called "Constants" of the Deflection Moment Equations.

The values of α and β of the combined loadings are the sum of individual constants due to various loadings.

$$\alpha = \alpha_a + \alpha_b + \alpha_c + \text{etc.} \qquad \beta = \beta_a + \beta_b + \beta_c + \text{etc.}$$

The individual values of the constants for the various loadings are given in many publications.

In conjunction with this subject the following books were used for the re-

ference.

(1) Manderla: Die Berechnung der Sekundarspannungen, welche im einfachen Fachwerke infolge starrer Knotenverbindungen entstehen. Allgemeine Bauzeitung, 1880.

(2) Winkler: Die Sekundarspannungen in Eisenkonstruktionen, Deutsche Bauzeitung, 1881.

(3) Landsberg: Beitrag zur Theorie der Fachwerke (Graphische Ermittelung der Sekundarspannungen infolge fester Knotenverbindungen der Gurtstäbe) Zeitschrift des Architekten-und Ingenieur-Vereins zu Hannover, 1885.

(4) Müller-Breslau: Zur Theorie der Biegungsspannungen in Fachwerktragen. Zeitschrift des Architekten-und Ingenieur-Vereins zu Hannover, 1886.

(5) Ritter: Anwendungen der graphischen Statik. II, Das Fachwerk. Zürich, 1890.

(6) Engesser: Die Zusatzkrafte und Nebenspannungen eiserner Fachwerkbrücken. I, Die Zusatzkrafte. Berlin, 1892. II, Die Nebenspannungen. Berlin, 1893.

(7) Mohr: Die Berechnung der Fachwerke mit starren Knotenverbindungen. Der Civil Ingenieur, 1892.

(8) Grimm: Secondary Stresses in Bridge Trusses, 1908.

(9) Kunz: Secondary Stresses, Engineering News, 1911.

(10) W. M. Wilson and G.A. Maney: Wind Stresses in the Steel Frames of Office Buildings, Illinois Bulletin No. 80, 1915.

(11) Others.

Deflection Moment Equations for One Hinged End.

From general equations (5) and (6)

$$A M_1 = -(2m_1+m_2) - A\alpha$$

$$A M_2 = -(2m_2+m_1) + A\beta$$

In this case it is obvious that

$M_2=0$ therefore $0=-(2m_2+m_1)+A\beta$

$$m_2 = -\frac{m_1}{2} + \frac{A}{2}\beta$$

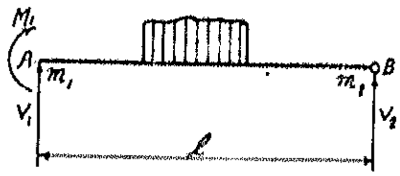


Fig. 3.

Substituting m_2 into M_1 equation

$$A M_1 = -\left(\frac{3}{2}\, m_1\right) - A\gamma \quad\text{.......}(7)$$

where $\gamma = \alpha + \frac{\beta}{2}$

(2) Angular Relations among Intersecting Members.

For the Angular Relations among members, let OA, OB, OC, OD and OE in the accompanying sketch, **Fig. 4**, represent the original and OA′, OB′, OC′, OD′ and OE′ the final directions of the straight line axes of intersecting members. Let OA_1, OB_1, OC_1, OD_1 and OE_1 represent the tangents at O of the elastic curves of these members after bending. Angles $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5$; m_1, m_2, m_3, m_4 and m_5 are respectively the deflection angles of the axes and the bending end deflection angles of the members with reference to these axes as deflected. As shown in the figure, these angles are positive or negative according to the convention that angles measured counter clockwise from the straight line axis in its final position are positive. The angular signs and the datum axes are consistent with the statement made when discussing Deflection Moment Equations.

The conditions of angular relations are such that the angles between the tangent lines of members intersecting at a joint having perfect rigidity remain the same before and after bending of members. Therefore the angles between the original direction of the members and the tangent line of the members at the joint after bending are all equal, i.e. $A\hat{O}A_1 = B\hat{O}B_1 = C\hat{O}C_1 = D\hat{O}D_1 = E\hat{O}E_1$.

We may write, according to geometrical relations:

$$m_1 + \varphi_1 = m_2 + \varphi_2 = m_3 + \varphi_3 = m_4 - \varphi_4 = m_5 - \varphi_5$$

Since from convention,

$m_1(+)$, $\varphi_1(-)$, $m_2(+)$, $\varphi_2(-)$, $m_3(+)$,
$\varphi_3(-)$, $m_4(+)$, $\varphi_4(+)$, $m_5(+)$, $\varphi_5(+)$,

whence

$$m_1 - \varphi_1 = m_2 - \varphi_2 = m_3 - \varphi_3 = m_4 - \varphi_4 = m_5 - \varphi_5 \quad \text{..(8)}$$

This is a general relation which may always be written for all intersecting members.

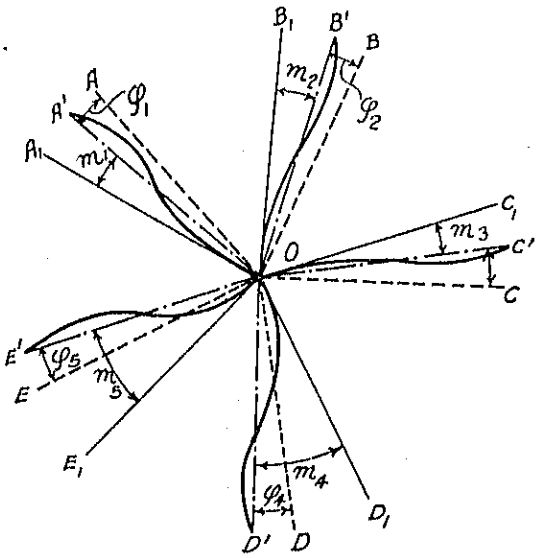


Fig. 4.

(3) Summation of Displacements of Joints between Supports.

For a condition in which the points of support remain in the same location, we may write $\Sigma\Delta H = 0$ and $\Sigma\Delta V = 0$ where ΔH and ΔV are respectively the horizontal and the vertical movements of one end of any member with reference to the other end. Thus in sketch, **Fig. 5**, we may practically write the following taking proper account of the sign of φ_1, φ_2 etc. and remembering that φ_1, φ_2 etc. are always very small:

$$\Sigma\Delta H = \varphi_1 y_1 + \varphi_2 y_2 - \varphi_3 y_3 - \varphi_4 y_4 = 0 \quad \text{......(9)}$$

and

$$\Sigma\Delta V = \varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 + \varphi_4 x_4 = 0 \quad \text{......(10)}$$

In (9), if we start at the left support, the terms are positive when the inclination of a member is up toward the right and negative when it is down toward the right. In (10), all terms are positive toward the right and negative toward the left.

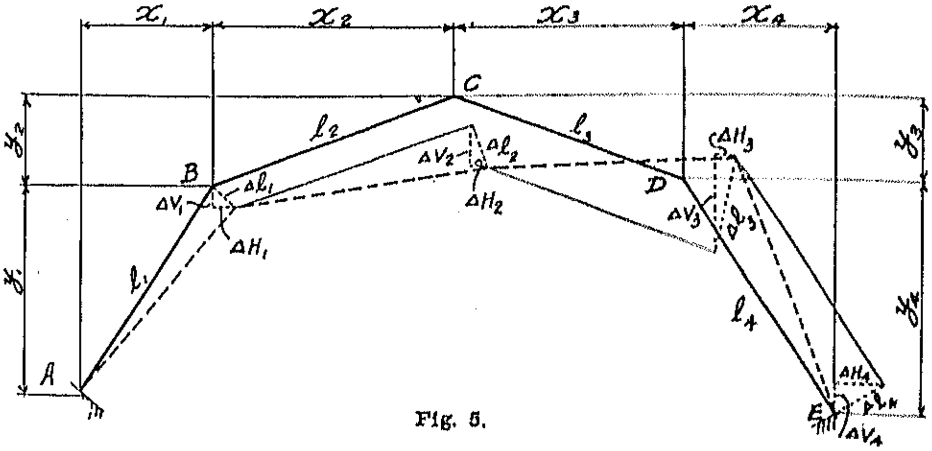


Fig. 5.

