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IMPORTANT PROBLEMS IN THE DESIGN OF REINFORCED CONCRETE FLOORS IN HIGHWAY BRIDGES

By Masaru Yasumi, C. E., Assoc. Member.

Synopsis

This paper is, so to speak, an extension of the Dr. Nádai's theories found in the "Elastische Platten". The author discusses the general nature of the stresses due to rectangular partial uniform loads in rectangular slabs. He has also taken into account the effect of the side support yielding.

Special attention is paid to the application of the author's theories to the design of reinforced concrete floors in highway bridges.

Contents

Chapter 1. General remarks.

1. Introductory note.

The problems of rectangular slabs seem to have been fully discussed by eminent scholars both in this country and abroad, but to apply the theories thus developed to particular cases, some more considerations are necessary.

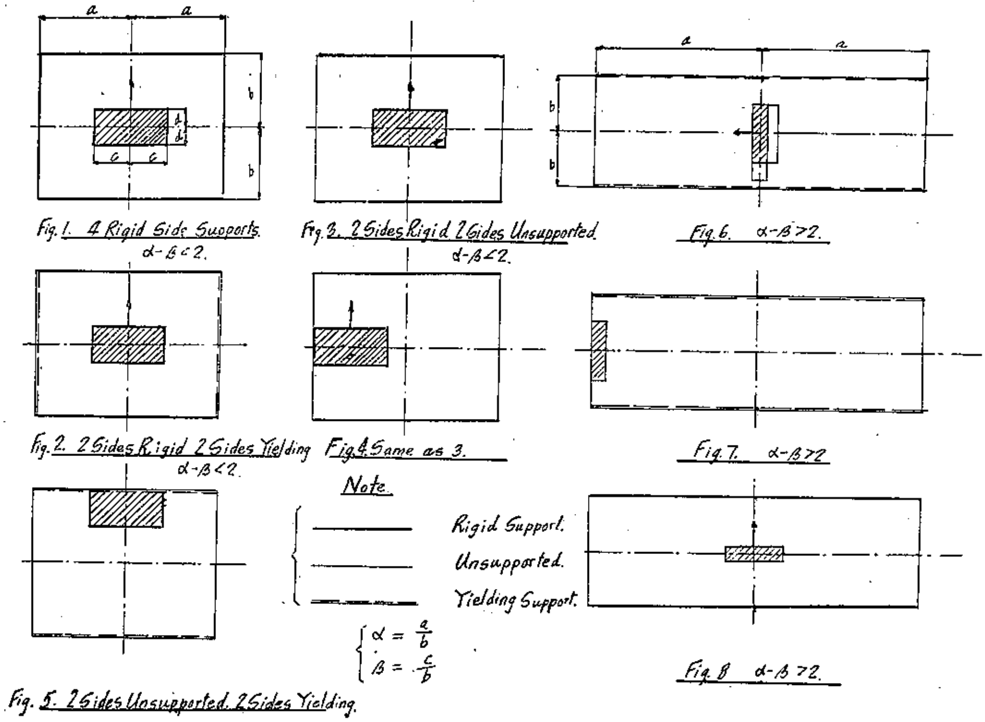
Formulas for stresses in slabs, whether theoretical or experimental, are for those on rigid unyielding side supports.

In bridge floors, for example, reinforced concrete slabs are supported by cross beams, stringers or main girders, and the effect of side support yielding is by no means negligible, and often leads to a considerable change of stresses.

In this connection a brief treatment is found in Mr. Marcus' book, "Theorie Elastischer Gewebe und ihre Anwendung auf die Berechnung biegsamer Platten". Hinted by this discussion I tried to get a more definite idea of the effect of side support yielding. To avoid confusion I obtained solutions for isotropic plates. The solutions are limited to the cases where Lévy's solution is applicable in simple forms.

The solutions for more complicated cases, which I omitted because I failed to obtain them in compact forms, are not at all impossible if we don't spare time and labour.

Fig. 1～8 Modes of Supporting Rectangular Slabs in Bridge Floors.



2. How the slabs are supported in bridge floors and how the support yielding affects stresses.

In bridge floors the rectangular slabs are supported by their four sides or by their two opposite sides, the other sides being unsupported. Let us call the former the "four side support slabs" and the latter the "two side support slabs".

These side supports have certain degrees of rigidity and their yielding affects stresses. In deck plate girders, if the slabs are supported directly by the main girders and the cross beams connecting them, they are "four side support slabs". In this case the sides supported by the main girders may be considered rigid and the yielding of the cross beams only affects the stresses. For four side support slabs see **Pl. 1** (**2**), (**3**).

3. Considerations on two side support slabs.

Pl. 1 (**1**), (**4**) are examples of "two side support slabs". The slabs are supported by cross beams only in **Pl. 1 (1)** and by the stringers in **Pl. 1 (4)**.

In this case if the side supports are rigid, the bending moment along the support will be zero, but if they yield and the side of the slabs follow them, some bending moment will be expected along the support line.

In practice little attention is paid to the so called "distributing bars".

In the most unfavourable load position shown by **Fig. 5,** the bending moment due to the yielding of the support may overstress the distributing bars.

If the side supports are rigid and the load is not concentrated between unsupported sides, no special consideration is necessary as to distributing bars, and a poor reinforcement will be all right, but in comparatively long slabs such as slab bridges or the floors of trusses and through plate girders, **Fig. 6, 7, 8** full knowledge of stress distribution is required for reasonable reinforcement parallel to the long sides. The bending moment in the direction of distributing bars caused by the concentration of the load is by no means negligible; they reach 20 to 80% of the main bending moment due to usual distributed wheel loads. See **Pl. 2,** and for particulars read Art. 1. Chapt. 4.

Some more considerations are necessary as to main stresses. In common practice dimensions of slabs are determined by the bending moment calculated with effective width obtained experimentally with concentrated loads at the centers of slabs.

But in two side support slabs the maximum bending moment will occur at the unsupported sides, so that the stresses thus calculated will be insufficient near the unsupported sides. The most unfavourable load position is shown in **Fig. 4.**

Chapter 2. On the nature of stresses in slabs.

1. Fundamental equations.

As general equations for deflections and stresses in plates we have,

$$\text{Bending moment}\quad \left\{\begin{aligned} M_x &= -N\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\\ M_y &= -N\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\end{aligned}\right\} \qquad \cdots\cdots(1)$$

Shearing moment $$M_{xy}=-(1-\nu)N\frac{\partial^2 w}{\partial x\partial y} \qquad \text{(2)}$$

Shear $$\left\{\begin{array}{l} S_x=-N\dfrac{\partial \Delta w}{\partial x} \\ S_y=-N\dfrac{\partial \Delta w}{\partial y} \end{array}\right\} \qquad \text{(3)}$$

where

$$\Delta w=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}$$

Reaction $$\left\{\begin{array}{l} R_x=-N\left\{\dfrac{\partial^3 w}{\partial x^3}+(2-\nu)\dfrac{\partial^3 w}{\partial x\partial y^2}\right\} \\ R_y=-N\left\{\dfrac{\partial^3 w}{\partial y^3}+(2-\nu)\dfrac{\partial^3 w}{\partial y\partial x^2}\right\} \end{array}\right\} \qquad \text{(4)}$$

Where

$N=$" Plattensteifigkeit " $=\dfrac{Eh^3}{12(1-\nu^2)}$

$\nu=$Poisson's ratio

$h=$thickness of the slab

and w is the solution of the partial differential equation,

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\Delta\Delta w=\frac{p}{N} \qquad \text{(5)}$$

p: load intensity, being a function of x & y.

The solution of the differential equation (5) is composed of a particular integral and a complimentary function, which is a solution of the partial differential equation.

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=0$$

In the present discussion the sine or cosine series are adopted as the particular integral which is a solution of the equation (5), namely

$$w_0=\frac{1}{N}\sum_n a_n\cos\frac{n\pi x}{2b} \qquad \text{(6)}$$

$n=1, 3, 5....$

for symmetrical loading with the origin at the center of the slab. (pt. A)

Or,

$$w_0=\frac{1}{N}\sum_n a_n\sin\frac{n\pi x}{2b} \qquad \text{(7)}$$

$n=1, 2, 3, 4, 5....$

with the origin as shown by **Fig. 9** (pt. B).

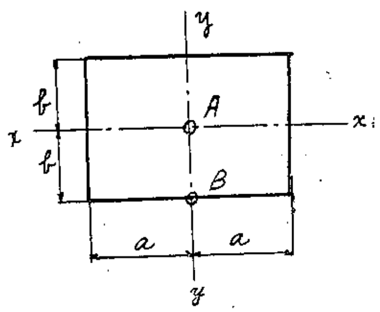


Fig. 9.

As the complementary function we have,

$$w_I = \frac{1}{N}\sum_n \left(A_n \cosh\frac{n\pi x}{2b} + B_n \frac{n\pi x}{2b}\sinh\frac{n\pi x}{2b} + C_n \sinh\frac{n\pi x}{2b} + D_n \frac{n\pi x}{2b}\cosh\frac{n\pi x}{2b}\right)\cos\frac{n\pi y}{2b} \quad \ldots(8)$$

A_n, B_n, C_n, D_n are constants to be determined by boundary conditions.

The notations and equations shown above are the same as are found in Dr. Nádai's "Elastische Platten".

2. The solution of the rectangular slabs simply supported by four rigid side supports.

The well known solution by Navier which is expressed with double sine series taking the origin at the corner of the rectangular plate is shown by eq. (9).

$$w = \sum_m \sum_n C_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad \ldots\ldots(9)$$

This is perhaps the most general solution for simply supported rectangular slabs. It evidently satisfies the eq. (5) and the boundary condition

$$\begin{cases} M_x = 0, \quad w = 0 \quad \text{at} \quad x = 0 \ \& \ x = a. \\ M_y = 0, \quad w = 0 \quad \text{at} \quad y = 0 \ \& \ y = b. \end{cases}$$

It is applicable to any loading p which can be expressed with double sine series. For the rectangular partial uniform loading in **Fig. 10,**

$$C_{mn} = \frac{16 p_0}{\pi^6 N} \cdot \frac{\sin\frac{m\pi\xi}{a} \sin\frac{m\pi u}{a} \sin\frac{n\pi\eta}{b} \sin\frac{n\pi v}{b}}{mn\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2}$$

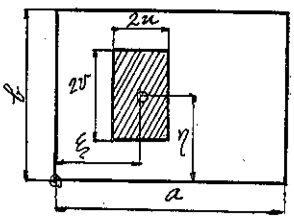


Fig. 10.

and for a full uniform load for which

$$\xi = u = \frac{a}{2} \quad \& \quad \eta = v = \frac{b}{2}$$

and the eq. (9) will be

$$w = \frac{16 p_0}{\pi^6 N} \sum_m \sum_n \frac{\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{mn\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \quad \ldots\ldots(10)$$

$$m, n = 1, 3, 5 \ldots.$$

This is a solution for a simply supported rectangular slab with full uniform load. This equation is simple in form but is not fit to get general idea of the nature of stresses.

It requires comparatively many terms to attain certain accuracy. For example, if we take three terms for both m & n the resultant number of terms will be nine.

Mr. Iguchi, in his excellent paper in our society's journal Vol. 17. No. 5. modified the Navier's solution into single series. This series is quickly convergent and is suited for stress calculation.

3. The solution for rectangular partial uniform loading.

The solution is composed of two systems of functions; the one is referred to the co-ordinate origin at A ($x=0$, $y=0$) and the other to the point B ($x=\pm c$, $y=0$). The loading is assumed to be symmetrical with regard to the Y-axis. See **Fig. 11.**

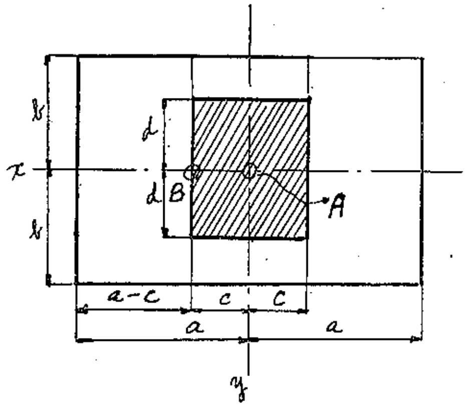


Fig. 11.

As the solution of the partial differential equation (eq. 5), in the range $c\geqq x\geqq -c$, referred to the origin A.

As particular integral

$$w_0=\frac{1}{N}\sum_n \frac{64b^4p_0}{n^5\pi^5}\sin\frac{n\pi d}{2b}\cos\frac{n\pi y}{2b} \quad\ldots\ldots(11)$$

$$n=1, 3, 5 \ldots.$$

and as complimentary function

$$w_{\mathrm{I}}=\frac{1}{N}\sum_n\left(A_n\cosh\frac{n\pi x}{2b}+B_n\frac{n\pi x}{2b}\sinh\frac{n\pi x}{2b}\right)\cos\frac{n\pi y}{2b} \quad\ldots\ldots(12)$$

In the range $a-c\geqq x\geqq 0$, referred to the origin at B,

$$w_{\mathrm{II}}=\frac{1}{N}\sum_n\Big[C_n\left\{\tanh\frac{n\pi(a-c)}{2b}\cosh\frac{n\pi x}{2b}-\sinh\frac{n\pi x}{2b}\right\}$$

$$+D_n\frac{n\pi(x-a+c)}{2b}\left\{\tanh\frac{n\pi(a-c)}{2b}\sinh\frac{n\pi x}{2b}-\cosh\frac{n\pi x}{2b}\right\}\Big]\cos\frac{n\pi y}{2b} \quad\ldots\ldots(13)$$

$$n=1, 3, 5 \ldots.$$

The equation (13) is a modification of eq. (8) and satisfies the diff. equation

$$\Delta\Delta w=0$$

and further at the boundary $x=a-c$, that is to say on the support, it satisfies that

$$w_{\mathrm{II}}=0, \qquad M_x=0$$

as referred to the origin at B.

The constants A_n, B_n, C_n & D_n are determined by the conditions of continuity at the common boundary at

$x=c$ & $x=0$

respectively with regard to the two origins of co-ordinates.

4. Conditions of continuity at the common boundary.

The conditions of continuity are as follows. See also "Elastische Gewebe,...." page 203.

(1) The deflection of the sides must be equal at the common boundary, that is to say,

$$(w_0+w_{\rm I})_{x=c}=(w_{\rm II})_{x=0} \quad \text{.......(14)}$$

(2) The slope of the two surfaces must be equal.

$$\left(\frac{\partial w_0}{\partial x}+\frac{\partial w_{\rm I}}{\partial x}\right)_{x=c}=\left(\frac{\partial w_{\rm II}}{\partial x}\right)_{x=0} \quad \text{.......(15)}$$

(3) The bending moment must be equal.

$$\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial^2 w_{\rm I}}{\partial x^2}\right)_{x=c}=\left(\frac{\partial^2 w_{\rm II}}{\partial x^2}\right)_{x=0} \quad \text{.......(16)}$$

In order that M_x be equal,

$$\left(\frac{\partial^2 w_0}{\partial x^2}+\nu\frac{\partial^2 w_0}{\partial y^2}\right)_{x=c}+\left(\frac{\partial^2 w_{\rm I}}{\partial x^2}+\nu\frac{\partial^2 w_{\rm I}}{\partial y^2}\right)_{x=c}=\left(\frac{\partial^2 w_{\rm II}}{\partial x^2}+\nu\frac{\partial^2 w_{\rm II}}{\partial y^2}\right)_{x=0}$$

But by the condition (1), the second derivatives of y are equal at the common boundary so that the relation shown by (3) is necessary.