These two conditions will be apparent upon careful consideration.

If we assume a known amount of end displacement due to any cause such as axial shortening or lengthening, movements of supports etc., then $\Sigma \Delta H$ and $\Sigma \Delta V$ have known values, i.e. $\Sigma \Delta H = \Delta x$ and $\Sigma \Delta V = \Delta y$.

Similar relations, i.e. $\Sigma \Delta H = 0$ and $\Sigma \Delta V = 0$ were expressed but in different ways.*

(4) Summation of Moments about a Joint.

The summation of moments in all members intersecting at any point is zero, or else has some known value in case of a known moment applied to the joint. This may result from an eccentric connection or other causes.

(5) Summation of Moments about a Section passed.

From the condition of equilibrium the summation of moments, external and internal forces, about a section must be zero.

STEPS IN DERIVING INFLUENCE EQUATIONS.

The Influence Equations for any structure will be derived step by step according to the above five fundamental conditions:

1. Establish deflection moment equations.

2. Transform and reduce the terms of "m" by angular relation, then eliminate the terms of "m".

3. Establish the relation of displacements of members and combine with the second step and eliminate the terms of "φ" entirely from the simultaneous equations.

* Bleich: Die Berechnung statisch unbestimmter Tragwerk nach der Methode des Viermomenten-satzes, 1925.

Thomas: Calcul des constructions continues a elements droits, 1925.

4. Eliminate and simplify the unknown terms of "M" in the above simultaneous equation by applying $\Sigma M=0$ at each joint.

We have then a series of simultaneous equations without terms of "m" and "φ" and these are called "Relational Moment Equations" which express the relations of the moments at joints in terms of dimensions and the elastic properties of the members, and for any given frame, are absolutely identical regardless of loading, so far as the terms containing internal moments are concerned.

They will differ only in the constant terms, α and β which contain the loading relations. The coefficients of these terms will also be the same for all loadings, being functions of the dimensions and elastic properties alone.

It is thus possible by the above method to derive "Relational Moment Equations" for any given form of structure, regardless of loading, and having these equations, their use, in conjunction with relations obtained from $\Sigma M=0$ as given below, will result in a set of "Influence Equations" for the structure in question.

5. Pass sections at each joint of members and establish series of moment equations applying $\Sigma M=0$ on the left hand side of each section passed.

The Relational Moment Equations with the series of moment equations established by the fifth step, $\Sigma M=0$ at sections, are called "Influence Equations" which express the variation of moment, shear, and thrust of the structure.

The above procedure will be illustrated with examples.

Member Ratio:

This is the Ratio between the length of the member and the product of the moment of inertia and the modulus of elasticity $\frac{l}{2EI}=A$ in the Deflection Moment Equations. If the material of the structure is the same then the $2E$ term can be eliminated from the Influence Equations, which makes the member ratio $A=\frac{l}{I}$. This ratio will be expressed in letters A, B, C and etc. from now on.

APPLICATION OF THE PRINCIPLE.

As a simplest case assume a bent of unsymmetrical shape with supports fixed at different elevations as shown in **Fig. 6.** Assume that any desired load is applied on each member. The directions of the reactions vary according to the loading therefore they have been omitted.

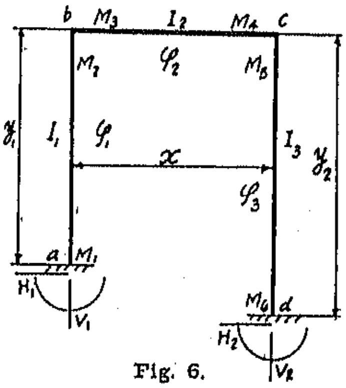


Fig. 6.

First; Deflection Moment Equations:

$$AM_1=-(2m_1+m_2)-A\alpha_{ab}$$
$$AM_2=-(2m_2+m_1)+A\beta_{ab}$$
$$BM_3=-(2m_3+m_4)-B\alpha_{bc}$$
$$BM_4=-(2m_4+m_3)+B\beta_{bc}$$
$$CM_5=-(2m_5+m_6)-C\alpha_{cd}$$
$$CM_6=-(2m_6+m_5)+C\beta_{cd}$$

Second ; Angular Relations :

$$m_1=\varphi_1,\ m_3=m_2-\varphi_1+\varphi_2,\ m_5=m_4-\varphi_2+\varphi_3,\ m_6=\varphi_3$$

Substituting these values into the first equations,

$$AM_1=-(2\varphi_1+m_2)-A\alpha_{ab}$$
$$AM_2=-(2m_2+\varphi_1)+A\beta_{ab}$$
$$BM_3=-(2m_2-2\varphi_1+2\varphi_2+m_4)-B\alpha_{bc}$$
$$BM_4=-(2m_4+m_2-\varphi_1+\varphi_2)+B\beta_{bc}$$
$$CM_5=-(2m_4-2\varphi_2+3\varphi_3)-C\alpha_{cd}$$
$$CM_6=-(m_4-\varphi_2+3\varphi_3)+C\beta_{cd}.$$

Transforming above equations.

Eq.	M_1	M_2	M_3	M_4	M_5	M_6	m_2	m_4	φ_1	φ_2	φ_3	α_{ab}	β_{ab}	α_{bc}	β_{bc}	α_{cd}	β_{cd}	Σax	Σay
1	A						1		2			A							
2		A					2		1				-A						
3			B				2	1	-2	2				B					
4				B			1	2	-1	1					-B				
5					C			2		-2	3					C			
6						C		1		-1	3						-C		

Eliminating " m " from equations (1), (2), (3), (4), (5) and (6).

7	2A	-A							3			2A	A						
8	A	-2A	2B	-B					-3	3		A	2A	2B	B				
9			B	-2B	2C	-C				-3	3			B	2B	2C	C		
10					C	-2C					-3					C	2C		

Third ; Summation of Displacements :

11									y_1		$-y_2$							1	
12										x									1

Eliminating " φ " from equations (7), (8), (9), (10) combining (11) and (12).

13	A	-A	B	-B	C	-C						A	A	B	B	C	C		
14	Ay_1	$-2Ay_1$	$3By_1$	$-3By_1$	$3Cy_1-Cy_2$	$-3Cy_1+2Cy_2$						Ay_1	$2Ay_1$	$3By_1$	$3By_1$	$3Cy_1-Cy_2$	$3Cy_1-2Cy_1$	3	
15	3A	-3A	B	B	-3C	3C						3A	3A	B	-B	-3C	-3C		$-\frac{6}{x}$

Fourth ; Moments about Joints :

$$M_2+M_3=0 \qquad M_4+M_5=0$$

Substituting these relations into equations (13), (14) and (15).

	M_1	M_2	M_4	M_6	H_1	V_1	M_0 at b	M_0 at c	M_0 at d	α_{ab}	β_{ab}	α_{bc}	β_{bc}	α_{cd}	β_{cd}	$\Sigma\Delta x$	$\Sigma\Delta y$
16	A	$-(A+B)$	$-(B+C)$	$-C$						A	A	B	B	C	C		
17	Ay_1	$-(2A+3B)y_1$	$-(3B+3C)y_1 + Cy_2$	$-3Cy_1 + 2Cy_2$						Ay_1	$2Ay_1$	$3By_1$	$3By_1$	$3Cy_1-Cy_2$	$3Cy_1-2Cy_2$		3
18	$3A$	$-(3A+B)$	$(B+3C)$	$3C$						$3A$	$3A$	B	$-B$	$-3C$	$-3C$	$-\frac{6}{x}$	

Fifth ; Moments about Sections :

Let M_0 be the moment about the right-hand end of the member under consideration due to all the external forces and external moments applied to the member between its ends.

From the condition of equilibrium of moments :

	M_1	M_2	M_4	M_6	H_1	V_1	M_0 at b	M_0 at c	M_0 at d	α_{ab}	β_{ab}	α_{bc}	β_{bc}	α_{cd}	β_{cd}	$\Sigma\Delta x$	$\Sigma\Delta y$
19			-1	1	y_2			-1	1								
20		-1	1			x	-1	1									
21	1	1			y_1		1										

Equations (16), (17) and (18) are "RELATIONAL MOMENT EQUATIONS". Equations (16), (17), (18) and (19), (20) (21) are "INFLUENCE EQUATIONS" because, for the case just considered, with these six equations the Influence Line of any point considered may be obtained by changing the known quantities.

Merit of Influence Equations :

For practical use these equations are more flexible than any others which have come to the writer's notice. Their apparent complexity may be readily reduced by the following considerations : If we have any loading on the left hand column alone then terms of α_{bc}, β_{bc}, α_{cd} and β_{cd} in Influence Equations disappear.

If we have any loading on the beam alone then terms α_{ab}, β_{ab}, α_{cd} and β_{cd} disappear.

In other words these Influence Equations can be used for any loading by simply changing the known quantities according to each case but coefficients of unknown quantities remain always the same for any loading.

The value of each unknown can be obtained by a process of elimination but for practical use Equations (16), (17), (18), (19), (20) and (21) are the most convenient forms.

From the character of equations (19), (20) and (21) it is seen that the final coefficents of M_1, H_1 and V_1 will be the sum of coefficients of M_2, M_4 and M_6.

After substituting numerical values, and combining equations

(16) and (19) (20) (21)

(17) and „ „ „

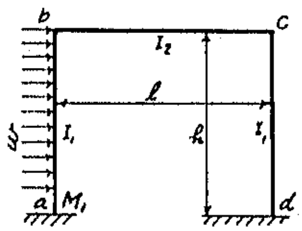
(18) and „ „ „

We get three final equations containing only the unknowns M_1, H_1 and V_1 whose values may thus be solved for.

The values of other unknowns will readily follow by substitution in (19), (20) and (21).

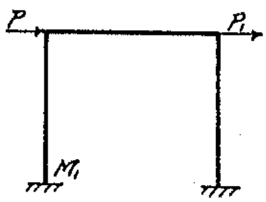
Examples ;

The following examples show the values of M_1 obtained by the foregoing method neglecting the axial deformation and displacement of points of support in order to compare with results obtained by the Least Work Theory in which these quantities were neglected.



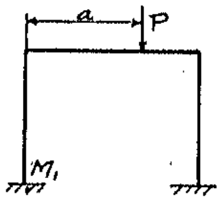
$$M_1 = -\frac{wh^2}{24}\cdot\frac{15l^2 I_1^2 + 73 I_1 I_2 hl + 30 I_2^2 h^2}{(I_1 l + 6h I_2)(2 I_1 l + h I_2)}$$

Fig. 7.



$$M_1 = -\left(\frac{P+P_1}{2}\right)\left(\frac{I_1 l + 3 I_2 h}{I_1 l + 6 I_2 h}\right) h$$

Fig. 8.



$$M_1 = -\frac{Pa(l-a)}{2}\cdot\left\{\frac{(l-2a)I_1}{l(lI_1+6hI_2)} - \frac{I_1}{(2lI_1+hI_2)}\right\}$$

Fig. 9.

These values of M_1 check against the results which others have obtained by the Least Work Theory.

It is not advisable to establish equations for different cases as shown above because such equations are cumbersome in their practical application. The Influence Equations have great flexibility in their use which will be illustrated by numerical examples as follow:

If the structure is of the same material then the term of "$2E$" disappears.

In this example $y_1 = y_2$, $A = C$ therefore the Influence Equations will be much simplified. Substituting these values into equations (16), (17), (18), (19), (20) and (21) and proceed as follows:

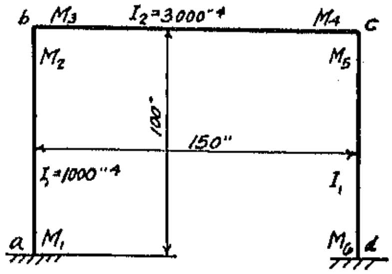


Fig. 10.

Eq.	M_1	M_2	M_4	M_6	H_1	V_1	M_{ob}	M_{oc}	M_{od}	α_{ab}	β_{ab}	α_{bc}	β_{bc}	α_{cd}	β_{cd}
1	.1	-.15	-.15	-.1						.1	.1	.05	.05	.1	.1
2	.1	-.35	-.35	-.1						.1	.2	.15	.15	.2	.1
3	.3	-.35	.35	.3						.3	.3	.05	-.05	-.3	-.3
4			-1	1	-100			-1	1						
5		-1	1			150	-1	1							
6	1	1			100		1								
7					1		.0050	.0050	-.0050	-.0050	.0013	.0038	.0038	.0013	-.0050
8	1						.2692	-.2692	.2692	.7308	.3558	-.0865	-.1635	-.1058	.2692
9						1	-.0036	.0036	.0031	-.0031	-.0031	-.0005	.0005	.0031	.0031

Table 1.

Equations (7), (8) and (9) are Influence Values for reactions H_1, M_1, V_1.

The following illustrate the use of Influence values of reactions (7), (8) and (9). Let the structure shown in **Fig. 10** have conditions of loading similar to that shown in **Fig. 7, 8** and **9.**

Just as in the case of the loading in **Fig. 7**, there are no loads on members bc and cd therefore α_{bc}, β_{bc}, α_{cd} and β_{cd} are zero.

And $\quad M_{ob} = M_{oc}, \quad M_{od} = \dfrac{w}{2}\, 100^2, \quad \alpha_{ab} = \beta_{ab} = \dfrac{w}{12}\, 100^2$

From equation (8)

$$M_1 + 0.2692\, M_{ob} - 0.2692\, M_{oc} + 0.2692\, \frac{w}{2}\, 100^2 + 0.7308\, \frac{w}{12}\, 100^2 + 0.3558\, \frac{w}{12}\, 100^2 = 0, \quad M_1 = -2251\, w''\#.$$

In the case of the loading in **Fig. 8**, we have no intervening load therefore α_{ab}, β_{ab}, α_{bc}, β_{bc}, α_{cd} and β_{cd} are zero. $\quad M_{ob} = M_{oc}$, $M_{od} = 100\,(P + P_1)$

From equation (8)

$$M_1 + 0.2692\, M_{ob} - 0.2692\, M_{oc} + 0.2692\, M_{od} = 0, \qquad M_1 = -26.92\,(P_1 + P)$$

In the case of the loading in **Fig. 9**, having $a = 50''$ then α_{ab}, β_{ab}, α_{cd} and β_{cd} are zero. $\quad M_{ob} = 0$, $M_{oc} = M_{od}$, $\alpha_{bc} = \dfrac{200}{9}\, P$, $\beta_{bc} = \dfrac{100}{9}\, P$

From equation (8)

$$M_1+0.2692\,M_{0b}-0.2692\,M_{0c}+0.2692\,M_{0d}-0.0865\,\alpha_{bc}-0.1635\,\beta_{bc}=0, \quad M_1=+3.759\,P$$

The values for H_1 and V_1 are obtained in the same manner. From these three illustrations we can see the flexibility of the Influence Equations.

The Influence values of reactions when a load of unity moves from a to d is easily obtained. These Influence Equations can be readily used for any loading.

Continuous Beam.

Application of Influence Equations for a continuous beam is very useful for practical purposes.

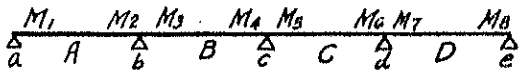


Fig. 11.

Deflection Moment Equations:

$$M_1=0$$

$$AM_2=-\left(\frac{3}{2}m_2\right)+A\gamma_{ab}$$

$$BM_3=-(2m_3+m_4)-B\alpha_{bc}$$

$$BM_4=-(2m_4+m_3)+B\beta_{bc}$$

$$CM_5=-(2m_5+m_6)-C\alpha_{cd}$$

$$CM_6=-(2m_6+m_5)+C\beta_{cd}$$

$$DM_7=-\left(\frac{3}{2}m_7\right)-D\gamma_{de}$$

$$M_8=0$$

From Angular Relations:

$$m_2=m_3, \quad m_4=m_5, \quad m_6=m_7, \quad \varphi=0$$

Then establish the Influence Equations by the same procedure as in the previous case.