(4) Shear is equal.

$$\left(\frac{\partial \varDelta w_0}{\partial x}+\frac{\partial \varDelta w_{\rm I}}{\partial x}\right)_{x=c}=\left(\frac{\partial \varDelta w_{\rm II}}{\partial x}\right)_{x=0}$$

$$\varDelta w=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}$$

But as eq. (15) implies the condition that $\frac{\partial^3 w}{\partial x \partial y^2}$ are equal, it will suffice that

$$\left(\frac{\partial^3 w_0}{\partial x^3}+\frac{\partial^3 w_{\rm I}}{\partial x^3}\right)_{x=c}=\left(\frac{\partial^3 w_{\rm II}}{\partial x^3}\right)_{x=0} \quad \text{.......(17)}$$

Thus we have got four relations by comparing the n-th terms of cosine series. From the above conditions we get the following relations by putting $\frac{a}{b}=\alpha$, $\frac{c}{b}=\beta$,

from (14) $$C_n\tanh\frac{n\pi(\alpha-\beta)}{2}+D_n\frac{n\pi(\alpha-\beta)}{2}=a_n+A_n\cosh\frac{n\pi\beta}{2}+B_n\frac{n\pi\beta}{2}\sinh\frac{n\pi\beta}{2}$$

from (15) $$-C_n-\left\{1+\frac{n\pi(\alpha-\beta)}{2}\tanh\frac{n\pi(\alpha-\beta)}{2}\right\}D_n$$

$$=A_n\sinh\frac{n\pi\beta}{2}+\left\{\sinh\frac{n\pi\beta}{2}+\frac{n\pi\beta}{2}\cosh\frac{n\pi\beta}{2}\right\}B_n$$

from (16) $C_n \tanh\frac{n\pi(\alpha-\beta)}{2} + D_n\left\{2\tanh\frac{n\pi(\alpha-\beta)}{2} + \frac{n\pi(\alpha-\beta)}{2}\right\}$

$$= A_n \cosh\frac{n\pi\beta}{2} + B_n\left\{2\cosh\frac{n\pi\beta}{2} + \frac{n\pi\beta}{2}\sinh\frac{n\pi\beta}{2}\right\}$$

from (17) $-C_n - \left\{3 + \frac{n\pi(\alpha-\beta)}{2}\tanh\frac{n\pi(\alpha-\beta)}{2}\right\}D_n$

$$= A_n \sinh\frac{n\pi\beta}{2} + B_n\left\{3\sinh\frac{n\pi\beta}{2} + \frac{n\pi\beta}{2}\cosh\frac{n\pi\beta}{2}\right\}$$

From the above simultaneous equations we get

$$\left.\begin{aligned}
A_n &= -\frac{\left\{\frac{n\pi\alpha}{2}\tanh\frac{n\pi\alpha}{2} - \frac{n\pi(\alpha-\beta)}{2}\tanh\frac{n\pi(\alpha-\beta)}{2} + 2\right\}a_n}{2\cosh\frac{n\pi\beta}{2}\left\{1+\tanh\frac{n\pi\beta}{2}\tanh\frac{n\pi(\alpha-\beta)}{2}\right\}}\\
B_n &= +\frac{a_n}{2\cosh\frac{n\pi\beta}{2}\left\{1+\tanh\frac{n\pi\beta}{2}\tanh\frac{n\pi(\alpha-\beta)}{2}\right\}}\\
C_n &= +\frac{a_n\tanh\frac{n\pi\beta}{2}\left\{\frac{n\pi(\alpha-\beta)}{2}\tanh\frac{n\pi\alpha}{2}+2\right\}}{2\left\{1+\tanh\frac{n\pi\beta}{2}\tanh\frac{n\pi(\alpha-\beta)}{2}\right\}}\\
&\quad -\frac{a_n\frac{n\pi\beta}{2}}{2\left\{1+\tanh\frac{n\pi\beta}{2}\tanh\frac{n\pi(\alpha-\beta)}{2}\right\}^2\cosh^2\frac{n\pi\beta}{2}}\\
D_n &= -\frac{a_n\tanh\frac{n\pi\beta}{2}}{2\left\{1+\tanh\frac{n\pi\beta}{2}\tanh\frac{n\pi(\alpha-\beta)}{2}\right\}}
\end{aligned}\right\}\quad\ldots\ldots\ldots(18)$$

$$a_n = \frac{64b^4p_0}{n^5\pi^5}\sin\frac{n\pi d}{2b} \qquad \alpha = \frac{a}{b}, \qquad \beta = \frac{c}{b}$$

If we put $\alpha=\beta$, $\frac{d}{b}=1$, namely for a full uniform load we get,

$$\left.\begin{aligned}
A_n &= -\frac{\left\{\frac{n\pi\alpha}{2}\tanh\frac{n\pi\alpha}{2}+2\right\}a_n}{2\cosh\frac{n\pi\alpha}{2}}\\
B_n &= +\frac{a_n}{2\cosh\frac{n\pi\alpha}{2}}\\
a_n &= \frac{64b^4p_0}{n^5\pi^5}(-1)^{\frac{n-1}{2}}
\end{aligned}\right\}\quad\ldots\ldots\ldots(19)$$

This is the same as shown in Dr. Nádai's "Elastische Platten".

5. A very important relation derived from the eq. (12), (13) and (18).

A very important relation is derived from the solutions give in the preceding article. Look at the coefficients shown by (18). If $(\alpha-\beta>2)$, the terms containing α will vanish, for if $(\alpha-\beta>2)$, $\tanh\frac{n\pi\alpha}{2}$ and $\tanh\frac{n\pi(\alpha-\beta)}{2}$ are very nearly 1.

For example,

$$\tanh\pi=0.99627,\quad \tanh 2\pi=0.99999,\quad \tanh 3\pi=1.00000$$

Thus the coefficients become independent of the side ratios α and it follows naturally that for side ratios $\alpha>2+\beta$ the stresses reach constant values only dependent upon β.

As is shown later in Chapter 3, about the effect of yielding of the side supports, for long rectangular slabs, the rigidity of the shorter side supports have little to do with the stresses near the center, so that the stresses given by the eq. (11), (12), (13) may be applicable to the calculation of two side support slabs with side ratios larger than $2+\beta$ or in case $(\alpha-\beta>2)$.

In this connection a full discussion will be found in the articles about "effective width".

In case $\alpha-\beta>2$,

$$\left.\begin{aligned}
A_n&=-\frac{\frac{n\pi\beta}{2}+2}{2\cosh\frac{n\pi\beta}{2}\left(1+\tanh\frac{n\pi\beta}{2}\right)}a_n\\
B_n&=+\frac{1}{2\cosh\frac{n\pi\beta}{2}\left(1+\tanh\frac{n\pi\beta}{2}\right)}a_n\\
D_n&=-\frac{\tanh\frac{n\pi\beta}{2}}{2\left\{1+\tanh\frac{n\pi\beta}{2}\right\}}a_n\\
C_n&=\frac{\tanh\frac{n\pi\beta}{2}\left\{\frac{n\pi(\alpha-\beta)}{2}+2\right\}a_n}{2\left\{1+\tanh\frac{n\pi\beta}{2}\right\}}-\frac{\frac{n\pi\beta}{2}a_n}{2\left\{1+\tanh\frac{n\pi\beta}{2}\right\}^2\cosh^2\frac{n\pi\beta}{2}}
\end{aligned}\right\}\quad\cdots\cdots\cdots(20)$$

C_n contains the term α, but by substituting these values in eq. (13) we can eliminate $\frac{n\pi(\alpha-\beta)}{2}$ and we get by rearranging eq. (13) remembering that $\tanh\frac{n\pi(\alpha-\beta)}{2}=1$.

$$w_{II}=\frac{1}{N}\sum\left\{C_{n_0}\left(\cosh\frac{n\pi x}{2b}-\sinh\frac{n\pi x}{2b}\right)-D_n\frac{n\pi x}{2b}\left(\cosh\frac{n\pi x}{2b}-\sinh\frac{n\pi x}{2b}\right)\right\}\cos\frac{n\pi y}{2b}\quad\ldots\ldots(21)$$

where

$$\left.\begin{aligned}C_{n_0}&=\frac{a_n\tanh\frac{n\pi\beta}{2}}{1+\tanh\frac{n\pi\beta}{2}}-\frac{a_n\frac{n\pi\beta}{2}}{2\left(1+\tanh\frac{n\pi\beta}{2}\right)^2\cos^2\mathrm{h}\frac{n\pi\beta}{2}}\\D_n&=-\frac{a_n\tanh\frac{n\pi\beta}{2}}{2\left(1+\tanh\frac{n\pi\beta}{2}\right)}\end{aligned}\right\}\quad\ldots\ldots(22)$$

also independent of α. You will see that w_{II} (eq. 21.) practically vanishes at $\frac{x}{b}>2$ as well as the stresses due to deflection, for

$$\cosh\frac{n\pi x}{2b}-\sinh\frac{n\pi x}{2b}\doteqdot 0.$$

6. Solution for linear loading. (in four side support slabs.)

Paying attention to eq. (11), (12) and values $(\alpha-\beta<2)$ of coefficients A_n & B_n shown by eq. (18);

In eq. (18) if β be very small,

$$\frac{n\pi\alpha}{2}\tanh\frac{n\pi\alpha}{2}-\frac{n\pi(\alpha-\beta)}{2}\tanh\frac{n\pi(\alpha-\beta)}{2}$$

$$=\frac{n\pi\beta}{2}d\left(\frac{n\pi\alpha}{2}\tanh\frac{n\pi\alpha}{2}\right)$$

$$=\frac{n\pi\beta}{2}\left(\tanh\frac{n\pi\alpha}{2}+\frac{n\pi\alpha}{2}\sec^2\mathrm{h}\frac{n\pi\alpha}{2}\right)$$

and $\cosh\frac{n\pi\beta}{2}=1,\quad \tanh\frac{n\pi\beta}{2}=\frac{n\pi\beta}{2},$

$$p_0=\frac{P}{4cd}\ \&\ \frac{c}{b}=\beta,\quad \frac{d}{b}=\gamma.$$

$$a_n=\frac{64p_0b^4}{n^5\pi^5}\sin\frac{n\pi d}{2b}=\frac{16b^2}{n^5\pi^5}\cdot\frac{P}{\beta\gamma}\sin\frac{n\pi\gamma}{2}$$

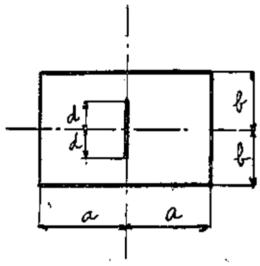


Fig. 12.

Hence the deflection at $x=0$, $y=0$ will be

$$(w)_{\substack{x=0\\y=0}}=\frac{1}{N}\sum_n a_n(1+A_n')\quad\text{and if}\quad\beta=0$$

$$=\frac{4Pb^2}{N\pi^4\gamma}\sum_n\frac{1}{n^4}\sin\frac{n\pi\gamma}{2}\left\{\tanh\frac{n\pi\alpha}{2}-\frac{n\pi\alpha}{2}\sec^2\mathrm{h}\frac{n\pi\alpha}{2}\right\}\quad\ldots\ldots(23)$$

$$(M_y)_{\substack{x=0\\y=0}}=(M_y)_{\substack{x=0\\y=0}}=\frac{(1+\nu)}{\gamma\pi^3}\sum_n\frac{1}{n^2}\sin\frac{n\pi\gamma}{2}\left\{\tanh\frac{n\pi\alpha}{2}-\frac{n\pi\alpha}{2}\sec^2\mathrm{h}\frac{n\pi\alpha}{2}\right\}\quad\ldots\ldots(24)$$

$n=1, 3, 5\ldots.$

We have obtained the same results as shown by Mr. Iguchi in our society's journal Vol. 17. No. 5.

7. Numerical example.

Problem: Calculate the bending moment at the center of a square slab simply supported on the four sides with a partial square uniform loading as shown by **Fig. 13,** where

$$a=b,\quad c=d=\frac{a}{2}$$

In the range $c\geqq x\geqq -c$, eq. (11) and eq. (12) are applicable, and we get for M_x & M_y at $x=0$, $y=0$,

$$M_x=\frac{16p_0b^2}{\pi^3}\sum_n\frac{1}{n^3}\sin\frac{n\pi d}{2b}\times\{(1-\nu)A_n'+2B_n'-\nu\}\quad\ldots(25)$$

$$n=1,\ 3,\ 5,\ 7\ldots.$$

$$M_y=\frac{16p_0b^2}{\pi^3}\sum_n\frac{1}{n^3}\sin\frac{n\pi d}{2b}\times\{1+(1-\nu)A_n'-2\nu B_n'\}\quad\ldots(26)$$

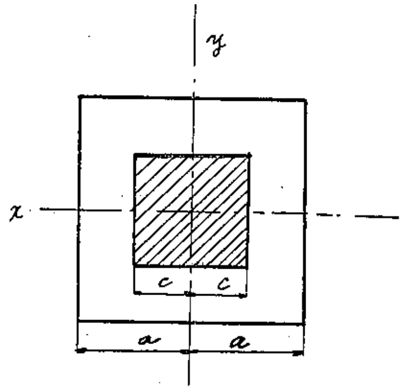


Fig. 13.

A_n' & B_n' are the constants in eq. (18) without the term a_n, namely

$$\begin{cases} A_n'=-\dfrac{\left\{\dfrac{n\pi\alpha}{2}\tanh\dfrac{n\pi\alpha}{2}-\dfrac{n\pi(\alpha-\beta)}{2}\tanh\dfrac{n\pi(\alpha-\beta)}{2}+2\right\}}{2\cosh\dfrac{n\pi\beta}{2}\left\{1+\tanh\dfrac{n\pi\beta}{2}\tanh\dfrac{n\pi(\alpha-\beta)}{2}\right\}} \\ B_n'=+\dfrac{1}{2\cosh\dfrac{n\pi\beta}{2}\left\{1+\tanh\dfrac{n\pi\beta}{2}\tanh\dfrac{n\pi(\alpha-\beta)}{2}\right\}} \end{cases}$$

here $\alpha=\dfrac{a}{b}=1,\quad \beta=\dfrac{c}{b}=0.5,\quad \gamma=\dfrac{d}{b}=0.5$

take $\nu=\dfrac{1}{10}$ for concrete.

For the values of $\cosh\dfrac{n\pi x}{2b}$, $\sinh\dfrac{n\pi x}{2b}$, $\tanh\dfrac{n\pi x}{2b}$ etc., see the table in the appendix, which was taken from Prof. Hayashi's "Fünfstellige Tafeln der Kreis- und Hyperbelfunktionen".

In this case,

n	$\frac{n\pi\alpha}{2}$	$\tanh\frac{n\pi\alpha}{2}$	$\frac{n\pi\beta}{2}$	$\tanh\frac{n\pi\beta}{2}$	$\frac{n\pi(\alpha-\beta)}{2}$	$\tanh\frac{n\pi(\alpha-\beta)}{2}$	$\cosh\frac{n\pi\beta}{2}$	$\sin\frac{n\pi\gamma}{2}$
1	1.5708	0.91717	0.78540	0.65579	0.78540	0.65579	1.32461	0.70711
3	4.71240	0.99984	2.35620	0.98219	2.35620	0.98219	5.32275	0.70711
5	7.8540	1.00000	3.92700	0.99922	3.92700	0.99922	25.38686	−0.70711
7	10.9560	1.00000	5.49780	0.99997	5.49780	0.99997	122.07758	−0.70711

$$\begin{cases} A_1' = -\dfrac{(1.5708 \times 0.91717 - 0.7854 \times 0.65579 + 2)}{2 \times 1.32461(1+0.65579^2)} = -0.77223 \\ B_1' = +\dfrac{1}{2 \times 1.32461(1+0.65579^2)} = +0.26395 \end{cases}$$

similarly

$$\begin{cases} A_3' = -0.21024 \\ B_3' = +0.047812 \end{cases} \quad \begin{cases} A_5' = -0.05844 \\ B_5' = +0.00986 \end{cases} \quad \begin{cases} A_7' = -0.01535 \\ B_7' = +0.00205 \end{cases}$$

By eq. (26) $(M_y)_{\substack{x=0\\y=0}}$ will be,

$$\frac{16}{3.1416^3} \times \frac{0.70711}{1} \times (1-0.9 \times 0.77223 - 0.2 \times 0.26395) p_0 b^2 = +0.09203 p_0 b^2$$

$$\frac{16}{3.1416^3} \times \frac{0.70711}{3^3} \times (1-0.9 \times 0.21024 - 0.2 \times 0.04781) p_0 b^2 = +0.01083 p_0 b^2$$

$$\frac{16}{3.1416^3} \times \frac{-0.70711}{5^3} \times (1-0.9 \times 0.05844 - 0.2 \times 0.00986) p_0 b^2 = -0.00276 p_0 b^2$$

$$\frac{16}{3.1416^3} \times \frac{-0.70711}{7^3} \times (1-0.9 \times 0.01535 - 0.2 \times 0.00205) p_0 b^2 = -0.00106 p_0 b^2$$

$$M_y = +0.09903 p_0 b^2$$

$$M_x = \frac{16}{3.1416^3} \times 0.70711(-0.9 \times 0.77223 + 2 \times 0.26395 - 0.1) p_0 b^2 = +0.09746 p_0 b^2$$

$$\frac{16}{3.1416^3} \times \frac{0.70711}{3^3} (-0.9 \times 0.21024 + 2 \times 0.04781 - 0.1) p_0 b^2 = +0.00262 p_0 b^2$$

$$\frac{16}{3.1416^3} \times \frac{-0.70711}{5^3} (-0.9 \times 0.05844 + 2 \times 0.00986 - 0.1) p_0 b^2 = -0.00039 p_0 b^2$$

$$\frac{16}{3.1416^3} \times \frac{-0.70711}{7^3} (-0.9 \times 0.01535 + 2 \times 0.00205 - 0.1) p_0 b^2 = -0.00011 p_0 b^2$$

$$M_x = +0.09958 p_0 b^2$$

$$\therefore \quad M_x = M_y = 0.099 p_0 b^2.$$

8. Note on Art. 3 in this chapter.

A little more general solution will be obtained with a modification of the solution in Art. 3. But the cases of symmetrical loading are principally dealt with in this paper to keep connection with the articles later on.

By taking the origin as shown by **Fig. 14** and replacing the cosines in eq. (11), (12), (13) with sines, we get

$$w_0 = \frac{1}{N} \sum_n \frac{64 b^4 p_0}{n^5 \pi^5} \sin \frac{n\pi\xi}{2b} \sin \frac{n\pi d}{2b} \sin \frac{n\pi y}{2b} \quad \ldots (27)$$

$n = 1, 2, 3, 4 \ldots$

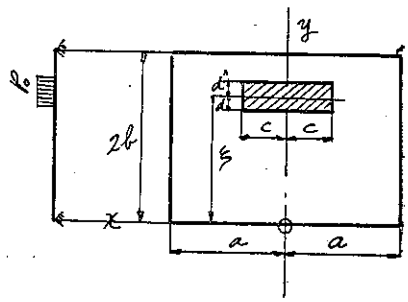


Fig. 14.

$$w_{\text{I}}=\frac{1}{N}\sum_n\left(A_n\cosh\frac{n\pi x}{2b}+B_n\frac{n\pi x}{2b}\sinh\frac{n\pi x}{2b}\right)\sin\frac{n\pi y}{2b} \quad\ldots\ldots(28)$$

$n=1,\ 2,\ 3,\ 4\ldots.$

$$w_{\text{II}}=\frac{1}{N}\sum_n\left\{C_n\left[\tanh\frac{n\pi(a-c)}{2b}\cosh\frac{n\pi x}{2b}-\sinh\frac{n\pi x}{2b}\right]\right.$$

$$\left.+D_n\frac{n\pi(x-a+c)}{2b}\left[\tanh\frac{n\pi(a-c)}{2b}\sinh\frac{n\pi x}{2b}-\cosh\frac{n\pi x}{2b}\right]\right\}\sin\frac{n\pi y}{2b} \quad\ldots\ldots(29)$$

With the same values of constants given by eq. (18)

N. B. The loading p will be expressed by Fourier's series,

$$p=\frac{4p_0}{\pi}\sum_n\frac{1}{n}\sin\frac{n\pi\xi}{2b}\sin\frac{n\pi d}{2b}\sin\frac{n\pi y}{2b}$$

and remembering that $N\dfrac{d^4w}{dy^4}$ gives p, we get the above equations.

Chapter 3. The effect of yielding of the two opposite sides, the other two being rigid.

1. Fundamental equations. (Symmetrical loading.)

(Case 1.) The side supports at $y=\pm b$ are rigid and the deflection of the sides at $x=\pm a$ is a known function of y.

(Case 2.) The side supports at $y=\pm b$ are rigid and the rigidity of the side supports at $x=\pm a$ is known. The loading is a function of y only.

(Case 3.) The side support conditions same as in Case 2., and the loading is a partial rectangular uniform loading.

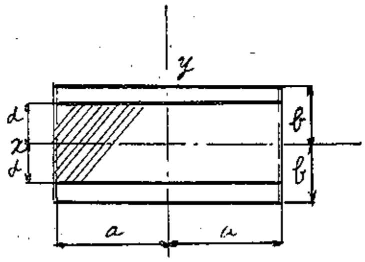


Fig. 15.

N. B. (Case 3.) is dealt with in Art. 4.

Case 1. is simpler than Case 2., but it will find little use in the calculation of bridge floors except when some allowance to the strength of the slabs is necessary in more complicated cases for the yielding of the side supports.

As particular integral we have as before,

$$w_0=\frac{1}{N}\sum_n a_n\cos\frac{n\pi y}{2b} \quad\ldots\ldots(30)$$

$n=1,\ 3,\ 5\ldots.$

$$a_n=\frac{64b^4p_0}{n^5\pi^5}(-1)^{\frac{n-1}{2}} \qquad \text{for full uniform load.}$$

$$a_n=\frac{64b^4p_0}{n^5\pi^5}\sin\frac{n\pi d}{2b} \qquad \text{for partial uniform load.}$$

and for complimentary functions,

$$w_{\text{I}}=\frac{1}{N}\sum_n\left\{A_n\cosh\frac{n\pi x}{2b}+B_n\frac{n\pi x}{2b}\sinh\frac{n\pi x}{2b}\right\}\cos\frac{n\pi y}{2b} \qquad \text{.......................(31)}$$

A_n & B_n are determined by the boundary conditions;

a) $(M_x)_{x=\pm a}=0$

b) The yielding of the sides at $x=\pm a$ is a known function of y or $\sum_n \mathfrak{A}_n\cos\frac{n\pi y}{2b}$ if expressed with cosine series.

As the condition (a), (b) we get by putting $\frac{a}{b}=\alpha$,

$$(1-\nu)\cosh\frac{n\pi\alpha}{2}A_n+\left\{2\cosh\frac{n\pi\alpha}{2}+(1-\nu)\frac{n\pi\alpha}{2}\sinh\frac{n\pi\alpha}{2}\right\}B_n-\nu a_n=0 \qquad \text{.......(32)}$$

$$a_n+\cosh\frac{n\pi\alpha}{2}A_n+B_n\frac{n\pi\alpha}{2}\sinh\frac{n\pi\alpha}{2}=\mathfrak{A}_n \qquad \text{..............................(33)}$$

By solving the simultaneous equations (32), (33) with regard to A_n & B_n

$$\left.\begin{aligned} A_n&=\frac{\mathfrak{A}_n\left\{2-(1-\nu)\frac{n\pi\alpha}{2}\tanh\frac{n\pi\alpha}{2}\right\}-a_n\left\{2+\frac{n\pi\alpha}{2}\tanh\frac{n\pi\alpha}{2}\right\}}{2\cosh\frac{n\pi\alpha}{2}} \\ B_n&=\frac{a_n-(1-\nu)\mathfrak{A}_n}{2\cosh\frac{n\pi\alpha}{2}} \end{aligned}\right\} \qquad \text{.................(34)}$$

Case 2. The rigidity of the side supports is known. Let us call the term $K=\frac{E_sI_s}{2bN}$ "side support rigidity".

Here,

E_s: modulous of elasticity of the supporting beam.

I_s: moment of inertia of the supporting beam.

E_b: modulous of elasticity of concrete.

h: depth of the slab.

ν: Poisson's ratio.

This value K is independent of units, and the effect of yielding is expressible in terms of K.

Using the equations (30), (31), as in case 1, and for the first boundary conditions we have, $(M_x)_{x=\pm a}=0$,

$$(1-\nu)\cosh\frac{n\pi\alpha}{2}A_n+\left\{2\cosh\frac{n\pi\alpha}{2}+(1-\nu)\frac{n\pi\alpha}{2}\sinh\frac{n\pi\alpha}{2}\right\}B_n-\nu a_n=0 \qquad \text{.......(35)}$$

For the second boundary conditions we have;

The reaction of the slab at $x = \pm a$ is equal to the external load to the supporting beam.

Let $f(y)$ be the yielding of the sides at $x = \pm a$, then as the external load to the supporting beam we have, $-E_s I_s \dfrac{d^4 f(y)}{dy^4}$ on one hand and on the other, the reaction for the slab

$$-N\left\{\frac{\partial^3(w_0+w_I)}{\partial x^3}+(2-\nu)\frac{\partial^3(w_0+w_I)}{\partial x\partial y^2}\right\}$$

These two values must be equal.

Here $$f(y)=\sum_n\left\{A_n\cosh\frac{n\pi\alpha}{2}+B_n\frac{n\pi\alpha}{2}\sinh\frac{n\pi\alpha}{2}+a_n\right\}\cos\frac{n\pi y}{2b}$$

We get the following relation by equating the n-th term,

$$\left\{(1-\nu)\sinh\frac{n\pi\alpha}{2}+\frac{n\pi E_s I_s}{2bN}\cosh\frac{n\pi\alpha}{2}\right\}A_n$$
$$+B_n\left\{-(1+\nu)\sinh\frac{n\pi\alpha}{2}+(1-\nu)\frac{n\pi\alpha}{2}\cosh\frac{n\pi\alpha}{2}+\frac{n^2\pi^2}{4}\alpha\frac{E_s I_s}{bN}\sinh\frac{n\pi\alpha}{2}\right\}$$
$$+\frac{n\pi}{2}\cdot\frac{E_s I_s}{bN}a_n=0 \quad\text{......(36)}$$

By solving (35), (36) with regard to A_n, B_n, and putting $K=\dfrac{E_s I_s}{2bN}$ we get,

$$\left.\begin{aligned}A_n&=-\frac{a_n\left\{\frac{n^2\pi^2\alpha}{2}K-\nu(1+\nu)\right\}\sinh\frac{n\pi\alpha}{2}+a_n\left\{2n\pi K+\nu(1-\nu)\frac{n\pi\alpha}{2}\right\}\cosh\frac{n\pi\alpha}{2}}{2n\pi K\cos^2\text{h}\frac{n\pi\alpha}{2}+(1-\nu)(3+\nu)\sinh\frac{n\pi\alpha}{2}\cosh\frac{n\pi\alpha}{2}-(1-\nu)^2\frac{n\pi\alpha}{2}}\\ B_n&=+\frac{a_n\left\{\nu(1-\nu)\sinh\frac{n\pi\alpha}{2}+n\pi K\cosh\frac{n\pi\alpha}{2}\right\}}{2n\pi K\cos^2\text{h}\frac{n\pi\alpha}{2}+(1-\nu)(3+\nu)\sinh\frac{n\pi\alpha}{2}\cosh\frac{n\pi\alpha}{2}-(1-\nu)^2\frac{n\pi\alpha}{2}}\end{aligned}\right\}\quad\text{.... (37)}$$

2. Solution for slabs with four rigid sides or with two rigid and two unsupported sides as the extreme cases for the solutions given in Art. 1, in this chapter.

Look at (37)

$$\left.\begin{aligned}\lim_{K=\infty}A_n&=-\frac{2+\frac{n\pi\alpha}{2}\tanh\frac{n\pi\alpha}{2}}{2\cosh\frac{n\pi\alpha}{2}}a_n\\ \lim_{K=\infty}B_n&=+\frac{1}{2\cosh\frac{n\pi\alpha}{2}}a_n\end{aligned}\right\}$$

If the rigidity K tends to infinity, the constants A_n and B_n will take the same values as are shown in the solutions for rigid four side supports.

See eq. (19) or "Elastische Platten".

Also for $K=0$,

$$\left.\begin{aligned} A_n&=-\frac{\nu}{1-\nu}\cdot\frac{(1-\nu)\frac{n\pi\alpha}{2}\cosh\frac{n\pi\alpha}{2}-(1+\nu)\sinh\frac{n\pi\alpha}{2}}{(3+\nu)\sinh\frac{n\pi\alpha}{2}\cosh\frac{n\pi\alpha}{2}-(1-\nu)\frac{n\pi\alpha}{2}} \\ B_n&=+\frac{\nu\sinh\frac{n\pi\alpha}{2}a_n}{(3+\nu)\sinh\frac{n\pi\alpha}{2}\cosh\frac{n\pi\alpha}{2}-(1-\nu)\frac{n\pi\alpha}{2}} \end{aligned}\right\} \quad\ldots\ldots(38)$$

Constants obtained for the rigidity $K=0$ are for slabs with two unsupported sides. This will be checked by independent calculation as follows. **(Check for the values given by eq. (38)).**

From the two equations (30) and (31) we get for A_n & B_n the following two relations with the boundary conditions;

a) $M_x=0$ at $x=\pm a$,

b) $R_x=0$ at $x=\pm a$.

Thus,

$$(1-\nu)\cosh\frac{n\pi\alpha}{2}A_n+\left\{2\cosh\frac{n\pi\alpha}{2}+(1-\nu)\frac{n\pi\alpha}{2}\sinh\frac{n\pi\alpha}{2}\right\}B_n-\nu a_n=0 \quad\ldots\ldots(39)$$

$$(1-\nu)\sinh\frac{n\pi\alpha}{2}A_n+\left\{-(1+\nu)\sinh\frac{n\pi\alpha}{2}+(1-\nu)\frac{n\pi\alpha}{2}\cosh\frac{n\pi\alpha}{2}\right\}B_n=0 \quad\ldots\ldots(40)$$

and we get

$$\left.\begin{aligned} A_n&=-\frac{\nu}{1-\nu}\cdot\frac{(1-\nu)\frac{n\pi\alpha}{2}\cosh\frac{n\pi\alpha}{2}-(1+\nu)\sinh\frac{n\pi\alpha}{2}}{(3+\nu)\sinh\frac{n\pi\alpha}{2}\cosh\frac{n\pi\alpha}{2}-(1-\nu)\frac{n\pi\alpha}{2}}a_n \\ B_n&=+\frac{\nu\sinh\frac{n\pi\alpha}{2}a_n}{(3+\nu)\sinh\frac{n\pi\alpha}{2}\cosh\frac{n\pi\alpha}{2}-(1-\nu)\frac{n\pi\alpha}{2}} \end{aligned}\right\}$$

This is exactly the same as eq. (38).

3. The extent of the effect of the side support yielding for full uniform load in bridge floors.

As is shown in Art. 2. in this chapter, the effect of yielding sides is dependent on the side support rigidity K.

Some examples of executed bridge floors are shown by the following examples with the corresponding values of K.

Ex. 1. (A cross beam of a deck plate girder.)

$2b=190\,\text{cm}$, $h=18\,\text{cm}$, $I_s=6\,060\,\text{cm}^4$(I-Beam 10″×5″ @ 29.#99)

$$\frac{E_s}{E_b}=15,\quad \nu=0.1,\quad N=\frac{h^3}{12}(1-\nu^2).$$

$$K=\frac{15\times 6\,060}{190\times\frac{18^3}{12}(1-0.1^2)}\doteqdot 1.0$$

Ex. 2. (A stringer of a truss bridge.)

$2b=415\,\text{cm}$, $h=15\,\text{cm}$, $I_s=13\,100\,\text{cm}^4$(I-Beam 12″ @ 44.#02)

$$K=\frac{15\times 13\,100}{415\times\frac{15^3}{12}(1-0.1^2)}\doteqdot 1.68$$

Ex. 3. (Same as above.)

$2b=570\,\text{cm}$, $h=15\,\text{cm}$, $I_s=30\,000\,\text{cm}^4$, and $K=2.7$

Bending moment at $x=0$, $y=0$ will be as before,

$$\left.\begin{aligned} M_b&=\frac{16p_0b^2}{\pi^3}\sum_n\frac{1}{n^3}\{1+(1-\nu)A_n'-2\nu B_n'\}(-1)^{\frac{n-1}{2}}\\ M_x&=\frac{16p_0b^2}{\pi^3}\sum_n\frac{1}{n^3}\{(1-\nu)A_n'+2B_n'-\nu\}(-1)^{\frac{n-1}{2}}\end{aligned}\right\}$$

Here A_n' & B_n' are the constants given by eq. (37) without the term a_n.

Max. moment in the supporting beam is

$$E_sI_s\left(\frac{d^2f(y)}{dy^2}\right)_{\substack{x=\pm a\\ y=0}}=\frac{E_sI_s}{N}\cdot\frac{16}{\pi^3}p_0b^2\sum_n\frac{1}{n^3}(-1)^{\frac{n-1}{2}}\left(1+A_n'\cosh\frac{n\pi\alpha}{2}+B_n'\frac{n\pi\alpha}{2}\sinh\frac{n\pi\alpha}{2}\right)$$

$$=\frac{E_sI_s}{2bN}\cdot\frac{32}{\pi^3}p_0b^3\sum_n\frac{1}{n^3}(-1)^{\frac{n-1}{2}}\left(1+A_n'\cosh\frac{n\pi\alpha}{2}+B_n'\frac{n\pi\alpha}{2}\sinh\frac{n\pi\alpha}{2}\right)\quad\ldots\ldots\ldots\ldots(41)$$

$\frac{E_sI_s}{2bN}=K=$side support rigidity.

The change of stresses due to support yielding for a square slab under full uniform load p_0 is tabulated below. You will find considerable change in stresses as well as in deflections. (The supports at $x=\pm a$ are yielding.)

See also **Pl. 8** for change of bending moment due to various values of K.

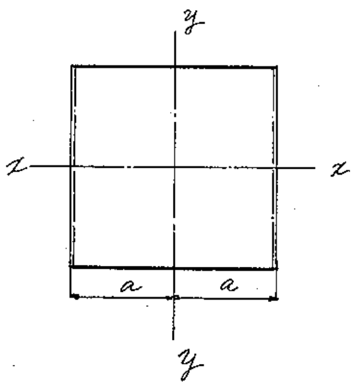


Fig. 16.

Side supt. rigidity	$K=\infty$ (rigid)	$K=1$ (yielding)	
B.M. at center of slab $(M_x)_{x=0,\,y=0}$	$0.159\ p_0b^2$	$0.129\ p_0b^2$	Note.
$(M_y)_{x=0,\,y=0}$	$0.159\ p_0b^2$	$0.253\ p_0b^2$	Side length $=2b$.
Deflection at center of slab	$0.0657\ \frac{p_0b^4}{N}$	$0.1044\ \frac{p_0l^4}{N}$	
Deflection of support	0	$0.0589\frac{p_0b^4}{N}$	
Reaction of slab	$0.82\ p_0b$	$0.59\ p_0b$	
Max. B. M. in supporting beam	$0.40\ p_0b^3$	$0.29\ p_0b^3$	

Note. In ordinary four side support slabs in bridge floors the side support rigidity $K=1$.

4. The solution for partial rectangular uniform load.

(Case 3 in Art. 1.)

In this case the rigidity of the side supports are known.

The effect of yielding at $x=\pm a$ will be added to the solution for rigid side supports shown in Art. 3, Chapt. 2.

By eq. (11), (12), (13), in the range $c\geqq x\geqq -c$, origin at A.

$$w_0=\frac{1}{N}\sum_n\frac{64b^4p_0}{n^5\pi^5}\sin\frac{n\pi d}{2b}\cos\frac{n\pi y}{2b}$$

$$w_{\rm I}=\frac{1}{N}\sum_n\left(A_n'\cosh\frac{n\pi x}{2b}+B_n\frac{n\pi x}{2b}\sinh\frac{n\pi x}{2b}\right)\cos\frac{n\pi y}{2b}$$

and in the range $a-c\geqq x\geqq 0$ with origin at B.

$$w_{\rm II}=\frac{1}{N}\sum_n\Big[C_n\left\{\tanh\frac{n\pi(a-c)}{2b}\cosh\frac{n\pi x}{2b}-\sinh\frac{n\pi x}{2b}\right\}$$
$$+D_n\frac{n\pi(x-a+c)}{2b}\left\{\tanh\frac{n\pi(a-c)}{2b}\sinh\frac{n\pi x}{2b}-\cosh\frac{n\pi x}{2b}\right\}\Big]\cos\frac{n\pi y}{2b}$$

The above three equations are the solutions for rigid side support slabs, for which the constants A_n, B_n, C_n, D_n are given by eq. (18).

Now let us add one more equation namely.

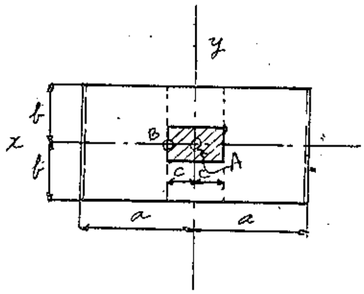


Fig. 17.

With origin at A in the range $a\geqq x\geqq -a$

$$w_{\rm III}=\frac{1}{N}\sum_n\left(E_n\cosh\frac{n\pi x}{2b}+F_n\frac{n\pi x}{2b}\sinh\frac{n\pi x}{2b}\right)\cos\frac{n\pi y}{2b}\quad\text{.........(42)}$$

By adding the both solutions we get the solution for yielding supports. In the range $c \geqq x \geqq -c$, with origin at A

$$w_0 = \frac{1}{N}\sum_n \frac{64b^4 p_0}{n^5\pi^5}\sin\frac{n\pi d}{2b}\cos\frac{n\pi y}{2b} \quad \text{.......(11)}$$

$$w_{\text{I}} + w_{\text{III}} = \frac{1}{N}\sum_n \left\{(A_n + E_n)\cosh\frac{n\pi x}{2b} + (B_n + F_n)\frac{n\pi x}{2b}\sinh\frac{n\pi x}{2b}\right\}\cos\frac{n\pi y}{2b} \quad \text{......(43)}$$

In the range $a - c \geqq x \geqq 0$, eq. (42) will be, by translating the axis of co-ordinates from A to B.

$$w_{\text{III}}' = \frac{1}{N}\sum_n \left\{E_n\cosh\frac{n\pi(x+c)}{2b} + F_n\frac{n\pi(x+c)}{2b}\sinh\frac{n\pi(x+c)}{2b}\right\}\cos\frac{n\pi y}{2b} \quad \text{.......(44)}$$

The resultant solution in the range $a - c \geqq x \geqq 0$ will be, $w_{\text{II}} + w_{\text{III}}'$ with origin at B.