Eq.	M_2	M_4	M_6	γ_{ab}	α_{bc}	β_{bc}	α_{cd}	β_{cd}	γ_{de}
1	2(A+B)	B		−2A	−2B	−B			
2	B	2(B+C)	C		−B	−2B	−2C	−C	
3		C	2(C+D)				−C	−2C	−2D

Table. 2.

The Influence Equations for any number of spans can be written by inspection.

Structure with Three Adjacent Spans.

The derivation of Influence Equations for this structure is done by the same procedure;

Establish Moment Equations and Angular Relations and eliminate "m".

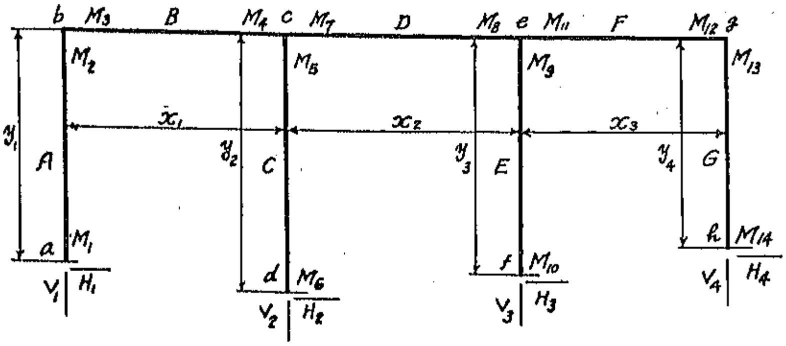


Fig. 12.

Then eliminate "φ" from equations by combining with the horizontal and the vertical displacement conditions, $\Sigma \Delta H=0$ and $\Sigma \Delta V=0$ which are taken between two points of supports such as a-d d-f and f-h, respectively.

In case certain values of displacement of supports or axial elongation or shortening are being considered, $\Sigma \Delta H=0$ and $\Sigma \Delta V=0$ become $\Sigma \Delta H \pm \Delta x=0$ and $\Sigma \Delta V \pm \Delta y=0$.

After the elimination of "m" and "φ" a set of Influence Equations will be established.

In practical cases the load is applied on the beam only, therefore, the terms of moment about each section on column, and Constants, α and β of columns disappear from the Influence Equations, therefore, the equations will be much simplified as shown on **Table 3.**

	M_1	M_6	M_{10}	M_{14}	H_1	V_1	H_2	V_2	H_3	V_3	H_4	V_4	M_o at c	M_o at e	M_o at g	M_o at h	H_0	V_0	α_{bc}	β_{bc}	α_{ce}	β_{ce}	α_{eg}	β_{eg}
1	$3A$	$-3CR_1$			Ay_1		$-CR_1y_2$																	
2	$6A$	$6C$			$3Ay_1$	$-Bx_1$	$3Cy_2$						$-B$						B	$-B$				
3	$2(A+B)$	$-2C$			$y_1(A+2B)$	Bx_1	$-Cy_2$						B						B	B				
4		$3C$	$-3ER_2$				Cy_2		$-ER_2y_3$															
5		$6C$	$6E$			$-Dx_2$	$3Cy_2$	$-Dx_2$	$3Ey_3$				D	$-D$							D	$-D$		
6	$2D$	$2(C+D)$	$-2E$		$2Dy_1$	$D(2x_1+x_2)$	$y_2(C+2D)$	Dx_2	$-Ey_3$				D	D							D	D		
7			$3E$	$-3GR_3$					Ey_3		$-GR_3y_4$													
8			$6E$	$6G$		$-Fx_3$		$-Fx_3$	$3Ey_3$	$-Fx_3$	$3Gy_4$			F	$-F$								F	$-F$
9	$2F$	$2F$	$2(E+F)$	$-2G$	$2Fy_1$	$F(2x_1+2x_2+x_3)$	$2Fy_2$	$F(2x_2+x_3)$	$y_3(E+2F)$	Fx_3	$-Gy_4$			F	F								F	F
10	1	1	1	1	y_1-y_4	$x_1+x_2+x_3$	y_2-y_4	x_2+x_3	y_3-y_4	x_3						1								
11					1		1		1		1						-1							
12						1		1		1		1						-1						

Table 3.

Where $R_1=\frac{y_2}{y_1}$, $R_2=\frac{y_3}{y_2}$, $R_3=\frac{y_4}{y_3}$

H_0 and V_0 are external horizontal and vertical load respectively. M_{oc}, M_{oe} etc. are moments about each section on the beams.

Numerical Example.

On the structure as shown in **Fig. 13** a load of unity travels from b to g. The influence values of reactions at supports a and d will be obtained from the Influence Equations.

In this case, A, B, C, D, E, F and G are 1, and Constants, α and β, on columns are zero.

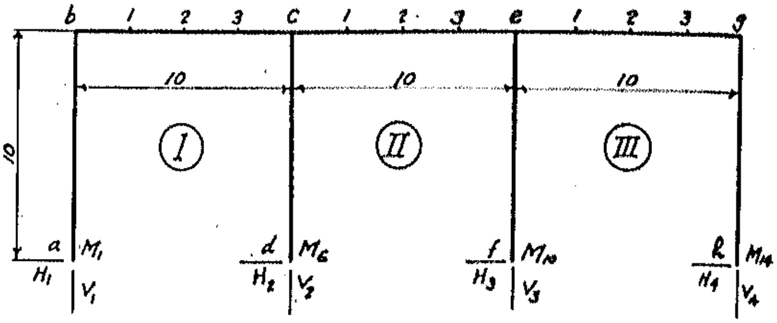


Fig. 13.

Substituting these numerical values into Equations on **Table 3** and solving we have the results as shown on **Table 4** and Curves. (**Fig. 14.**)

Curved Beams.

The principle which has been introduced is readily applicable not only to frames of straight line members but also applicable to arches, other curved beams and closed rings. The Relational Moment Equations in any such problem is always similar and may be written easily by inspection.

H. Müller-Breslau has made a study of this problem in his book "Die Graphische Statik der Baukonstruktionen, Band II, 11 Abt. 1908," but he attacks it in a different way, expressing the deflection angle "w" in terms of "τ."

A curve may be considered as a series of straight line elements. Therefore in the problem to follow the arch is divided into several sections in accordance with the usual practice.

As the most general case the writer took arches which are of unsymmetrical shape and which have supports on different elevations.

Fixed Arch.

In order to make the solution easy all loads were assumed to be applied at each pannel point, thus eliminating the terms, α and β, in the Deflection Moment Equations.

Now divide the arch into 7 segments and derive the Influence Equations.

Refering to **Fig. 16** H_1, H_2, V_1 and V_2 are the horizontal and vertical reactions at supports a and h respectively.

x and y are the horizontal and vertical projections of each segment.

Table 4.

Influence Values of Reaction at Supports.

	b	(I)1	(I)2	(I)3	c	(II)1	(II)2	(II)3	e	(III)1	(III)2	(III)3
M_1		+287	+329	+206		-107	-66	+9		-8	+66	+108
M_6		-263	-342	-250		+221	+263	+174		-71	+13	+90
H_1		-99	-106	-59		+27	+20	+3		-1	-13	-19
H_2		+66	+96	+78		-71	-79	-47		+19	+3	-14
V_1	+1000	+758	+462	+185		-65	-59	-24		+8	+18	+19
V_2		+269	+602	+884	+1000	+864	+559	+225		-78	-83	-47