E_n & F_n are determined by the two boundary conditions;

(a) $M_x = 0$ at the support.

(b) Reaction at support is equal to the external load to the supporting beam.

Condition (a) will be rendered by

$$E_n(1-\nu)\cosh\frac{n\pi\alpha}{2} + F_n\left\{2\cosh\frac{n\pi\alpha}{2} + (1-\nu)\frac{n\pi\alpha}{2}\sinh\frac{n\pi\alpha}{2}\right\} = 0 \quad \text{.............(45)}$$

Other terms vanish as they are fixed by the same conditions. Condition (b) will be,

$$N\left[\frac{\partial^3}{\partial x^3} + (2-\nu)\frac{\partial^3}{\partial x\partial y^2}\right]\{(w_{\text{II}})_{x=a-c} + (w_{\text{III}})_{x=a}\} = E_s I_s \frac{d^4}{dy^4}(w_{\text{III}})_{x=a}$$

We get from (45) and from condition (b) the following two relations.

$$E_n = -\frac{F_n}{1-\nu}\left\{2 + (1-\nu)\frac{n\pi\alpha}{2}\tanh\frac{n\pi\alpha}{2}\right\} \quad \text{.................................(46)}$$

$$\frac{\{(1-\nu)C_n - (1+\nu)D_n\}}{\cosh\frac{n\pi(\alpha-\beta)}{2}} + \left[-\frac{n\pi\alpha}{2}\cdot\frac{(1-\nu)}{\cosh\frac{n\pi\alpha}{2}} + (3+\nu)\sinh\frac{n\pi\alpha}{2}\right]F_n$$

$$= -\frac{2n\pi K}{1-\nu}\cosh\frac{n\pi\alpha}{2} \quad n \quad \text{...(47)}$$

here $K = \frac{E_s I_s}{2bN}$ side support rigidity.

Thus the constants are,

$$F_n = -\frac{\dfrac{\{(1-\nu)C_n - (1+\nu)D_n\}}{\cosh\frac{n\pi(\alpha-\beta)}{2}}}{\left[-\frac{n\pi\alpha}{2}\cdot\frac{(1-\nu)}{\cosh\frac{n\pi\alpha}{2}} + (3+\nu)\sinh\frac{n\pi\alpha}{2} + \frac{2n\pi K}{1-\nu}\cosh\frac{n\pi\alpha}{2}\right]} \quad \text{.........(48)}$$

$$E_n = -\frac{F_n}{1-\nu}\left\{2+(1-\nu)\frac{n\pi\alpha}{2}\tanh\frac{n\pi\alpha}{2}\right\}$$

C_n, D_n are given by eq. (18).

5. Numerical example.

Ex. Calculate the bending moment at the center of a square slab 140 cm × 140 cm × 15 cm of which the two opposite sides, are supported by the

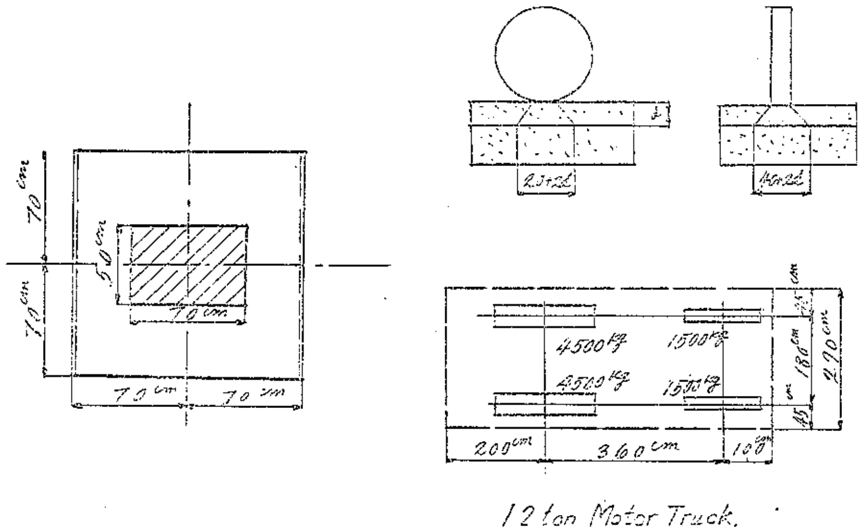


Fig. 13.

main girders and the other two by the I-Beams 200×100 @ 25.95 kg. Take as live load one wheel of a 12 ton motor truck as specified by the Department of Home Affairs. The depth of surface filling of the slab is 15 cm.

As is stated in previous chapters, the sides supported by the main girders are considered rigid. The distributed area of one wheel will be 70 cm × 50 cm.

a) Side support rigidity K (for calculation of dead load stresses take $\frac{K}{2}$ as side support rigidity, if we take into account the effect of the adjoining span.

Side support rigidity $K = \dfrac{15\times 2\,175}{\frac{1}{12}\times 140\times 15^3(1-0.1^2)} = 0.84$

$I_s = 2\,175\,\text{cm}^4$, $\nu = 0.1$, $2b = 140\,\text{cm}$, $h = 15\,\text{cm}$

b) Bending mt. at $x=0$, $y=0$.

By eq. (11), (43), (44).

$$
\left.
\begin{aligned}
(M_x)_{x=0,\ y=0}&=\frac{16p_0b^2}{\pi^3}\sum_n\frac{\sin\frac{n\pi d}{2b}}{n^3}\{(1-\nu)(A_n'+E_n')+2(B_n'+F_n')-\nu\}\\
(M_y)_{x=0,\ y=0}&=\frac{16p_0b^2}{\pi^3}\sum_n\frac{\sin\frac{n\pi d}{2b}}{n^3}\{1+(1-\nu)(A_n'+E_n')-2\nu(B_n'+F_n')\}
\end{aligned}
\right\}\quad\ldots(49)
$$

A_n', B_n', E_n', F_n' are the constants in eq. (18), (48), without the term a_n.

For practical purpose only the first terms will give sufficiently accurate values. In this case three terms were taken.

By eq. (18), (48), we get,

(the values A_n' & B_n' are found in page 13)

$$
\begin{cases} A_1'=-0.77223\\ A_3'=-0.21024\\ A_5'=-0.05844\end{cases}
\qquad
\begin{cases} B_1'=0.26395\\ B_3'=0.04781\\ B_5'=0.00986\end{cases}
$$

By eq. (18),

$$
\beta=0.5\quad
\begin{cases}
D_1'=-\dfrac{0.65579}{2(1+0.65579^2)}=-0.22929\\[2ex]
D_3'=-\dfrac{0.98219}{2(1+0.98219^2)}=-0.24995\\[2ex]
D_5'=-\dfrac{0.99922}{2(1+0.99922^2)}=-0.2500
\end{cases}
$$

also by eq. (18)

$$
C_n'=-D_n'\left[\frac{n\pi(\alpha-\beta)}{2}\tanh\frac{n\pi\alpha}{2}+2\right]-B_n'^2n\pi\beta
$$

$$
\begin{cases} C_1'=0.40488\\ C_3'=1.06719\\ C_5'=1.48160\end{cases}
$$

By eq. (48) we get

$$
\text{for } K=0.84\quad
\begin{cases} E_1'=0.08022\\ E_3'=0.00139\\ E_5'=0.00002\end{cases}
\qquad
\begin{cases} F_1'=-0.02187\\ F_3'=-0.00020\\ F_5'=-0.00000\end{cases}
$$

If $K=\infty$, E_n' & F_n' are zero.

$$
\text{If } K=0\quad
\begin{cases} E_1'=0.25992\\ E_3'=0.09333\\ E_5'=0.00014\end{cases}
\qquad
\begin{cases} F_1'=-0.07086\\ F_3'=-0.00135\\ F_5'=-0.00002\end{cases}
$$

And we get $(M_x)_{x=0,\ y=0}$ & $(M_y)_{x=0,\ y=0}$ by eq. (49)

Let us compare the results obtained for $K=0$ (unsupported.) $K=0.84$ (yielding) and $K=\infty$ (rigid), respectively

$$
\text{For }\ K=0\quad
\begin{cases} M_x=0.05119\ p_0b^2\\ M_y=0.15334\ p_0b^2\end{cases}
$$

$$
\text{For }\ K=0.84\quad
\begin{cases} M_x=0.06835\ p_0b^2\\ M_y=0.10544\ p_0b^2\end{cases}
$$

For $K=\infty$ $\begin{cases} M_x=0.07595\ p_0b^2 \\ M_y=0.08477\ p_0b^2 \end{cases}$

Due to the yielding of the supports $(M_y)(K=\infty)$ is increased by 24%, while $(M_x)(K=\infty)$ is decreased by 10%.

Chapter 4. On the "effective width" and some important problems connected with it.

1. General nature of the problem.

In the preceding chapter I have shown that the two side support slabs can be treated as special case of slabs with yielding sides, the rigidity of the side support being zero. Also I have mentioned in Chapter 2, Art. 5 that the solution for four rigid side supports are applicable to two side support slabs in case $(\alpha-\beta\geqq2)$ or $a-c$ exceeds $2b$. Here $\alpha=\frac{a}{b}$, $\beta=\frac{c}{b}$ & $\gamma=\frac{d}{b}$.

The word "effective width" involves some ambiguity as to its definition, but let us consider it as a simple means of calculating stresses in two side support slabs. The formulas for effective widths specified by the Dept. of Home Affairs, (see Appendix) perhaps have the same origin as those found in p. 358 of Hool's "Reinforced concrete construction"., Vol. III.

In this book, according to Mr. Slater, as the limit for application of the formula given there, the total width of the slab must be greater than twice the span.

This formula agrees well with the theoretical results within certain limits. The limitation is, rigorously speaking, $(\alpha-\beta)\geqq2$ instead of $\alpha\geqq2$ as was shown by Mr. Slater. See **Pl. 6.**

As far as I know, nothing particular is specified in ordinary regulations as to "distributing bars". In **Pl. 2** the ratio $M_x : M_y$ at $x=0$, $y=0$ is shown, in case $(\alpha-\beta>2)$ obtained from the formula given in Art. 5., Chapter 2.

According to this diagram, if $\beta=0.2$ which is not at all seldom in bridge floors especially in slab bridges, the ratio reaches 80%!. Fortunately, in bridge floors the depth of slabs are governed by dead load for such concentrated loadings and much steel is not needed as distributing bars.

It should be born in mind that, if we have heavy live load as compared with the dead load, due attention should be paid to distributing bars.

Also a small amount of negative bending moment occurs in the direction of distributing bars, and sometimes a certain amount of reinforcement may be necessary in the upper side of the slab near the center of the span. See **Pl. 10.**

In designing slab bridges the effect of the adjoining two wheels which can not be considered as a single rectangular loading should be considered. The method of calculation will be shown in the following articles.

Some special calculation is necessary when there is possibility of the load coming near the unsupported sides. The limitations added to the formulas for effective width found in the specification of the Dept. of Home Affairs show fairly good results for slabs with $\alpha-\beta<2$ if compared with the theoretical results. See **Pl. 5.,** also Art. 9, Chapt. 2.

2. Calculation of the two side support slabs in case ($\alpha-\beta>2$).

The solution given in Art. 5, Chapter 2 applies to this case.

In the range $c\geqq x\geqq -c$,

origin at A.

$$\begin{cases} w_0=\dfrac{1}{N}\sum\limits_n \dfrac{64b^4p_0}{n^5\pi^5}\sin\dfrac{n\pi d}{2b}\cos\dfrac{n\pi y}{2b} \quad \ldots(11) \\ \qquad n=1,\,3,\,5\ldots\ldots \\ w_{\mathrm{I}}=\dfrac{1}{N}\sum\limits_n\Big(A_n\cosh\dfrac{n\pi x}{2b} \\ \qquad +B_n\dfrac{n\pi x}{2b}\sinh\dfrac{n\pi x}{2b}\Big)\cos\dfrac{n\pi y}{2b} \\ \qquad n=1,\,3,\,5\ldots\ldots \quad \ldots(12) \end{cases}$$

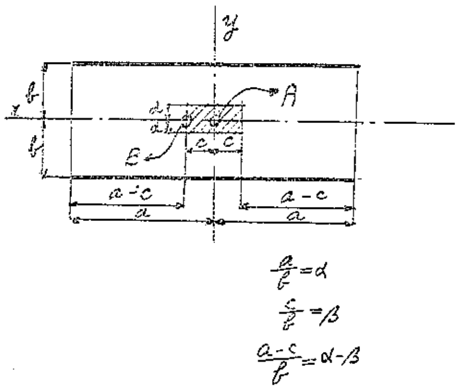


Fig. 19.

and in the range $a-c\geqq x\geqq 0$, with the origin at B.

$$w_{\mathrm{II}}=\frac{1}{N}\sum_n\left\{C_{n0}\left(\cosh\frac{n\pi x}{2b}-\sinh\frac{n\pi x}{2b}\right)-D_n\frac{n\pi x}{2b}\left(\cosh\frac{n\pi x}{2b}-\sinh\frac{n\pi x}{2b}\right)\cos\frac{n\pi y}{2b}\right. \quad \ldots(21)$$

$$\left.\begin{aligned} A_n&=-\frac{\dfrac{n\pi\beta}{2}+2}{2\cosh\dfrac{n\pi\beta}{2}\left(1+\tanh\dfrac{n\pi\beta}{2}\right)}a_n \\ B_n&=+\frac{1}{2\cosh\dfrac{n\pi\beta}{2}\left(1+\tanh\dfrac{n\pi\beta}{2}\right)}a_n \end{aligned}\right\} \quad \ldots\ldots(20)$$

$$\left.\begin{aligned} C_{n} &= \frac{a_n \tanh\frac{n\pi\beta}{2}}{1+\tanh\frac{n\pi\beta}{2}} - \frac{a_n\frac{n\pi\beta}{2}}{2\left(1+\tanh\frac{n\pi\beta}{2}\right)^2 \cosh^2\frac{n\pi\beta}{2}} \\ D_n &= -\frac{a_n \tanh\frac{n\pi\beta}{2}}{2\left(1+\tanh\frac{n\pi\beta}{2}\right)}, \quad a_n = \frac{64b^4 p_0}{n^5\pi^5}\sin\frac{n\pi d}{2b} \end{aligned}\right\} \qquad \text{(22)}$$

The above solution is for four rigid side supports, but from the reasons that the term

$$\left(\cosh\frac{n\pi x}{2b} - \sinh\frac{n\pi x}{2b}\right)$$

in w_{II} soon vanishes for the values $\frac{x}{b}>2$ and that the stresses near the center of the slab are little influenced by the side support rigidity, they are applicable to two side support slabs in case $\alpha-\beta>2$. Refer to the table of hyperbolic functions.

The bending moment at $x=0$, $y=0$ will be as before

$$\left.\begin{aligned} M_x &= \frac{16p_0b^2}{\pi^3}\sum_n \frac{\sin\frac{n\pi d}{2b}}{n^3}\{(1-\nu)A_n' + 2B_n' - \nu\} \\ M_y &= \frac{16p_0b^2}{\pi^3}\sum_n \frac{\sin\frac{n\pi d}{2b}}{n^3}\{1+(1-\nu)A_n' - 2\nu B_n'\} \end{aligned}\right\} \qquad \text{(50)}$$

3. Comparison of the prevalent formula for "effective width" with the theoretical results in case ($\alpha-\beta>2$).

In **Pl. 6.** the moment ratio, that is to say, the ratio of the theoretical bending moment to the simple beam moment

$$p_0b^2\left(\gamma-\frac{\gamma^2}{2}\right), \quad \left(\gamma=\frac{d}{b}\right)$$

is shown by the diagram. The moment ratio obtained from the prevalent formula will be,

$$\frac{c}{\frac{2}{3}b+c} = \frac{\beta}{\frac{2}{3}+\beta}, \quad \frac{c}{b}=\beta. \qquad \text{See **Fig. 19.**}$$

This formula without the limitations as specified by the Dept. of Home Affairs agrees very well with the theoretical values within ordinary limit of load concentration in bridge floors. See **Pl. 6.** The theo-

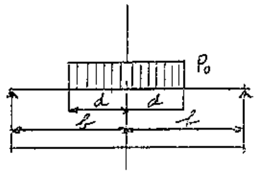


Fig. 20.

retical moment ratio is in this case within the range denoted by the two thick lines. The thin line denotes.

$$\frac{\beta}{\frac{2}{3}+\beta}$$

I wonder whether the limitation "$e \leqq 2\text{m}$" is necessary in this case in the common formula. This limitation is too much on the safe side. It may be that this limitation is provided for the possibility of the load coming near the unsupported sides (see **Fig. 4, 7.**), and still it seems to me unadvisable to construct the most part of the floor over strong.

It will be more economical and free from stress ambiguities to support comparatively long slabs as in truss bridge floors on the four sides instead of the two. If the load is not expected on the unsupported sides, as in the case of a slab bridge with foot paths and carriage ways, bending moment may be calculated with the moment ratio $\dfrac{\beta}{\frac{2}{3}+\beta}$ or with the theoretical formulas. See **Pl. 3, 4.**

(Explanation as to the deduction of the moment ratio from the prevalent formula).

Let us first consider the general results obtained from the formula given in Article 29 of the specification of the Dept. of Home Affairs.

According to Art. 29, the effective width e will be

$$e=\frac{2}{3}(2b)+2c, \quad \text{but} \quad e \leqq 2\,\text{m}, \quad e \leqq 2a.$$

The moment ratio is

$$\frac{2c}{e}=\frac{c}{\frac{2}{3}b+c}=\frac{\frac{c}{b}}{\frac{2}{3}+\frac{c}{b}}$$

If we denote $\dfrac{c}{b}$ with β & $\dfrac{a}{b}$ with α, the ratio will be

$$\frac{\beta}{\frac{2}{3}+\beta} \quad \text{.............................(i)}$$

also

$$e \leqq 2a \quad \text{or} \quad \frac{2}{3}+\beta \leqq \alpha \quad \text{..................(ii)}$$

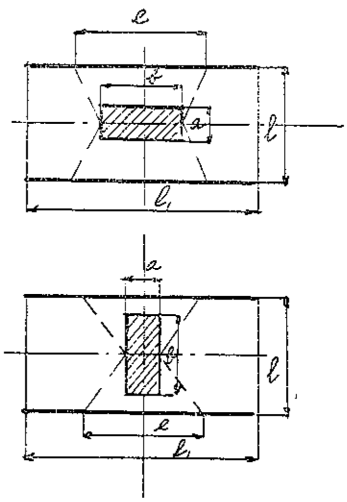


Fig. 21.

The distributed area of one wheel of a 12 ton motor truck, (see **Fig. 18.**) with the depth of filling 15 cm. will be, 70×50 cm. The mean depth of filling consisting of wooden block pavement is usually 15 cm.

As is specified, the max. effective width is 2 m. and the limiting moment ratio will be $\frac{70}{200}$ or $\frac{50}{200}$.

We may express Art. 29 as follows; the moment ratio, or

$$\text{the ratio, } \frac{\text{the moment calculated with effective width}}{\text{simple beam moment (without taking eff. } w.)}$$

will be denoted by

$$\left.\begin{array}{l} \dfrac{\beta}{\frac{2}{3}+\beta} \geqq \dfrac{70}{200} \quad \text{or} \quad \dfrac{50}{200} \\ \frac{2}{3}+\beta \leqq \alpha \end{array}\right\} \text{..(iii)}$$

If the above relation is not satisfied simultaneously, the moment ratio will be simply,

$$\frac{\beta}{\alpha}$$

but not less than $\frac{70}{200}$ or $\frac{50}{200}$.

For square slabs the relation (iii) will be

$$\left.\begin{array}{l} \frac{2}{3}+\beta \leqq 1 \\ \dfrac{\beta}{\frac{2}{3}+\beta} \geqq \dfrac{50}{200} \end{array}\right\} \quad \therefore \ \frac{1}{3} \geqq \beta \geqq \frac{2}{9}$$

We see that the formula is effective in the range $\frac{1}{3} \geqq \beta \geqq \frac{2}{9}$ and outside it the moment ratio will be β ($\alpha = 1$), but not less than $\frac{50}{200}$ or 0.25. See **Pl. 5.**

4. Numerical Examples.

Ex. 1. (Stress due to a single wheel.)

Calculate the bending moment near the center of a long slab as shown by **Fig. 22.**, for one wheel of a 12 ton motor truck. The depth of filling is 15 cm. The long sides are assumed to be rigid and their length is larger than 3 m. The short sides are either supported or unsupported.

If $2a > 3\,\text{m}$, $(\alpha-\beta)>2$ and we can use the solution given in Art. 2. in this chapter. The stress near the center is independent of the rigidity of the short sides.

a) **Dead load intensity.**

			t/m2
Wooden blocks 7.5 cm thick		$=1\times0.075$	$=0.075$
Cushion mortar 3 cm	„	$=1.7\times0.03$	$=0.051$
Filling concrete 4.5 cm	„	$=2.2\times0.045$	$=0.099$
Reinforced conc. slab 16 cm	„	$=2.4\times0.16$	$=0.384$
			0.609

b) **Live load intensity.**

The weight of one wheel $=4.5$ tons

Coefficient of impact $=30\%$.

$$\text{Live load intensity}=\frac{4.5\times1.3}{0.5\times0.7}=16.7\ \text{t/m}^2$$

c) **Live load moment at** $x=0$, $y=0$,

By eq. (50) and (20), As $\gamma=0.6$ & $\beta=0.4$,

n	$\frac{n\pi\beta}{2}$	$\cosh\frac{n\pi\beta}{2}$	$\tanh\frac{n\pi\beta}{2}$	$\sin\frac{n\pi\gamma}{2}$
1	0.62832	1.20376	0.55667	0.80874
3	1.88496	3.36909	0.95493	0.30943
5	3.1416	11.59195	0.99627	-1

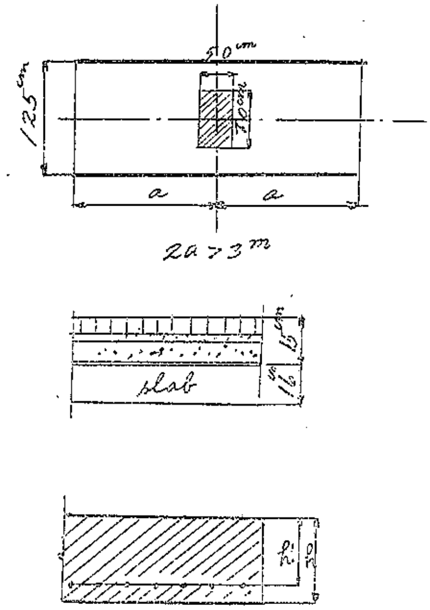


Fig. 22.

$$A_1'=-\frac{2+0.62837}{2\times1.20376(1+0.55667)}=-0.70133$$

$$B_1'=+\frac{1}{2\times1.20376(1+0.55667)}=+0.26682$$

Similarly,

$$\begin{cases}A_3'=-0.29493\\ B_3'=+0.07591\end{cases}\qquad\begin{cases}A_5'=-0.11110\\ B_5'=+0.02161\end{cases}$$

$\nu=0.1$

$(M_y)_{x=0,\,y=0}$

$$\frac{16\,p_0b^2}{3.1416^3}\times\frac{0.80874}{1^3}(1-0.9\times0.70133-0.2\times0.26682)=\quad 0.13164\,p_0b^2$$

$$\frac{16\,p_0b^2}{3.1416^3}\times\frac{0.30943}{3^3}(1-0.9\times0.29493-0.2\times0.07591)=\quad 0.00425\,p_0b^2$$

$$\frac{16\,p_0b^2}{3.1416^3}\times\frac{1}{5^3}(1-0.9\times0.11110-0.2\times0.02161)\quad=-0.00370\,p_0b^2$$

$$\overline{0.13219\,p_0b^2}$$

$(M_y)_{x=0,\,y=0}$ according to the Dept. H. A. formula will be

$$\text{simple beam moment}\times\frac{\beta}{\frac{2}{3}+\beta}=p_0b^2\left(\gamma-\frac{\gamma^2}{2}\right)\frac{\beta}{\frac{2}{3}+\beta}=0.153\,p_0b^2$$

The difference is about 20%.

Similarly

$(M_x)_{x=0,\,y=0}$

$$p_0b^2\times 0.41731(-0.63120+0.53364-0.1)=\quad 0.08246\,p_0b^2$$
$$p_0b^2\times 0.00591(-0.26544+0.15180-0.1)=\quad 0.00126\,p_0b^2$$
$$-p_0b^2\times 0.00413(-0.09999+0.01230-0.1)=-0.00006\,p_0b^2$$
$$\overline{0.08366\,p_0b^2}$$

Thus for $\nu=0.1$

$$\begin{cases}(M_y)_{x=0,\,y=0}=0.132\,p_0b^2\\(M_x)_{x=0,\,y=0}=0.083\,p_0b^2\end{cases}$$

Live load moment plus impact $=0.132\times 16.7\times 0.625^2=0.860$ tm

Dead load moment $=0.50\times 0.60\times 0.625^2=0.117$

$\overline{0.977}$ tm

Required $h'=0.375\sqrt{\dfrac{M}{b}}=0.375\sqrt{977}=11.7$ cm ($\sigma_e=1\,200$ kg/cm², $\sigma_b=45$ kg/cm²). Take $h=15$ cm.

Amount of main reinforcement.

$$f_s=0.253\sqrt{\frac{M}{b}}=7.91\ \text{cm}^2\ \text{(13 mm rods at 15 cm spc.)}$$

Amount of distributing bars.

$$M_x=0.083\times 16.7\times 0.625^2=0.54\ \text{tm}$$

$$f_s'\doteqdot\frac{54\,000}{1\,200\times 11.7\times\dfrac{7}{8}}=4.39\ \text{(13 mm rods at 25 cm spc.)}$$

You see that the common practice to use 9 mm rods at large spacings is insufficient to develop full strength of the slab.

d) **Max. negative mt. in the direction of X-axis.**

In the second range with the origin at B, negative M_x will be calculated with the first term only.

$$(M_x)_{y=0}=\frac{16\,b^2p_0}{\pi^3}\sin\frac{\pi d}{2b}\Big\{[(1-\nu)C_{10}'+2D_1']$$
$$-(1-\nu)D_1'\frac{\pi x}{2b}\Big\}\left(\cosh\frac{\pi x}{2b}-\sinh\frac{\pi x}{2b}\right)$$

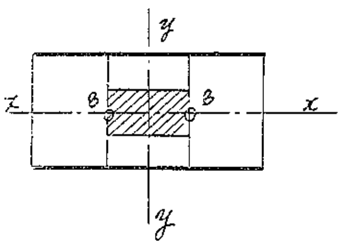


Fig. 23.

from eq. (20) for $\beta=0.4$ we get $C_{10}'=0.27814$, $D_1'=-0.17889$ and as $\gamma=0.6$

$$(M_x)_{y=0}=p_0b^2\times 0.51602$$
$$\times 0.80874\left(0.10727-0.16692\frac{\pi x}{2b}\right)\left(\cosh\frac{\pi x}{2b}-\sinh\frac{\pi x}{2b}\right)$$

put $\dfrac{d(M_x)_{y=0}}{dx}=0$ we get $\dfrac{x}{b}=1.06$

The max. negative mt. occurs at $x\doteqdot b$ from the origin B in this case.

The max. negative mt. $=-0.0127\,p_0b^2$.

Max. negative mt. along X-axis will be

$$-16.7\times 0.0127\times 0.625^2=0.083\ \text{tm}.$$

This amount is very small and need not be considered.

Ex. 2. (Stress due to a pair of wheels.)

Calculate the max. bending mt. for a pair of the front wheels of a 12 ton motor truck. The span of the slab bridge is 3.5 m and its width is more than 9.5 m. The depth of filling is 15 cm.

In this case the effects of the two wheels 1.8 m apart are superposed.

In order that the stresses are independent of the yielding of the short sides, each load must have a clearance from the unsupported sides at least equal to the span. See Fig. 24.

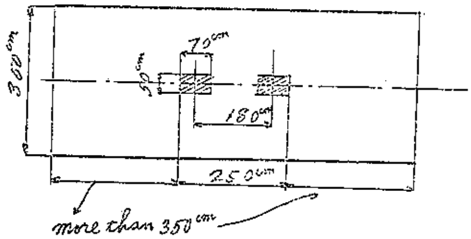


Fig. 24.

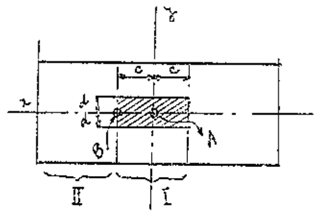


Fig. 25.

(Stress due to one wheel.)

From eq. (12) and (21) we get $(M_x)_{y=0}$, $(M_y)_{y=0}$, taking the first term only.

In the domain (I) with the origin at A,

$$(M_x)_{y=0}=\frac{16b^2p_0}{\pi^3}\sin\frac{\pi d}{2b}\left[(1-\nu)A_1'\cosh\frac{\pi x}{2b}+B_1'\left\{2\cosh\frac{\pi x}{2b}+(1-\nu)\frac{\pi x}{2b}\sinh\frac{\pi x}{2b}\right\}-\nu\right] \qquad \text{.........(51)}$$

In the domain (II) with origin at B,

$$(M_x)_{y=0}=\frac{16b^2p_0}{\pi^3}\sin\frac{\pi d}{2b}\left[\{(1-\nu)C_{10}'+2D_1'\}-(1-\nu)D_1'\frac{\pi x}{2b}\right]\left(\cosh\frac{\pi x}{2b}-\sinh\frac{\pi x}{2b}\right) \qquad \text{.........(52)}$$

Similarly for $(M_y)_{y=0}$

$$(M_y)_{y=0}=\frac{16b^2p_0}{\pi^3}\sin\frac{\pi d}{2b}\left[1+(1-\nu)A_1'\cosh\frac{\pi x}{2b}+B_1'\left\{-2\nu\cosh\frac{\pi x}{2b}+(1-\nu)\frac{\pi x}{2b}\sinh\frac{\pi x}{2b}\right\}\right] \qquad \text{.........(53)}$$

$$(M_y)_{y=0}=\frac{16b^2p_0}{\pi^3}\sin\frac{\pi d}{2b}\left\{(1-\nu)C_{10}'-2\nu D_1'-(1-\nu)D_1'\frac{\pi x}{2b}\right\}\left(\cosh\frac{\pi x}{2b}-\sinh\frac{\pi x}{2b}\right) \qquad \text{.........(54)}$$

$(\beta=0.2,\ \ \gamma=0.15)$

Thus we get by 20) and 22)

$$\begin{cases} A_1'=-0.84527, \\ B_1'=0.36526, \end{cases} \qquad \begin{cases} C_{10}'=0.14934 \\ D_1'=-0.11659 \end{cases}$$

And in the domain (I) with the origin at A,

$$(M_x)_{y=0}=0.12065\, p_0 b^2\left(-0.03022\cosh\frac{\pi x}{2b}+0.32373\frac{\pi x}{2b}\sinh\frac{\pi x}{2b}-0.1\right)$$

in the domain (II) ....(a)

$$(M_x)_{y=0}=0.12065\, p_0 b^2\left(-0.09877+0.10493\frac{\pi x}{2b}\right)\left(\cosh\frac{\pi x}{2b}-\sinh\frac{\pi x}{2b}\right)$$

in the domain (I)

$$(M_y)_{y=0}=0.12065\, p_0 b^2\left(1-0.83379\cosh\frac{\pi x}{2b}+0.32873\frac{\pi x}{2b}\sinh\frac{\pi x}{2b}\right)$$

in the domain (II) ....(b)

$$(M_y)_{y=0}=0.12065\, p_0 b^2\left\{0.15773+0.10493\frac{\pi x}{2b}\right\}\left(\cosh\frac{\pi x}{2b}-\sinh\frac{\pi x}{2b}\right)$$

The curves (a) (b) are plotted according to the values of $\frac{x}{b}$ for each wheel and added together. See Pl. 9.

Max. (M_y) for one wheel is $0.020\, p_0 b^2$. If the effect of the adjoining wheel is considered it becomes $0.031\, p_0 b^2$, about 55% increase!

Max. (M_x) is on the otherhand hardly affected by the adjacent load, and is decreased by 7%.

Required depth of the slab.

Max. $(M_y)_{y=0}=0.03021\, p_0 b^2=0.03021\times 16.7\times 1.75^2=1.55$ tm

l. l. m. = 1.55 tm (live load intensity 16.7 t/m²)

d. l. m. = 1.53 (dead load intensity 1 t/m²)

3.08

Req. $h'=0.441\sqrt{3\,080}=24.4$ cm

$\sigma_e=1\,200$ kg/cm², $\sigma_b=40$ kg/cm².

Amount of main reinforcement.

$f_e=0.228\sqrt{3\,080}=12.65$ cm² = 15 mm rods at 14 cm spacing.

Amount of distribution bars.

Max. $(M_x)_{y=0}=0.01457\, p_0 b^2$

$=16.7\times 0.01457\times 1.75^2=0.74$ tm.

Req. amount of steel

$\doteqdot 3.04$ cm².

1 cm rods at 25 cm spacing = 3.14 cm².

See Pl. 9.

5. Two side support slabs in case $(\alpha-\beta<2)$.

As the two side support slabs are nothing but the slabs with yielding side supports in their special cases, nothing will be stated here as to the method of calculation.

Let us compare again the theoretical results with those obtained from the specification of the Dept. H. A.

One wheel of a 12 ton motor truck with the distributed load area 70 cm ×50 cm is considered. Moment ratio according to the specification is denoted by the thick lines for side ratios $\alpha=0.5$, 1, 1.5, 2 etc. (See **Pl. 5.**)

The curve 1 denotes the theoretical moment ratio for $\alpha=1$. It agrees well with the thick line except the horizontal part involving the limitation that " $e\leqq 2\text{m}$ ".

The same will be said to the curves 2 & 3. The curve 2 is for $\alpha=1.5$ and the curve 3 is for the case $(\alpha-\beta\geqq 2)$, that is to say for long slabs.

6. Two side support slabs with rectangular uniform load on the unsupported side. $(\alpha-\beta>1)$.

It is needless to say that the case $(\alpha-\beta<1)$ should be considered, but this case is rather complicated, and as we may expect that the effect of the extreme position of the load will not be so great as in case $(\alpha-\beta>1)$, the case $(\alpha-\beta<1)$ was omitted to avoid complication.

The solution will be given in Art. 8.

7. The solution for slabs with four rigid side supports with extreme load position as basis of the calculation in the following articles.

Taking the origins at A and B, in the domains I & II respectively, we have,

In the domain I, origin at A.

$$w_{\text{I}}=\frac{1}{N}\sum_{n}a_n\cos\frac{n\pi y}{2b} \quad \ldots\ldots(55)$$

$$n=1, 3, 5, \ldots$$

$$w_{\text{II}}=\frac{1}{N}\sum_{n}\left\{\frac{1}{2}a_n\left(\frac{n\pi x}{2b}\sinh\frac{n\pi x}{2b}-2\cosh\frac{n\pi x}{2b}\right)+A_n\sinh\frac{n\pi x}{2b}+B_n\frac{n\pi x}{2b}\cosh\frac{n\pi x}{2b}\right\}\cos\frac{n\pi y}{2b} \quad \ldots(56)$$

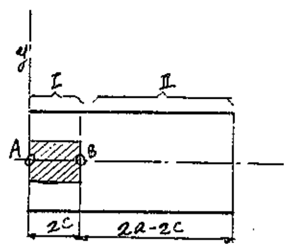


Fig. 26.