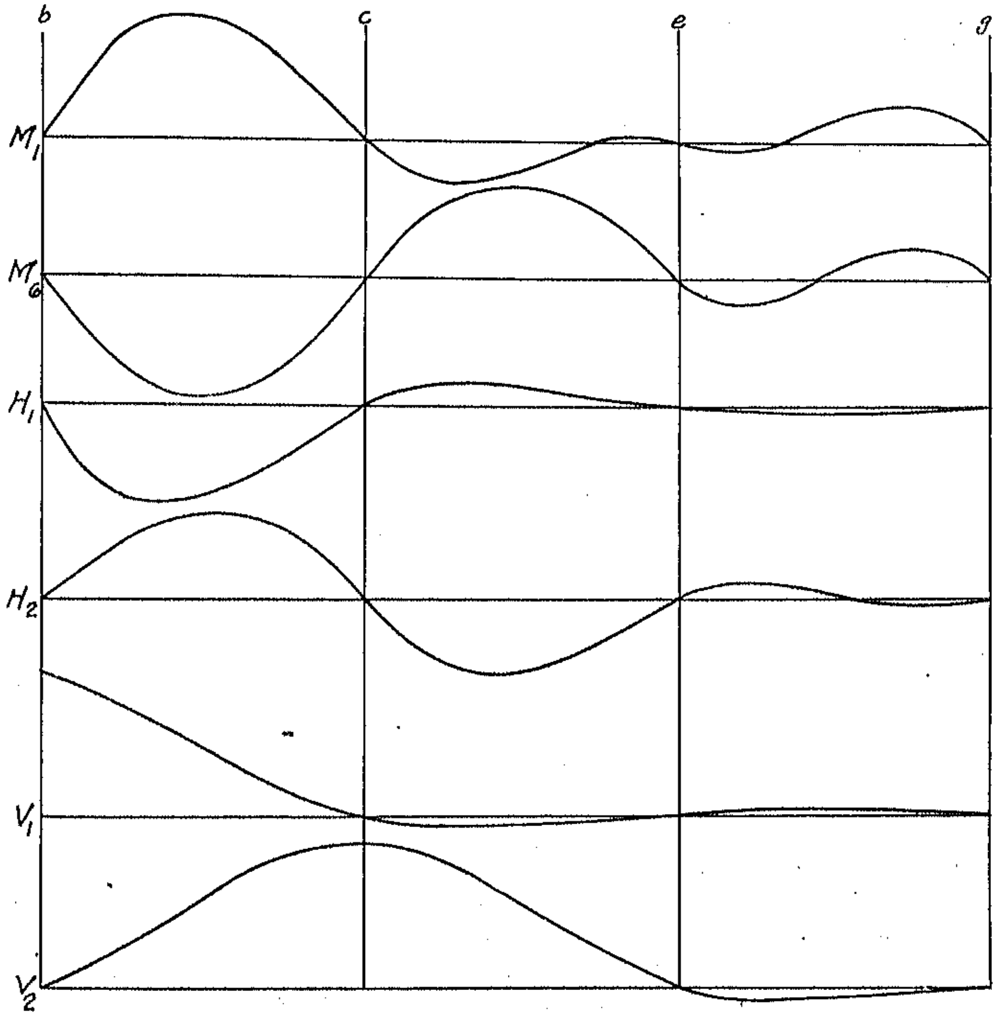


Fig. 14.

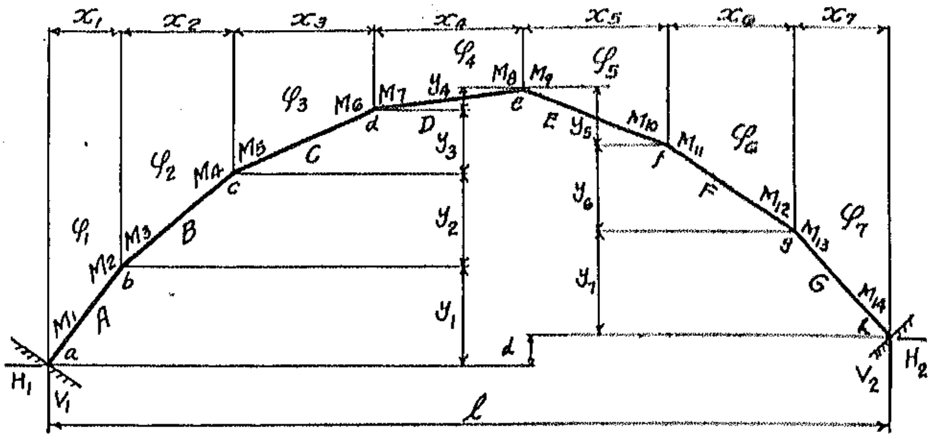


Fig. 15.

From the Deflection Moment Equations,

$$AM_1=-(2m_1+m_2) \qquad DM_8=-(2m_8+m_7)$$
$$AM_2=-(2m_2+m_1) \qquad EM_9=-(2m_9+m_{10})$$
$$BM_3=-(2m_3+m_4) \qquad EM_{10}=-(2m_{10}+m_9)$$
$$BM_4=-(2m_4+m_3) \qquad FM_{11}=-(2m_{11}+m_{12})$$
$$CM_5=-(2m_5+m_6) \qquad FM_{12}=-(2m_{12}+m_{11})$$
$$CM_6=-(2m_6+m_5) \qquad GM_{13}=-(2m_{13}+m_{14})$$
$$DM_7=-(2m_7+m_8) \qquad GM_{14}=-(2m_{14}+m_{13})$$

From the angular relations,

$$m_1=\varphi_1 \qquad m_9=m_8-\varphi_4+\varphi_5$$
$$m_3=m_2-\varphi_1+\varphi_2 \qquad m_{11}=m_{10}-\varphi_5+\varphi_6$$
$$m_5=m_4-\varphi_2+\varphi_3 \qquad m_{13}=m_{12}-\varphi_6+\varphi_7$$
$$m_7=m_6-\varphi_3+\varphi_4 \qquad m_{14}=\varphi_7$$

Substituting these angular relations into the above moment equations and eliminating the "m" then eliminating "φ" combining with $\Sigma\Delta H=0$ and $\Sigma\Delta V=0$ we then get Relational Moment Equations.

$$\Sigma\Delta H=\varphi_1y_1+\varphi_2y_2+\varphi_3y_3+\varphi_4y_4-\varphi_5y_5-\varphi_6y_6-\varphi_7y_7\pm wt\Sigma_1^7x-\Delta u\pm\Delta S_h$$
$$\Sigma\Delta V=\varphi_1x_1+\varphi_2x_2+\varphi_3x_3+\varphi_4x_4+\varphi_5x_5+\varphi_6x_6+\varphi_7x_7\pm wt\Sigma_1^7y-\Delta v\pm\Delta S_v$$

Where $wt\Sigma_1^7x$ and $wt\Sigma_1^7y$ are the horizontal and vertical distortions of the arch due to the temperature change.

Δu and Δv are the effects due to the arch shortening.

ΔS_h and ΔS_v are the effects due to the relative movement of the supports due to various causes, such as earthquake shock, imperfect foundation work, etc.

After obtaining the Relational Moment Equations, which are more simplified by applying $\Sigma M=0$ at each pannel point, combine with statical moment equations and establish the Influence Equations of the arch, as shown on **Table 5.**

Eq.	M_1	M_2	M_4	M_6	M_8	M_{10}	M_{12}	M_{14}	H_1	V_1	Temp. $GE t \omega$	Short. GE	Movement GE	M_n at b	c	d	e	f	g	h
1	$-A$	$(A+B)$	$(B+C)$	$(C+D)$	$(D+E)$	$(E+F)$	$(F+G)$	G												
2	$-A\phi_1$	$A\phi_2+B\phi_3$	$B\phi_4+C\phi_5$	$C\phi_6+D\phi_7$	$D\phi_8+E\phi_9$	$E\phi_{10}+F\phi_{11}$	$F\phi_{12}+G\phi_{13}$	$G\phi_{14}$			$\mp\Sigma_1^7 x$	$+\Delta_H$	$\mp\Delta S_h$							
3	$-A\theta_1$	$A\theta_2+B\theta_3$	$B\theta_4+C\theta_5$	$C\theta_6+D\theta_7$	$D\theta_8+E\theta_9$	$E\theta_{10}+F\theta_{11}$	$F\theta_{12}+G\theta_{13}$	$G\theta_{14}$			$\mp\Sigma_1^7 y$	$+\Delta_V$	$\mp\Delta S_v$							
4	$+1$	1							y_1	x_1				1						
5		-1	1						y_2	x_2				-1	1					
6			-1	1					y_3	x_3					-1	1				
7				-1	1				y_4	x_4						-1	1			
8					-1	1			$-y_5$	x_5							-1	1		
9						-1	1		$-y_6$	x_6								-1	1	
10							-1	1	$-y_7$	x_7									-1	1

Table 5.

$\phi_1 = y_1 \quad \phi_2 = 2y_1 \quad \phi_3 = 3y_1+y_2 \quad \phi_4 = 3y_1+2y_2 \quad \phi_5 = 3\Sigma_1^2 y+y_3 \quad \phi_6 = 3\Sigma_1^2+2y_3 \quad \phi_7 = 3\Sigma_1^3 y+y_4$

$\phi_8 = 3\Sigma_1^3 y+2y_4 \quad \phi_9 = 3\Sigma_1^4 y-y_5 \quad \phi_{10} = 3\Sigma_1^4 y-2y_5 \quad \phi_{11} = 3\Sigma_1^5 y-y_6 \quad \phi_{12} = 3\Sigma_1^5 y-2y_6 \quad \phi_{13} = 3\Sigma_1^6 y-y_7$

$\phi_{14} = 3\Sigma_1^6 y-2y_7$

$\theta_1 = x_1 \quad \theta_2 = 2x_1 \quad \theta_3 = 3x_1+x_2 \quad \theta_4 = 3x_1+2x_2 \quad \theta_5 = 3\Sigma_1^2 x+x_3 \quad \theta_6 = 3\Sigma_1^2 x+2x_3 \quad \theta_7 = 3\Sigma_1^3 x+x_4 \quad \theta_8 = 3\Sigma_1^3 x+2x_4$

$\theta_9 = 3\Sigma_1^4 x+x_5 \quad \theta_{10} = 3\Sigma_1^4 x+2x_5 \quad \theta_{11} = 3\Sigma_1^5 x+x_6 \quad \theta_{12} = 3\Sigma_1^5 x+2x_6 \quad \theta_{13} = 3\Sigma_1^6 x+x_7 \quad \theta_{14} = 3\Sigma_1^6 x+2x_7$

Generally $\phi_{(2n-1)} = 3\Sigma_1^{(n-1)} y \pm y_n \quad \phi_{2n} = 3\Sigma_1^{(n-1)} y \pm 2y_n \quad \theta_{2n-1} = 3\Sigma_1^{(n-1)} x+x_n \quad \theta_{2n} = 3\Sigma_1^{(n-1)} x+2x_n$

Numerical Example of Fixed Arch.

The writer took the same numerical example which was adopted and analyzed by Prof. C.W. Hudson shown on pages 80 to 119 in his book (Deflections and Statically Indeterminate Stresses).

The writer divided the arch into 10 sections with equal horizontal distances instead of 20 sections and applied a load of unity at each pannel point on the right

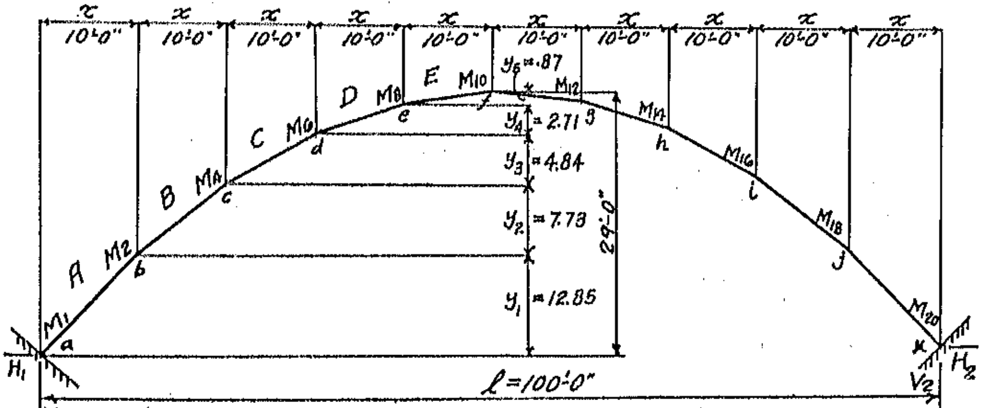


Fig. 16.

	M_1	M_2	M_4	M_6	M_8	M_{10}	M_{12}	M_{14}	M_{16}	M_{18}	M_{20}	H_1	V_1	Temp GEtω	Short GE	M_0 Load at Pt. f	g	h	i	j
1	$-A$	$A+B$	$B+C$	$C+D$	$D+E$	$2E$	$D+E$	$C+D$	$B+C$	$A+B$	A									
2	$-A\phi_1$	$A\phi_2+B\phi_3$	$B\phi_4+C\phi_5$	$C\phi_6+D\phi_7$	$D\phi_8+E\phi_9$	$2E\phi_{10}$	$D\phi_8+E\phi_9$	$C\phi_6+D\phi_7$	$B\phi_4+C\phi_5$	$A\phi_2+B\phi_3$	$A\phi_1$			$\mp l$	$+\Delta_H$					
3	$-A\theta_1$	$A\theta_2+B\theta_3$	$B\theta_4+C\theta_5$	$C\theta_6+D\theta_7$	$D\theta_8+E\theta_9$	$E\theta_{10}+E\theta_{11}$	$E\theta_{12}+D\theta_{13}$	$D\theta_{14}+C\theta_{15}$	$C\theta_{16}+B\theta_{17}$	$B\theta_{18}+A\theta_{19}$	$A\theta_{20}$									
4	+1	1										y_1	x							
5		−1	1									y_2	x							
6			−1	1								y_3	x							
7				−1	1							y_4	x							
8					−1	1						y_5	x							
9						−1	1					$-y_5$	x			x				
10							−1	1				$-y_4$	x			x	x			
11								−1	1			$-y_3$	x			x	x	x		
12									−1	1		$-y_2$	x			x	x	x	x	
13										−1	1	$-y_1$	x			x	x	x	x	x

Table 6.

half of the arch.

All figures for the data are taken from Prof. Hudson's book. The computation and the Influence values of each part are given on **Table 8.**

Before the numerical substitution establish the Influence Equations for the symmetrical arch with 10 segments which can be done by inspection.

	l	I	Member Ratio l/I		ϕ	ϕ	Devide Eq.(3) by x. θ			
1	16.29	4.446	A	3.665	$\phi_1=12.85$	$\phi_6=71.42$	$\theta_1=1$	$\theta_6=8$	$\theta_{11}=16$	$\theta_{16}=23$
2	12.64	2.820	B	4.484	$\phi_2=25.70$	$\phi_7=78.97$	$\theta_2=2$	$\theta_7=10$	$\theta_{12}=17$	$\theta_{17}=25$
3	11.11	1.813	C	6.128	$\phi_3=46.28$	$\phi_8=81.68$	$\theta_3=4$	$\theta_8=11$	$\theta_{13}=19$	$\theta_{18}=26$
4	10.35	1.236	D	8.374	$\phi_4=54.01$	$\phi_9=85.26$	$\theta_4=5$	$\theta_9=13$	$\theta_{14}=20$	$\theta_{19}=28$
5	10.04	1.007	E	9.965	$\phi_5=66.58$	$\phi_{10}=86.13$	$\theta_5=7$	$\theta_{10}=14$	$\theta_{15}=22$	$\theta_{20}=29$

Table 7.

Substituting the numerical values of **Table 7** into equations of **Table 6** we have **Table 8.** Equations (14), (15), (16) and (17) are the Influence values of Vertical and Horizontal reactions, and End and Center Moments, all of which check against Prof. Hudson's results, as shown with curves in **Fig. 17.**

Two Hinged Arch.

The solution of the Two Hinged Arch is the same as that of the fixed arch except that $M_1=M_{14}=0$, "m_1" is not equal to "φ_1" and "m_{14}" is not equal to "φ_7".