In the domain II, origin at B.

$$w_{\text{III}}=\frac{1}{N}\sum_{n}\left[C_n\left\{\tanh n\pi(\alpha-\beta)\cosh\frac{n\pi x}{2b}-\sinh\frac{n\pi x}{2b}\right\}+D_n\left(\frac{n\pi x}{2b}-n\pi(\alpha-\beta)\right)\left\{\tanh n\pi(\alpha-\beta)\sinh\frac{n\pi x}{2b}-\cosh\frac{n\pi x}{2b}\right\}\right]\cos\frac{n\pi y}{2b} \quad \ldots(57)$$

The above three equations are the solutions for four rigid side supports with extreme load position.

With origin at A, $w_{\mathrm{I}}+w_{\mathrm{II}}$ evidently satisfies the condition,

$$\left.\begin{aligned}(w_{\mathrm{I}}+w_{\mathrm{II}})&=0\\ \text{and}\quad \frac{\partial^2}{\partial x^2}(w_{\mathrm{I}}+w_{\mathrm{II}})&=0\end{aligned}\right\}\quad \text{at}\quad x=0.$$

And with origin at B. w_{III} satisfies the same conditions at the boundary $x=2a-2c$.

The constants A_n, B_n, C_n, D_n are to be determined by the condition of continuity at the common boundary of the two surfaces.

By equating the n-th terms of the series according to the conditions of continuity we get.

a) $(w_{\mathrm{I}}+w_{\mathrm{II}})_{x=2c}=(w_{\mathrm{III}})_{x=0}$

$$\frac{1}{2}a_n\{n\pi\beta\sinh n\pi\beta+2(1-\cosh n\pi\beta)\}+A_n\sinh n\pi\beta+B_n n\pi\beta\cosh n\pi\beta$$
$$=C_n\tanh n\pi(\alpha-\beta)+D_n n\pi(\alpha-\beta) \quad\text{.........................(58)}$$

b) $\left(\frac{\partial w_{\mathrm{I}}}{\partial x}+\frac{\partial w_{\mathrm{II}}}{\partial x}\right)_{x=2c}=\left(\frac{\partial w_{\mathrm{III}}}{\partial x}\right)_{x=0}$

$$\frac{1}{2}a_n\{n\pi\beta\cosh n\pi\beta-\sinh n\pi\beta\}+A_n\cosh n\pi\beta+B_n(\cosh n\pi\beta+n\pi\beta\sinh n\pi\beta)$$
$$=-C_n-D_n\{1+n\pi(\alpha-\beta)\tanh n\pi(\alpha-\beta)\} \quad\text{.....................(59)}$$

c) $\left(\frac{\partial^2 w_{\mathrm{I}}}{\partial x^2}+\frac{\partial^2 w_{\mathrm{II}}}{\partial x^2}\right)_{x=2c}=\left(\frac{\partial^2 w_{\mathrm{III}}}{\partial x^2}\right)_{x=0}$

$$\frac{1}{2}a_n\, n\pi\beta\sinh n\pi\beta+A_n\sinh n\pi\beta+B_n(2\sinh n\pi\beta+n\pi\beta\cosh n\pi\beta)$$
$$=C_n\tanh n\pi(\alpha-\beta)+D_n\{2\tanh n\pi(\alpha-\beta)+n\pi(\alpha-\beta)\} \quad\text{...........(60)}$$

d) $\left(\frac{\partial^3 w_{\mathrm{I}}}{\partial x^3}+\frac{\partial^3 w_{\mathrm{II}}}{\partial x^3}\right)_{x=2c}=\left(\frac{\partial^3 w_{\mathrm{III}}}{\partial x^3}\right)_{x=0}$

$$\frac{1}{2}a_n(\sinh n\pi\beta+n\pi\beta\cosh n\pi\beta)+A_n\cosh n\pi\beta+B_n(3\cosh n\pi\beta+n\pi\beta\sinh n\pi\beta)$$
$$=-C_n-\{3+n\pi(\alpha-\beta)\tanh n\pi(\alpha-\beta)\}D_n \quad\text{......................(61)}$$

By solving the above equations with regard to A_n, B_n, C_n, D_n and putting $(\alpha-\beta>1)$, we get

$$\left.\begin{aligned}
A_n&=-\frac{(n\pi\beta+2)a_n}{2(\cosh n\pi\beta+\sinh n\pi\beta)}+a_n\\
B_n&=\frac{a_n}{2(\cosh n\pi\beta+\sinh n\pi\beta)}-\frac{a_n}{2}\\
C_n&=-n\pi\alpha D_n+a_n(1-\cosh n\pi\beta)+A_n\sinh n\pi\beta\\
D_n&=-\frac{(\cosh n\pi\beta-1)a_n}{2(\cosh n\pi\beta+\sinh n\pi\beta)},\qquad a_n=\frac{64l^4 p_0}{n^5\pi^5}\sin\frac{n\pi d}{2b}
\end{aligned}\right\}\quad\text{..................(62)}$$

8. Stress near the yielding side support for the loading given in the preceding articles.

To calculate the effect of the side support yielding at $x=0$ in the domain I, one more equation is necessary which is as follows.

Also it must be remembered that the solution given by eq. (55) & (56), (57) was obtained as basis of calculation of stresses near the yielding support of a long slab, of which the rigidity of the opposite side is independent of the stresses above mentioned ($\alpha-\beta>1$).

As an equation to be added to eq. (55), (56), (57), in the range $2a \geqq x \geqq 0$

$$w_{IV}=\frac{1}{N}\sum_n E_n\left(\frac{2}{1-\nu}+\frac{n\pi x}{2b}\right)\left(\cosh\frac{n\pi x}{2b}-\sinh\frac{n\pi x}{2b}\right)\cos\frac{n\pi y}{2b} \quad \dots\dots(63)$$

(63) satisfies the condition, $(M_x)_{x=0}=0$.

Also in case ($\alpha-\beta>1$) this equation rapidly vanishes with increasing values of x, and it satisfies

$$(w_{IV})_{x=2a}=0, \quad \left(\frac{\partial^2 w_{IV}}{\partial x^2}\right)_{x=2a}=0 \quad \text{for slabs with side ratios } \alpha-\beta>1$$

The eq. (63) together with eq. (55), (56), (57) can be used for the calculation of stresses near the yielding side of a long slab.

To determine E_n in eq. (63) we put the following boundary condition,

$$\left(\frac{\partial^3}{\partial x^3}+(2-\nu)\frac{\partial^3}{\partial x\partial y^2}\right)(w_I+w_{II})_{x=0}+\left(\frac{\partial^3}{\partial x^3}+(2-\nu)\frac{\partial^3}{\partial x\partial y^2}\right)(w_{IV})_{x=0}=E_sI_s\frac{d^4}{dy^4}(w_{IV})_{x=0}$$

from which we get

$$E_n=\frac{(1-\nu)A_n-(1+\nu)B_n}{(3+\nu)+\dfrac{2n\pi K}{1-\nu}} \quad \dots\dots(64)$$

K=side support rigidity.

in the domain (I)

$$(M_y)_{x=0,\,y=0}=\frac{16\,p_0b^2}{\pi^3}\sum_n\frac{1}{n^3}\sin\frac{n\pi d}{2b}\,2(1+\nu)E_n' \quad \dots\dots(65)$$

$$n=1,\,3,\,5\dots.$$

$$E_n'=\frac{(1-\nu)A_n'-(1+\nu)B_n'}{(3+\nu)+\dfrac{2n\pi K}{1-\nu}}$$

A_n' & B_n' are the const. without the term a_n given in eq. (62).

The bending mt. in the supporting beam at $y=0$ is,

$$\text{Support mt.}=K\,\frac{32b^3p_0}{\pi^3}\sum_n\frac{1}{n^3}\sin\frac{n\pi d}{2b}\,\frac{2}{1-\nu}\,E_n' \quad \dots\dots(66)$$

9. Numerical example.

Ex. In the two side support slab calculate the max. bending mt. at the unsupported side, when the rectangular uniform load is on it. The load and the dimension of the slab are exactly the same as in Ex. 1, Chapt. 4. (p. 27)

$$\beta=\frac{c}{b}=0.4,\quad \gamma=\frac{d}{b}\doteqdot 0.6 \;\&\; K=0 \quad \text{(unsupported.)}$$

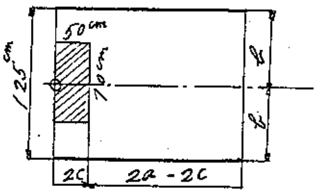


Fig. 27.

By eq. (62), (64), (65)

$$\begin{cases} A_1'=0.53674 \\ A_3'=0.93350 \\ A_5'=0.99225 \end{cases} \quad \begin{cases} B_1'=-0.35775 \\ B_3'=-0.48848 \\ B_5'=-0.49907 \end{cases} \quad \begin{cases} E_1'=0.28277 \\ E_3'=0.44434 \\ E_5'=0.46516 \end{cases}$$

By eq. (65)

$$(M_y)_{x=0,\ y=0}=0.245\,p_0b^2.$$

The live load mt. shows 85% increase as compared with the results in Ex. 1, Chapt. 4.

In the present example,

$$p_0=16.7\ \text{t/m}^2,\qquad b=0.625$$

$$\text{l. l. m.}=16.7\times0.245\times0.625^2=1.61\ \text{tm}$$

$$\text{d. l. m.}=0.60\times0.50\times0.625^2=0.12$$

$$1.73$$

$$h'=0.375\sqrt{1\,730}=15.6\ \text{cm}$$

$h=18$ cm is used.

In this slab with rectangular load moving along the long side the required depth are 18 cm & 15 cm in the range (II) & (I) respectively. See **Fig. 28.**

10. Note on Art. 8, Chapt. 4.

The moment ratio at the unsupported side (see **Fig. 26**), that is to say the ratio of the theoretical moment to the simple beam mt. will be denoted approximately by

$$2(1+\nu)E_1'$$

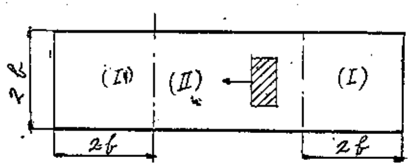


Fig. 28.

For, the theoretical moment is shown by eq. (65) and the simple beam moment is

$$p_0b^2\left(\gamma-\frac{\gamma^2}{2}\right)=\frac{16\,p_0}{\pi^3}\sum_n\frac{1}{n^3}\sin\frac{n\pi\gamma}{2}$$

For the moment ratio in this case see **Pl. 7** and compare with **Pl. 6.** You will find considerable change in mt. ratio at the unsupported side.

11. Practical rules for the design of two side support slabs.

Judging from the results obtained in the preceding articles I have reached the following conclusion.

a) If a rectangular slab shown by **Fig. 19** is supported at $y=\pm b$ and unsupported at $x=\pm a$, the bending mt. shall be calculated with the following moment ratio to provide for the load on the unsupported side.

The proposed moment ratio is $\frac{\beta}{0.3+\beta}$ or $\frac{\beta}{\alpha}$ instead of $\frac{\beta}{\frac{2}{3}+\beta}$ which is commonly used in practice.

In other words the effective width will be,

$$e=0.3\,l+a\leqq l_1$$

If we use the same notation as in the specification of the Dept., H. A.

b) In slabs with long width, when it is possible that the rectangular load comes near the unsupported sides, the portions of slabs at least equal to the span or $2b$ from the unsupported sides shall be proportioned with the calculation shown in Art. 8, or with the moment ratio, $\frac{\beta}{0.3+\beta}$. See **Fig. 28.**

This moment ratio shall preferably be used for values of β less than 0.5 to expect accuracy.

The remaining portion shall be proportioned by the calculation in Art. 2, Chapt. 4, or by means of the usual formula.

$$\frac{\beta}{\frac{2}{3}+\beta}$$

c) In point of economy as well as simplicity in calculation, four side support slabs shall be preferred to the two side support slabs especially in long slabs.

12. Solutions for point and linear loadings ($\alpha-\beta>2$).

In the previous chapters we have considered the nature of stresses due to rectangular uniform loading with const. intensity p_0 and observed how they are affected by the load sizes.

Now let us take a concentrated load P and inspect the change of stress due to load distribution.

The concentrated load P is distributed uniformly over rectangular areas. **Fig. 29** (**2**), (**3**), (**4**) are its special cases.

a) **The solution for linear loading** (**2**) in case $(\alpha-\beta>2)$.

Look at the solution given by eq. (11), (12), (20), (21) in Art. 5, Chapt. 2 of which substitute p_0 with $\frac{P}{2c\times 2d}$, $\frac{c}{b}=\beta$, $\frac{d}{b}=\gamma$.

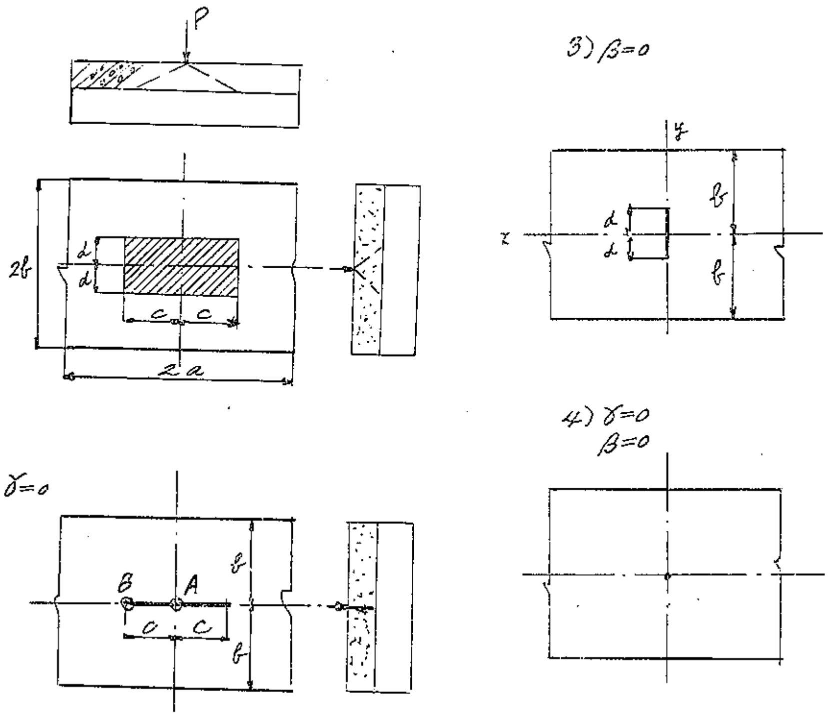


Fig. 29.

if γ tends to zero, as

$$a_n=\frac{64b^4p_0}{n^5\pi^5}\sin\frac{n\pi d}{2b}=\frac{16b^2}{n^5\pi^5}\cdot\frac{P}{\beta\gamma}\sin\frac{n\pi\gamma}{2}$$

$$\lim_{\gamma=0}a_n=\frac{8b^2P}{n^4\pi^4\beta}\quad\cdots\cdots(67)$$

A_n', B_n', C_n', D_n' remain unchanged.

b) **Linear loading** (3).

Look at eq. (21), (22)

$$C_{n0}=\frac{a_n\tanh\frac{n\pi\beta}{2}}{1+\tanh\frac{n\pi\beta}{2}}-\frac{a_n\;\frac{n\pi\beta}{2}}{2\left(1+\tanh\frac{n\pi\beta}{2}\right)^2\cos^2\text{h}\,\frac{n\pi\beta}{2}}$$

$$D_n=-\frac{a_n\tanh\frac{n\pi\beta}{2}}{2\left(1+\tanh\frac{n\pi\beta}{2}\right)}$$

$$a_n=\frac{16b^2}{n^5\pi^5}\cdot\frac{P}{\beta\gamma}\sin\frac{n\pi\gamma}{2}$$

and,

$$\left.\begin{aligned}\lim_{\beta=0}C_{n0}&=\frac{4b^2}{n^4\pi^4}\cdot\frac{P}{\gamma}\sin\frac{n\pi\gamma}{2}\\ \lim_{\beta=0}D_n&=\frac{-4b^2}{n^4\pi^4}\cdot\frac{P}{\gamma}\sin\frac{n\pi\gamma}{2}\end{aligned}\right\}\quad\ldots\ldots(68)$$

The solution is given in the simple form, in the range $a\geqq x\geqq 0$.

$$w=\frac{4b^2P}{N\pi^4\gamma}\sum\frac{1}{n^4}\sin\frac{n\pi\gamma}{2}\left(1+\frac{n\pi x}{2b}\right)\times\left(\cosh\frac{n\pi x}{2b}-\sinh\frac{n\pi x}{2b}\right)\cos\frac{n\pi y}{2b}\quad\ldots(69)$$

$$n=1,\ 3,\ 5,\ldots.$$

The deflection at $x=0$, $y=0$ is, for the loading shown by **Fig. 30.**

$$(w)_{x=0,\,y=0}=\frac{4b^2P}{N\pi^4}\sum_n\frac{1}{n^4}(-1)^{\frac{n-1}{2}}\quad\ldots\ldots(70)$$

$$n=1,\ 3,\ 5,\ldots.$$

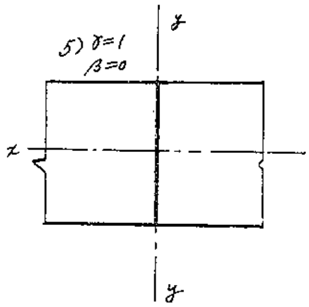


Fig. 30.

We have obtained the same result as were done by Dr. Nádai in "Elastische Platten" S. 81.