Eq.	M_1	M_2	M_4	M_6	M_8	M_{10}	M_{12}	M_{14}	M_{16}	M_{18}	M_{20}	H_1	V_1	Temp eff.	Arch Short.	Load of Unity at Pt. M_0 f	g	h	i	j
1	-3.665	8.149	10.61	14.50	18.34	19.93	18.34	14.50	10.61	8.149	3.665									
2	-47.08	301.6	650.1	1099.	1534.	1716.	1534.	1099.	650.1	301.6	47.08			∓4752	+840					
3	-3.665	25.27	65.31	132.8	221.7	298.9	328.5	302.2	253.0	219.2	106.3									
4	+1	1										1285	10							
5		-1	1									7.73	10							
6			-1	1								4.84	10							
7				-1	1							2.71	10							
8					-1	1						.87	10							
9						-1	1					-.87	10			10				
10							-1	1				-2.71	10			10	10			
11								-1	1			-4.84	10			10	10	10		
12									-1	1		-7.73	10			10	10	10	10	
13										-1	1	-1285	10			10	10	10	10	10
Solving above Eqs. We have.																				
14													1			.501	.337	.192	.078	.022
15												1		±.236	∓.042	-.911	-.826	-.602	-.331	-.107
16											1			±5.43	∓.960	-5.92	-1.52	+3.67	+7.32	+6.15
17						1								∓1.41	±.251	-4.46	-.720	+.740	+.830	+.360

Table 8.

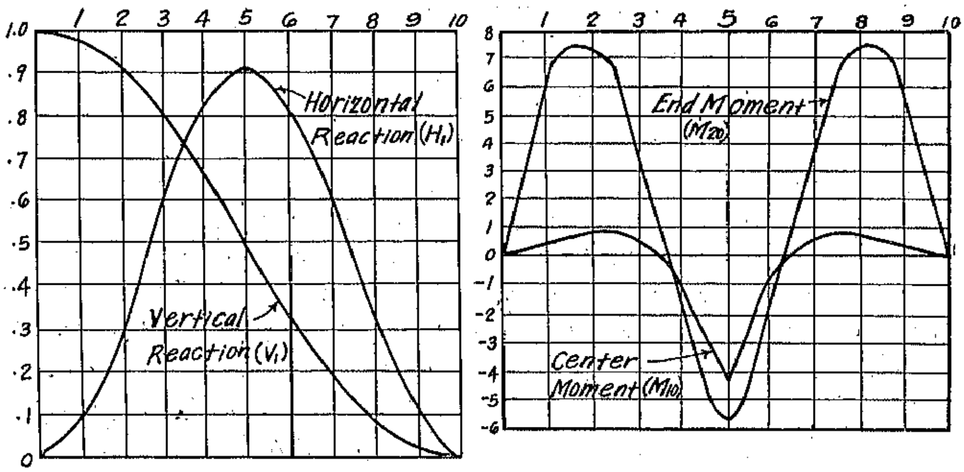


Fig. 17.

And the values of V_1 and V_2 are statically determinate therefore at support only one quantity is statically indeterminate that is horizontal reaction H. If " H " is found then rest of unknowns are statically determinate.

Therefore the most important thing in this problem is how to find the Influence values of " H ".

In the numerical example the effects of temperature, arch shortening and movement of supports will not be taken into consideration because the process of

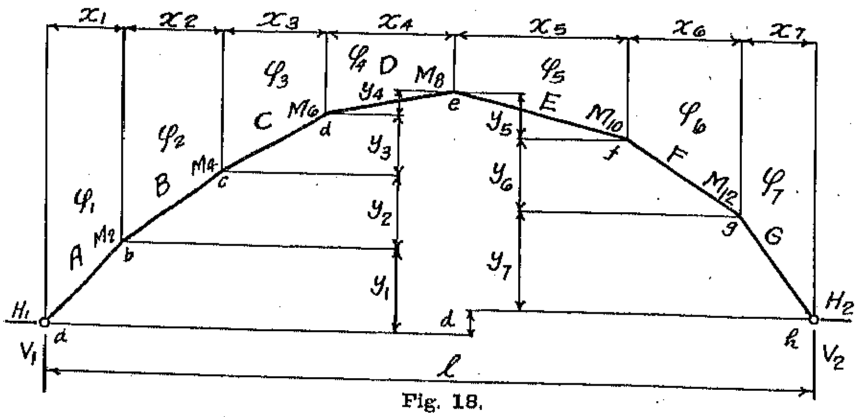


Fig. 18.

solution for these is the same as for a fixed arch.

From the Deflection Moment Equations,

$$M_1=0 \qquad DM_8=-(2m_8+m_7)$$
$$AM_2=-\left(\frac{3}{2}m_2\right) \qquad EM_9=-(2m_9+m_{10})$$
$$BM_3=-(2m_3+m_4) \qquad EM_{10}=-(2m_{10}+m_9)$$
$$BM_4=-(2m_4+m_3) \qquad FM_{11}=-(2m_{11}+m_{12})$$
$$CM_5=-(2m_5+m_6) \qquad FM_{12}=-(2m_{12}+m_{11})$$
$$CM_6=-(2m_6+m_5) \qquad GM_{13}=-\left(\frac{3}{2}m_{13}\right)$$
$$DM_7=-(2m_7+m_8) \qquad M_{14}=0$$

From the angular relations

$$m_3=m_2-\varphi_1+\varphi_2 \qquad m_5=m_4-\varphi_2+\varphi_3 \qquad m_7=m_6-\varphi_3+\varphi_4$$
$$m_9=m_8-\varphi_4+\varphi_5 \qquad m_{11}=m_{10}-\varphi_5+\varphi_6 \qquad m_{13}=m_{12}-\varphi_6+\varphi_7$$

Substituting these values into above moment equations and by the same manner as in a fixed arch eliminate " m " and " φ " from the equations and get " Relational Moment Equations " and " Influence Equations ".

Eq.	M_2	M_4	M_6	M_8	M_{10}	M_{12}	H_1	V_1	Temp. 6EWt	Load at Pt. M_0				
										b	c	d	e	f
1	$A\phi_2+B\phi_3$	$B\phi_4+C\phi_5$	$C\phi_6+D\phi_7$	$D\phi_8+E\phi_9$	$E\phi_{10}+F\phi_{11}$	$F\phi_{12}+G\phi'_{13}$			$\mp l$					
2	1						y_1	x_1						
3	−1	1					y_2	x_2		x_2				
4		−1	1				y_3	x_3		x_3	x_3			
5			−1	1			y_4	x_4		x_4	x_4	x_4		
6				−1	1		$-y_5$	x_5		x_5	x_5	x_5	x_5	
7					−1	1	$-y_6$	x_6		x_6	x_6	x_6	x_6	x_6

Table 9.

The values of " ϕ " are the same as that of a fixed arch except $\phi'_{13}=2\Sigma_1^6 y$ or $\phi'_{2i-1}=2\Sigma_1^{n-1}y$.

Rings.

This principle is also applicable to the solution of rings which are circular, oval or any unsymmetrical shape.

To illustrate the principle an unsymmetrical ring under any loading, as shown in **Fig. 19** and **Fig. 20** will be employed and the general equations for rings and closed frames will be established.

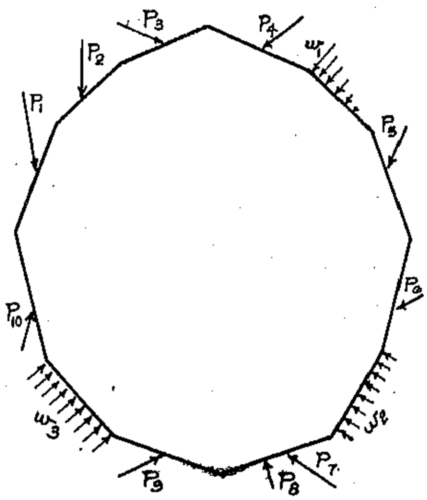


Fig. 19.

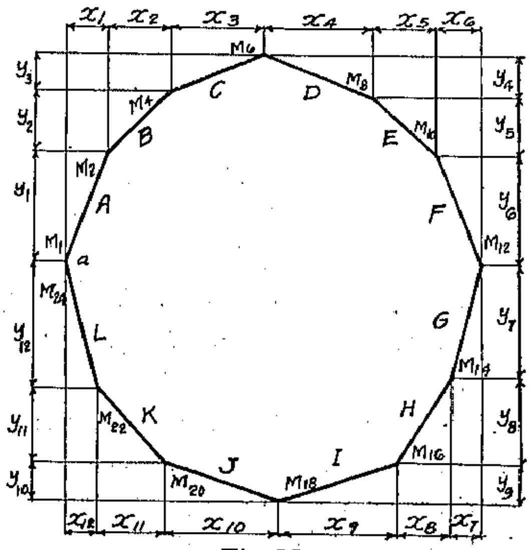


Fig. 20.