N. B. In the book above mentioned it is given that

$$w_0=\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right)\frac{Pa^2}{\pi^4N}$$

but this is, $w_0=\left(1-\frac{1}{3^4}+\frac{1}{5^4}-\ldots\right)\frac{Pa^2}{\pi^4N}$

as is clear by eq. (60) in the same page.

e) Point loading.

$$a_n=\frac{16b^2}{n^5\pi^5}\cdot\frac{P}{\beta\gamma}\sin\frac{n\pi\gamma}{2}$$

$$\left.\begin{aligned}\lim_{\substack{\beta=0\\ \gamma=0}}C_{n0}&=\frac{2b^2P}{n^4\pi^4}\\ \lim_{\substack{\beta=0\\ \gamma=0}}D_n&=-\frac{2b^2P}{n^4\pi^4}\end{aligned}\right\}\quad\ldots\ldots(71)$$

and the solution is, in the range $a \geqq x \geqq 0$,

$$w=\frac{2b^2P}{N\pi^3}\sum_n \frac{1}{n^3}\left(1+\frac{n\pi x}{2b}\right)\left(\cosh\frac{n\pi x}{2b}-\sinh\frac{n\pi x}{2b}\right)\cos\frac{n\pi y}{2b} \qquad (72)$$

$n=1, 3, 5, \ldots$

The deflection at $x=0$, $y=0$

$$(w)_{x=0,\, y=0}=\frac{2b^2P}{\pi^3}\sum_n \frac{1}{n^3} \qquad (73)$$

$n=1, 3, 5, \ldots$

The same will be found in "El. Pn." S. 86.

For the values $(M_x)_{x=0,\, y=0}$ & $(M_y)_{x=0,\, y=0}$ due to a concentrated load P distributed over rectangular areas see **Pl. 4.** and compare with **Pl. 3.** which was prepared with the ordinary practical formula. Observe the limit of application of the prevalent formula.

N. B. In the linear loading (b) eq. (69)

$$(M_x)_{x=0,\, y=0}=(M_y)_{x=0,\, y=0}=\frac{P(1+\nu)}{\pi^2}\sum_n \frac{1}{n^2}\cdot\frac{\sin\frac{n\pi\gamma}{2}}{\gamma} \qquad (74)$$

For the point loading

$$(M_x)_{x=0,\, y=0}=(M_y)_{x=0,\, y=0}=\frac{P(1+\nu)}{2\pi}\sum_n \frac{1}{n} \qquad (75)$$

This series is divergent.

Chapter 5. Continuous slabs on yielding supports.

1. General remarks.

In this chapter some simplest cases are dealt with. In deck plate girders the slabs are often separated by the main girders, the tops of which are flush with the surface of the slabs, thus minimumizing the road surface elevation and increasing lateral stiffness of the girders. See **Pl. 1.**

As a simple case we shall consider a continuous strip of a slab supported by the cross beams and the main girders which are considered rigid, and we shall put into account the yielding of the cross beams.

2. The solution to a three span continuous strip, Center field loaded.

Assumptions:

(1) The center field is loaded with symmetrical uniform load p which is a function of y only.

(2) The supports are rigid except those at $x=\pm a$.

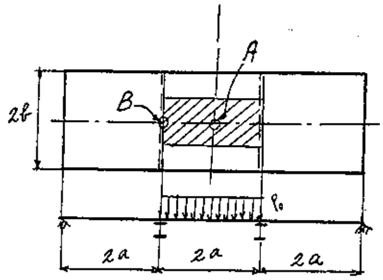


Fig. 31.

Solution.

In the same way as in Art. 2, Chapt. 2, we get.

In the range $a\geqq x\geqq -a$ with origin A at,

$$\left.\begin{aligned} w_{\mathrm{I}}&=\frac{1}{N}\sum_n a_n \cos\frac{n\pi y}{2b} \\ w_{\mathrm{II}}&=\frac{1}{N}\sum_n\left(A_n\cosh\frac{n\pi x}{2b}+B_n\frac{n\pi x}{2b}\sinh\frac{n\pi x}{2b}\right)\cos\frac{n\pi y}{2b} \end{aligned}\right\} \qquad \cdots\cdots(76)$$

And in the range $2a\geqq x\geqq 0$, with origin at B.

$$\begin{aligned} w_{\mathrm{III}}=\frac{1}{N}\sum_n\Big\{&C_n\left(\tanh n\pi\alpha\cosh\frac{n\pi x}{2b}-\sinh\frac{n\pi x}{2b}\right) \\ &+D_n\left(\frac{n\pi x}{2b}-n\pi\alpha\right)\left(\tanh n\pi\alpha\sinh\frac{n\pi x}{2b}-\cosh\frac{n\pi x}{2b}\right)\Big\}\cos\frac{n\pi y}{2b} \end{aligned} \qquad \cdots\cdots(77)$$

$$\frac{a}{b}=\alpha.$$

w_{III} satisfies the condition

$$(w_{\mathrm{III}})_{x=2a}=0,\quad \left(\frac{\partial^2 w_{\mathrm{III}}}{\partial x^2}\right)_{x=2a}=0$$

A_n, B_n, C_n, D_n, are the constants to be determined by the boundary conditions at the common supports.

To simplify calculation we shall consider the case $\alpha>1$ or $\tanh n\pi\alpha=1$.

3. Boundary conditions at the common supports.

With regard to w,

$$C_n+n\pi\alpha\, D_n=a_n+A_n\cosh\frac{n\pi\alpha}{2}+B_n\frac{n\pi\alpha}{2}\sinh\frac{n\pi\alpha}{2} \qquad \cdots\cdots(78)$$

With regard to $\dfrac{\partial w}{\partial x}$,

$$-C_n-(1+n\pi\alpha)\,D_n=A_n\sinh\frac{n\pi\alpha}{2}+\left\{\sinh\frac{n\pi\alpha}{2}+\frac{n\pi\alpha}{2}\cos\frac{n\pi\alpha}{2}\right\}B_n \qquad \cdots\cdots(79)$$

With regard to $\dfrac{\partial^2 w}{\partial x^2}$,

$$C_n+(2+n\pi\alpha)\,D_n=A_n\cosh\frac{n\pi\alpha}{2}+\left\{2\cosh\frac{n\pi\alpha}{2}+\frac{n\pi\alpha}{2}\sinh\frac{n\pi\alpha}{2}\right\}B_n \qquad \cdots\cdots(80)$$

As the last boundary condition we have,

$$\left(\frac{\partial^3 w_{\mathrm{II}}}{\partial x^3}\right)_{x=a}-\left(\frac{\partial^3 w_{\mathrm{III}}}{\partial x^3}\right)_{x=0}=\text{external load for the supporting beam.}$$

$$-(1-n\pi K)C_n-\{3+n\pi\alpha(1-n\pi K)\}D_n=A_n\sinh\frac{n\pi\alpha}{2}$$
$$+\left(3\sinh\frac{n\pi\alpha}{2}+\frac{n\pi\alpha}{2}\cosh\frac{n\pi\alpha}{2}\right)B_n \quad\text{......(81)}$$

The terms $\frac{\partial^3 w}{\partial x\,\partial y^2}$ vanish because it is clear from the second boundary condition that $\left(\frac{\partial^3 w_{II}}{\partial x\,\partial y^2}\right)_{x=a}=\left(\frac{\partial^3 w_{III}}{\partial x\,\partial y^2}\right)_{x=0}$

$$K=\frac{E_sI_s}{2bN}$$

We get

$$\left.\begin{aligned}
A_n&=-\left(\frac{n\pi\alpha}{2}+1\right)B_n-\frac{a_n}{2\left(\cosh\frac{n\pi\alpha}{2}+\sinh\frac{n\pi\alpha}{2}\right)}\\
B_n&=\frac{a_nn\pi K\left\{1-\frac{1}{2\left(1+\tanh\frac{n\pi\alpha}{2}\right)}\right\}+a_n}{n\pi K\left\{\left(\frac{n\pi\alpha}{2}+1\right)\cosh\frac{n\pi\alpha}{2}-\frac{n\pi\alpha}{2}\sinh\frac{n\pi\alpha}{2}\right\}+2\left(\sinh\frac{n\pi\alpha}{2}+\cosh\frac{n\pi\alpha}{2}\right)}\\
C_n&=a_n+A_n\cosh\frac{n\pi\alpha}{2}+B_n\frac{n\pi\alpha}{2}\sinh\frac{n\pi\alpha}{2}-n\pi\alpha\,D_n\\
D_n&=-\frac{a_n}{2}+\cosh\frac{n\pi\alpha}{2}\,B_n\\
a_n&=\frac{64b^4p_0}{n^5\pi^5}\sin\frac{n\pi d}{2b}
\end{aligned}\right\}\quad\text{..(82)}$$

If $K=0$ (No support)

$$\left.\begin{aligned}
A_n&=-\frac{\frac{n\pi\alpha}{2}+2}{2\cosh\frac{n\pi\alpha}{2}\left(1+\tanh\frac{n\pi\alpha}{2}\right)}a_n\\
B_n&=\frac{a_n}{2\cosh\frac{n\pi\alpha}{2}\left(1+\tanh\frac{n\pi\alpha}{2}\right)}\\
D_n&=-\frac{\tanh\frac{n\pi\alpha}{2}}{2\left(1+\tanh\frac{n\pi\alpha}{2}\right)}a_n,\ \text{etc.}
\end{aligned}\right\}$$

These values of constants are the same as in eq. (18), Art. 2, Chapt. 2 if we put $\alpha=3\alpha$ & $\beta=\alpha$.

If $K=\infty$ or the supports are rigid,

$$B_n=\frac{a_n\left\{1-\frac{1}{2\left(1+\tanh\frac{n\pi\alpha}{2}\right)}\right\}}{\left(\frac{n\pi\alpha}{2}+2\right)\cosh\frac{n\pi\alpha}{2}-\frac{n\pi\alpha}{2}\sinh\frac{n\pi\alpha}{2}}\quad\text{......(83)}$$

Other consts. will be obtained from eq. (82).

See **Pl. 11** for the effect of the support yielding.

4. Note to Art. 3, Chapt. 5.

Though it is a little difficult in case ($\alpha<1$) to give the values of A_n, B_n, C_n, D_n, we can obtain them by solving the following simultaneous equations.

$$C_n \tanh n\pi\alpha + D_n n\pi\alpha = a_n + A_n \cosh\frac{n\pi\alpha}{2} + B_n \frac{n\pi\alpha}{2} \sinh\frac{n\pi\alpha}{2} \quad \ldots\ldots\ldots\ldots(84)$$

$$-C_n - (1 + n\pi\alpha \tanh n\pi\alpha) D_n = A_n \sinh\frac{n\pi\alpha}{2} + \left(\sinh\frac{n\pi\alpha}{2} + \frac{n\pi\alpha}{2} \cosh\frac{n\pi\alpha}{2}\right) B_n \quad \ldots\ldots\ldots\ldots(85)$$

$$C_n \tanh n\pi\alpha + (2 \tanh n\pi\alpha + n\pi\alpha) D_n = A_n \cosh\frac{n\pi\alpha}{2} + \left(2 \cosh\frac{n\pi\alpha}{2} + \frac{n\pi\alpha}{2} \sinh\frac{n\pi\alpha}{2}\right) B_n \quad \ldots\ldots\ldots\ldots(86)$$

$$(n\pi K \tanh n\pi\alpha - 1) C_n + \{(n\pi K - \tanh n\pi\alpha) n\pi\alpha - 3\} D_n$$
$$= A_n \sinh\frac{n\pi\alpha}{2} + \left\{3 \sinh\frac{n\pi\alpha}{2} + \frac{n\pi\alpha}{2} \cosh\frac{n\pi\alpha}{2}\right\} B_n \quad \ldots\ldots\ldots\ldots(87)$$

P. S. This chapter is not complete, and as we have still many unsolved problems to fit the "Platten Theorie" for practice, I hope that more comprehensive treatment by some able engineers will appear in this journal.

In the end I must beg excuse for being so presumptuous as to give self-decided opinions.

The end.

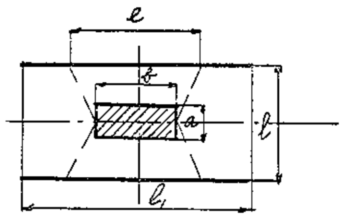
June, 1931.

附 錄 內務省道路構造に關する細則案拔萃

第29條 自動車荷重及び輾壓機荷重を負載する鐵筋コンクリート版の有效幅は第一號に在りては a, 第二號に在りては b が 2 メートルを超過する場合を除くの外次の各式に依り之れを算出すべし。

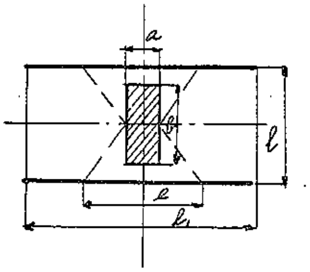
1. 縱桁を有する版

$$e=\frac{2l}{3}+a$$
$$\gtrless 2$$
$$\gtrless l_1$$



2. 橫桁を有する版

$$e=\frac{2l}{3}+b$$
$$\gtrless 2$$
$$\gtrless l_1$$



a：分布面の車輛進行の方向に於ける長（メートル）

b：分布面の車輛進行と直角の方向に於ける長（メートル）

e：版の有效幅（メートル）

l：版の徑間（メートル）

l_1：版の幅（メートル）

第30條 短徑間 l_1 長徑間 l_2 とを兩邊とする矩形版が網狀鐵筋又は縱橫の鐵筋を有し其の四邊に於て支承さるゝ場合に在りては左の定に依り其の荷重を兩徑間に分配すべし。

1. 長徑間が短徑間の2倍を超過せざるときは荷重が短徑間に働く割合は $\left(1.5-\frac{l_1}{l_2}\right)$ にして長徑間に働く割合は $\left(\frac{l_1}{l_2}-0.5\right)$ と假定すべし。

2. 長徑間が短徑間の2倍を超過するときは全荷重が短徑間のみに働くものと假定すべし。

Table. Values of $\cosh\frac{n\pi x}{2b}$, $\sinh\frac{n\pi x}{2b}$, $\tanh\frac{n\pi x}{2b}$ etc.

I. $n = 1$.

$\frac{x}{b}$	$\frac{n\pi x}{2b}$	$\cosh\frac{n\pi x}{2b}$	$\sinh\frac{n\pi x}{2b}$	$\tanh\frac{n\pi x}{2b}$	$\cos\frac{n\pi x}{2b}$	$\sin\frac{n\pi x}{2b}$
0.1	0.157	1.01235	0.15765	0.15572	0.98770	0.15636
0.2	0.314	1.04970	0.31919	0.30407	0.95111	0.30887
0.3	0.471	1.11299	0.48861	0.43901	0.89111	0.45378
0.4	0.628	1.20376	0.67010	0.55667	0.80920	0.58753
0.5	0.785	1.32461	0.86867	0.65579	0.70711	0.70711
0.6	0.942	1.47748	1.08763	0.73614	0.58817	0.80874
0.7	1.100	1.66852	1.33565	0.80050	0.45360	0.89121
0.8	1.257	1.89968	1.61518	0.85023	0.30867	0.95117
0.9	1.414	2.17584	1.93242	0.88813	0.15714	0.98758
1.0	1.571	2.50918	2.30130	0.91717	0	1.
1.1	1.728	2.90351	2.72587	0.93882		
1.2	1.885	3.36909	3.21726	0.95493		
1.3	2.042	3.91789	3.78812	0.96698		
1.4	2.199	4.56345	4.45254	0.97570		
1.5	2.356	5.32275	5.22797	0.98219		
1.6	2.513	6.21146	6.13044	0.98696		
1.7	2.670	7.25461	7.18536	0.99046		
1.8	2.827	8.47695	8.41776	0.99302		
1.9	2.985	9.91852	9.86798	0.99490		
2.0	3.142	11.59195	11.54874	0.99627		
2.1	3.299	13.57476	13.53788	0.99728		
2.2	3.456	15.92420	15.89277	0.99808		
2.3	3.613	18.49655	18.46950	0.99854		
2.4	3.770	21.70156	21.67851	0.99894		
2.5	3.927	25.38686	25.36716	0.99922		

These tables were taken from Prof. Hayashi's "Fünfstellige Tafeln der Kreis- u. Hyperbel Funktionen."

II. $n = 3$

$\frac{x}{b}$	$\frac{n\pi x}{2b}$	$\cosh\frac{n\pi x}{2b}$	$\sinh\frac{n\pi x}{2b}$	$\tanh\frac{n\pi x}{2b}$	$\cos\frac{n\pi x}{2b}$	$\sin\frac{n\pi x}{2b}$
0.1	0.471	1.11299	0.48861	0.43901	0.89111	0.45378
0.2	0.942	1.47748	1.08763	0.73614	0.58817	0.80874
0.3	1.414	2.17777	1.93460	c.88834	0.15615	0.98773
0.4	1.885	3.36909	3.21726	c.95493	-0.30906	0.95104
0.5	2.356	5.32275	5.22797	0.98219	-0.70711	0.70711
0.6	2.827	8.47695	8.41776	0.99302	-0.95092	0.30943
0.7	3.299	13.57476	13.53788	0.99728	-0.99748	-0.15775
0.8	3.770	21.70156	21.67851	0.99894	-0.80896	-0.58786
0.9	4.241	34.71113	34.69672	0.99958	-0.45501	-0.89048
1.0	4.712	55.66338	55.65440	0.99984	0	-1
1.1	5.184	88.84422	88.83859	0.99994	0.45076	-0.89265
1.2	5.655	142.14749	142.14397	0.99998	0.80615	-0.59172
1.3	6.126	229.71917	229.71699	0.99999	0.98829	-0.15259
1.4	6.597	367.54827	367.54691	1.00000	0.95023	0.31154
1.5	7.069	587.24202	587.24117	1.00000	0.70711	0.70711

III. $n = 5$

$\frac{x}{b}$	$\frac{n\pi x}{2b}$	$\cosh\frac{n\pi x}{2b}$	$\sinh\frac{n\pi x}{2b}$	$\tanh\frac{n\pi x}{2b}$	$\cos\frac{n\pi x}{2b}$	$\sin\frac{n\pi x}{2b}$
0.1	0.785	1.32461	0.86867	0.65579	0.70711	0.70711
0.2	1.571	2.50918	2.30130	0.91717	0	1
0.3	2.356	5.32275	5.22797	0.98219	-0.70711	0.70711
0.4	3.142	11.59195	11.54874	0.99627	-1	0
0.5	3.927	25.38686	25.36716	0.99922	-0.70711	-0.70711
0.6	4.712	55.66338	55.65440	0.99984	0	-1
0.7	5.498	122.07758	122.07348	0.99997	0.70711	-0.70711
0.8	6.283	267.74676	267.74489	0.99999	1.	0
0.9	7.069	587.24202	587.24117	1.	0.70711	0.70711
1.0	7.854	1287.98544	1287.98505	1.	0	1

$n = 7$.