Divide the ring into 12 segments and as in other cases establish Deflection Moment Equations and Angular Relations and eliminate "m".

Then eliminate "φ" from the equations by combining with the horizontal and

	M_1	M_2	M_4	M_6	M_8	M_{10}	M_{12}	M_{14}	M_{16}	M_{18}	M_{20}	M_{22}	H_a	V_a	M_o 2	3	4	5	6	7	8	9	10	11	12	α_1	β_1	α_2	β_2	α_3	β_3	α_4	β_4	α_5	β_5	α_6	β_6	α_7	β_7	α_8	β_8	α_9	β_9	α_{10}	β_{10}	α_{11}	β_{11}	α_{12}	β_{12}
1	-A-L	A+B	B+C	C+D	D+E	E+F	F+G	G+H	H+I	I+J	J+K	K+L														-A	-A	-B	-B	-C	-C	-D	-D	-E	-E	-F	-F	-G	-G	-H	-H	-I	-I	-J	-J	-K	-K	-L	-L
2	$-A\phi'_1 - L\phi'_{24}$	$A\phi'_2 + B\phi'_3$	$B\phi'_4 + C\phi'_5$	$C\phi'_6 + D\phi'_7$	$D\phi'_8 + E\phi'_9$	$E\phi'_{10} + F\phi'_{11}$	$F\phi'_{12} + G\phi'_{13}$	$G\phi'_{14} + H\phi'_{15}$	$H\phi'_{16} + I\phi'_{17}$	$I\phi'_{18} + J\phi'_{19}$	$J\phi'_{20} + K\phi'_{21}$	$K\phi'_{22} + L\phi'_{23}$														$-A\phi_1$	$-A\phi_2$	$-B\phi_3$	$-B\phi_4$	$-C\phi_5$	$-C\phi_6$	$-D\phi_7$	$-D\phi_8$	$-E\phi_9$	$-E\phi_{10}$	$-F\phi_{11}$	$-F\phi_{12}$	$-G\phi_{13}$	$-G\phi_{14}$	$-H\phi_{15}$	$-H\phi_{16}$	$-I\phi_{17}$	$-I\phi_{18}$	$-J\phi_{19}$	$-J\phi_{20}$	$-K\phi_{21}$	$-K\phi_{22}$	$-L\phi_{23}$	$-L\phi_{24}$
3	$-A\theta_1 - L\theta'_{24}$	$A\theta_2 + B\theta_3$	$B\theta_4 + C\theta_5$	$C\theta_6 + D\theta_7$	$D\theta_8 + E\theta_9$	$E\theta_{10} + F\theta_{11}$	$F\theta_{12} + G\theta_{13}$	$G\theta_{14} + H\theta_{15}$	$H\theta_{16} + I\theta_{17}$	$I\theta_{18} + J\theta_{19}$	$J\theta_{20} + K\theta_{21}$	$K\theta_{22} + L\theta_{23}$														$-A\theta_1$	$-A\theta_2$	$-B\theta_3$	$-B\theta_4$	$-C\theta_5$	$-C\theta_6$	$-D\theta_7$	$-D\theta_8$	$-E\theta_9$	$-E\theta_{10}$	$-F\theta_{11}$	$-F\theta_{12}$	$-G\theta_{13}$	$-G\theta_{14}$	$-H\theta_{15}$	$-H\theta_{16}$	$-I\theta_{17}$	$-I\theta_{18}$	$-J\theta_{19}$	$-J\theta_{20}$	$-K\theta_{21}$	$-K\theta_{22}$	$-L\theta_{23}$	$-L\theta_{24}$
4											-1	1	y_{11}	$-x_{11}$										-1	1																								
5										-1	1		y_{10}	$-x_{10}$									-1	1																									
6									-1	1			$-y_9$	$-x_9$								-1	1																										
7								-1	1				$-y_8$	$-x_8$							-1	1																											
8							-1	1					$-y_7$	$-x_7$						-1	1																												
9						-1	1						$-y_6$	x_6					-1	1																													
10					-1	1							$-y_5$	x_5				-1	1																														
11				-1	1								$-y_4$	x_4			-1	1																															
12			-1	1									y_3	x_3		-1	1																																
13		-1	1										y_2	x_2	-1	1																																	
14	+1	1											y_1	x_1	1																																		

Values of ϕ and θ are the same as in Arch Solution. $\phi'_{2n-1} = 2\Sigma_1^{(n-1)} y$ $\quad \phi'_{2n} = \Sigma_1^{(n-1)} y$

Table 10. $\theta'_{2n-1} = 2\Sigma_1^{(n-1)} x$ $\quad \theta'_{2n} = \Sigma_1^{(n-1)} x$

vertical displacement conditions, $\Sigma\Delta H=0$ and $\Sigma\Delta V=0$.

$$\Sigma \Delta H=0$$

$$\varphi_1 y_1+\varphi_2 y_2+\varphi_3 y_3-\varphi_4 y_4-\varphi_5 y_5-\varphi_6 y_6-\varphi_7 y_7-\varphi_8 y_8-\varphi_9 y_9+\varphi_{10} y_{10}+\varphi_{11} y_{11}+\varphi_{12} y_{12}=0$$

$$\Sigma \Delta V=0$$

$$\varphi_1 x_1+\varphi_2 x_2+\varphi_3 x_3+\varphi_4 x_4+\varphi_5 x_5+\varphi_6 x_6-\varphi_7 x_7-\varphi_8 x_8-\varphi_9 x_9-\varphi_{10} x_{10}-\varphi_{11} x_{11}-\varphi_{12} x_{12}=0$$

After the elimination of "m" and "φ" a set of Influence Equations, which are the general Influence Equations for rings, will be established as shown in **Table 10.**

The Influence Equations for any ring can easily be written by inspection.

Egg-shaped Ring.

For the application of the principle take an egg-shaped ring of uniform thickness and dimensions as shown in **Fig. 21** which is subjected to uniform and concentrated loads. The solution of the problem will be done by following the usual procedure.

Divide the ring into several sections:

6 sections between a-c,

5 sections between a-e and

2 sections between e-d.

Establish the Influence Equations for this structure from the general equations by inspection and substitute the numerical values into the equations and solve.

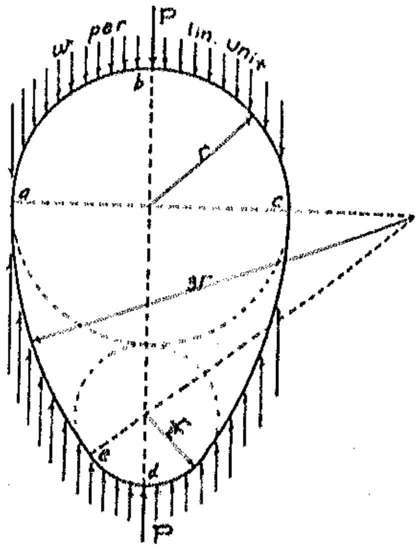


Fig. 21.

Then we have

$$M_a=-(0.15909\,rP+0.10423\,r^2w)$$
$$H_a=0.02282\,P+0.03470\,rw$$
$$V_a=0.49997\,P+0.08300\,rw$$
$$M_b=-(0.30370\,rP+0.31005\,r^2w)$$
$$M_d=-(0.29344\,rP+0.20301\,r^2w)$$

In case we have concentrated loades P only then we have

$M_a=-0.15909\,rP$ $\quad$ $M_b=-0.30370\,rP$

$H_a=0.02282\,P$ $\quad$ $M_d=-0.29344\,rP$

$V_a=0.49907\,P$

In case we have uniform loads only then we have

$M_a=-0.10423\,r^2w$ $\quad$ $M_b=-0.31005\,r^2w$

$H_a=0.03470\,rw$ $\quad$ $M_d=-0.20301\,r^2w$

$V_a=0.08900\,rw$

The above values check with the statement made by Dr. Hibi (Reinforced concrete Vol. III) regarding approximate values for these quantities obtained by experiments.

The applications of this principle were illustrated with few examples but the applications for the solution of statically indeterminate structures are unlimitted; i.e. bridges, building frames and any other framed structures with curved or straight line members.