$\frac{x}{b}$	$\frac{n\pi x}{2b}$	$\cosh\frac{n\pi x}{2b}$	$\sinh\frac{n\pi x}{2b}$	$\tanh\frac{n\pi x}{2b}$	$\cos\frac{n\pi x}{2b}$	$\sin\frac{n\pi x}{2b}$
0.1	1.099	1.66852	1.33565	0.80050	0.45360	0.89121
0.2	2.199	4.56345	4.45254	0.97570	-0.58769	0.80908
0.3	3.299	13.57476	13.53788	0.99728	-0.98748	-0.15775
0.4	4.398	40.73157	40.71930	0.99970	-0.30733	-0.95160
0.5	5.498	122.07758	122.07549	0.99997	0.70711	-0.70711
0.6	6.597	367.54827	367.54671	1.00000	0.95023	0.31154
0.7	7.697	1104.17422	1104.17377	1.00000	0.15337	0.98817
0.8	8.796	3317.12208	3317.12193	1.00000	-0.81109	0.58492
0.9	9.896	9965.18524	9965.18517	1.00000	-0.88919	-0.45754

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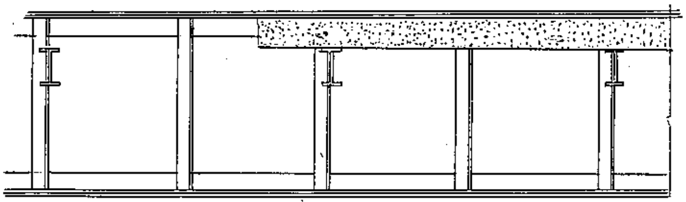
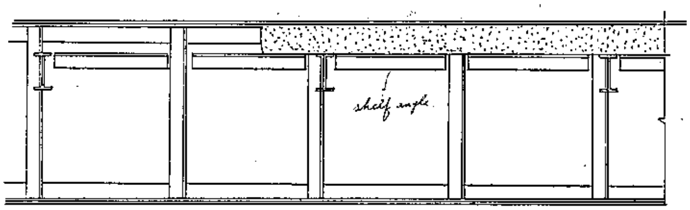
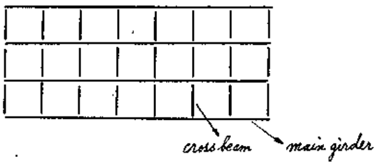
Pl. 1. Two & Four Side Support Slabs.

(1)

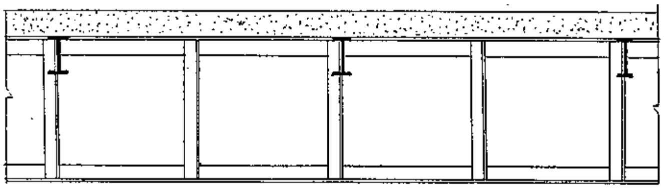


(2)

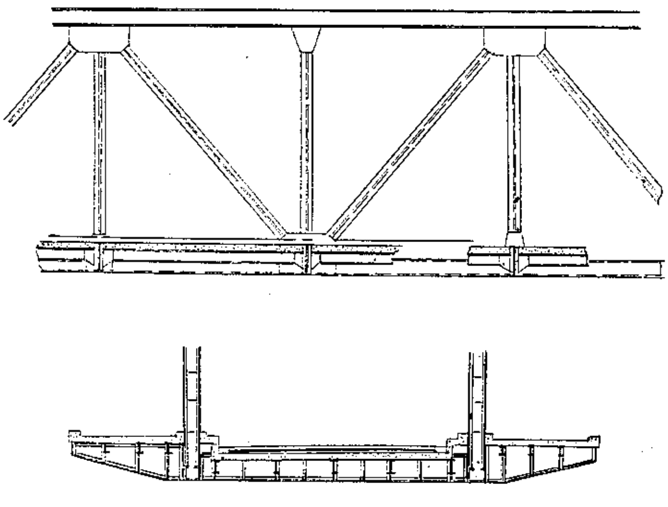




(3)



(4)



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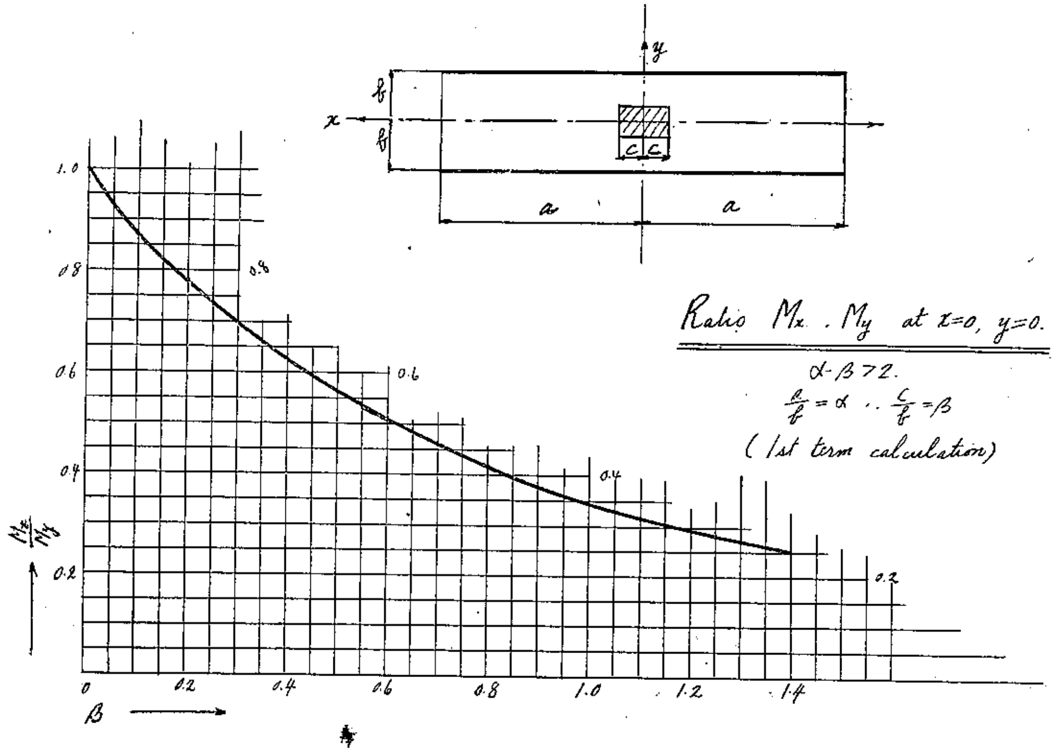
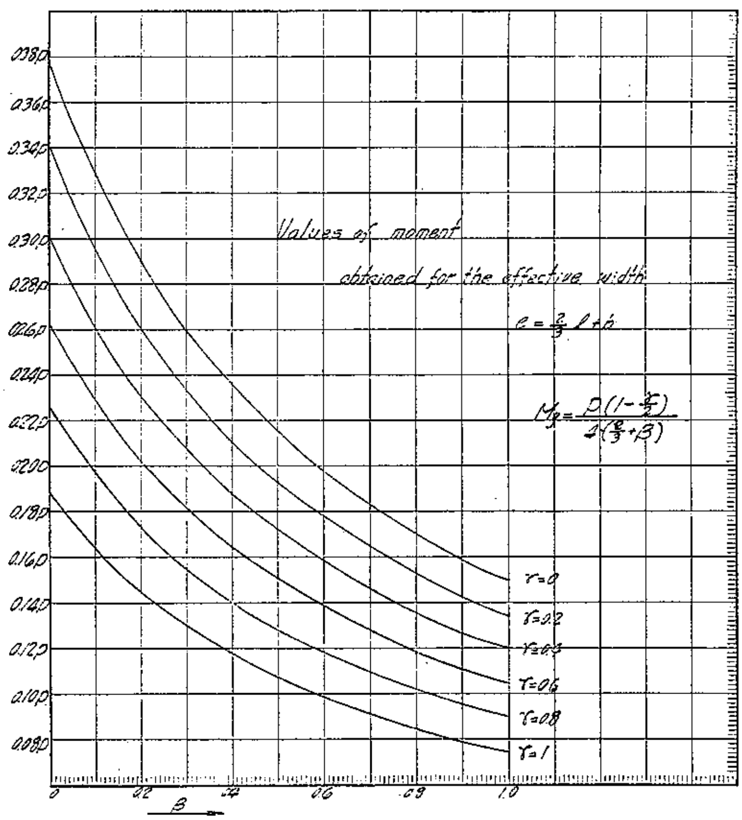
Pl. 2.



Pl. 3.



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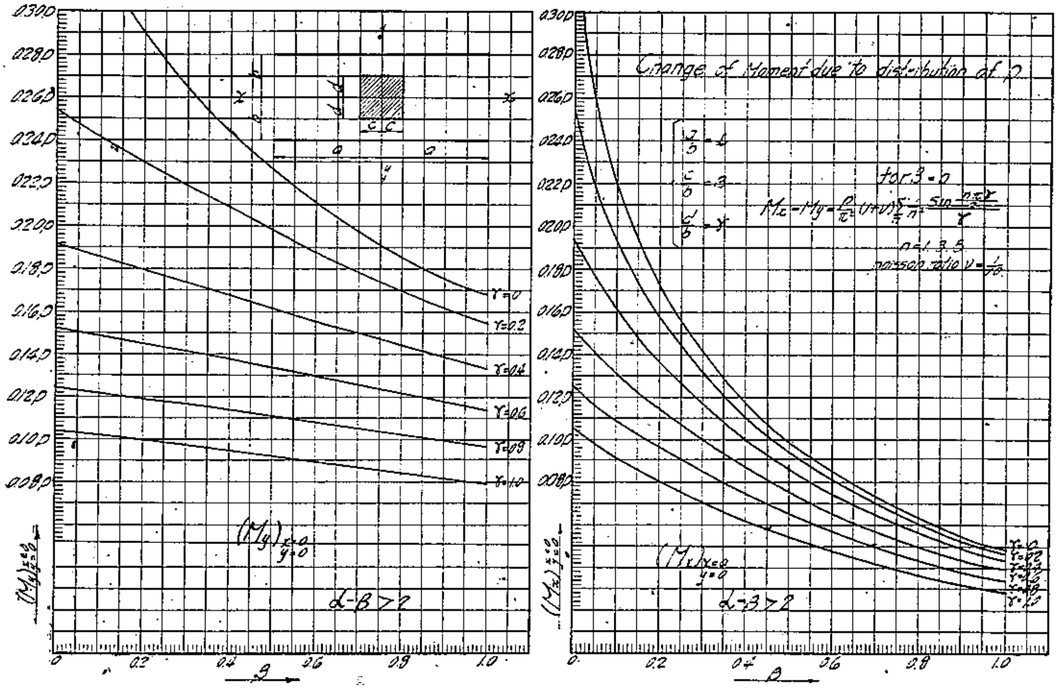
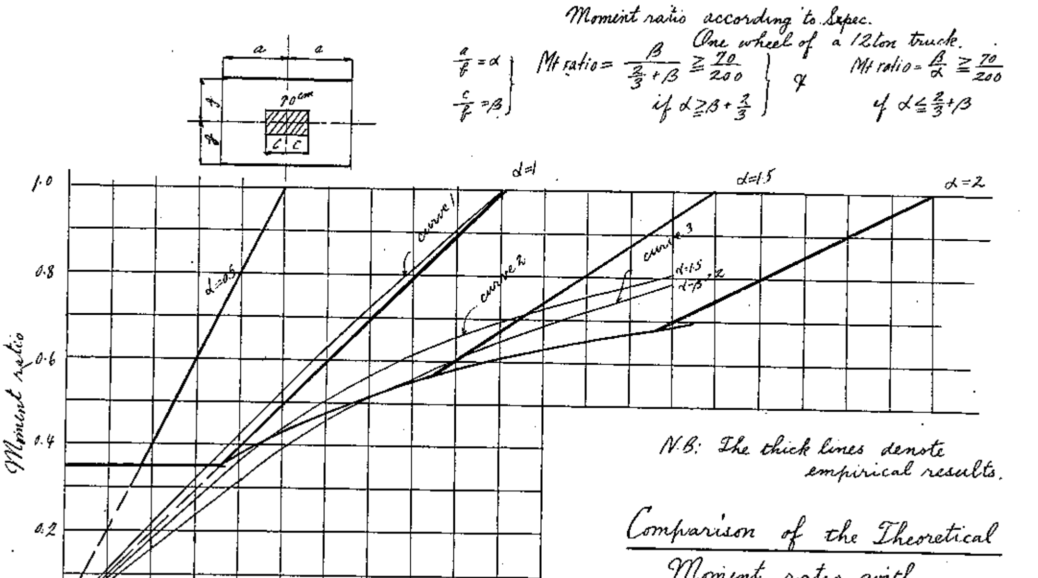
Pl. 4.

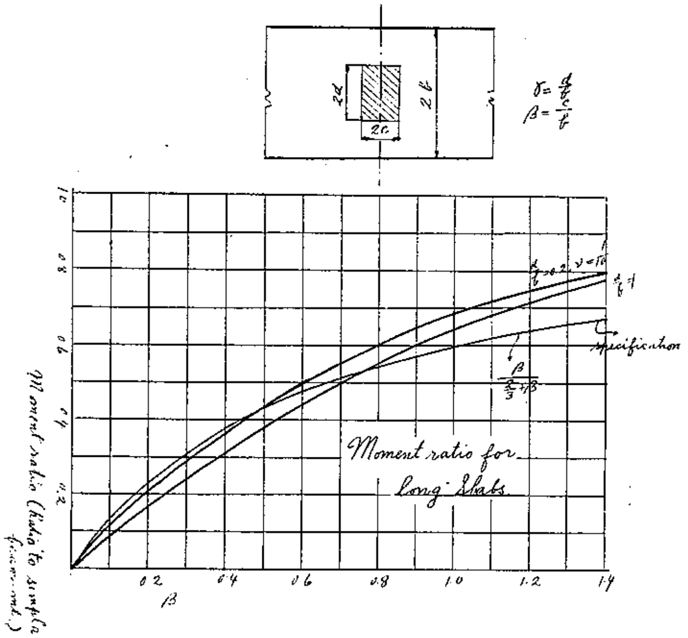


Pl. 5.

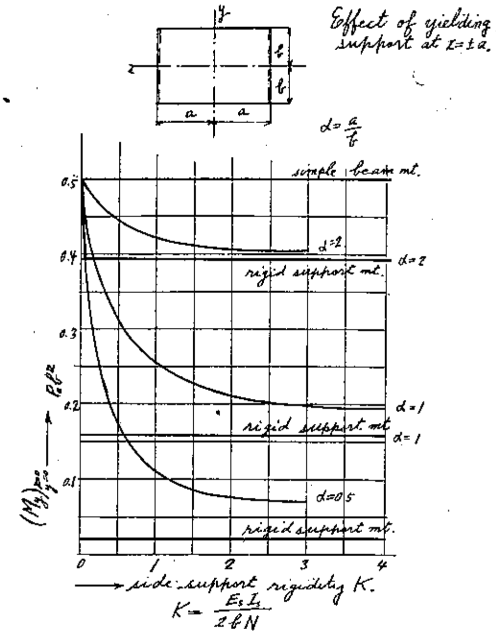


Comparison of the Theoretical Moment ratio with the Empirical one.

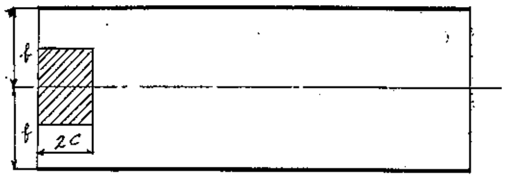
Pl. 6.



Pl. 8.



Pl. 7.



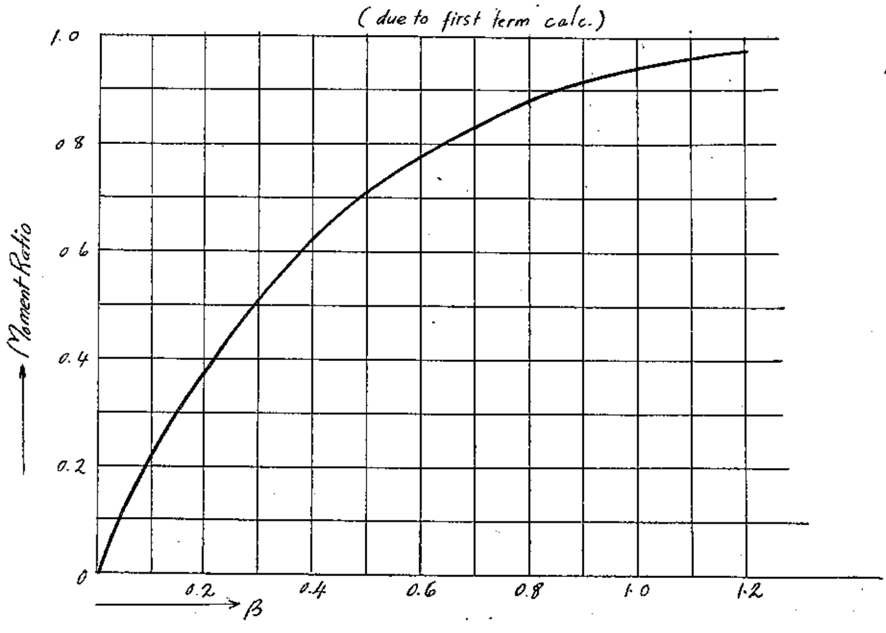
$\frac{c}{b} = \beta$

$\alpha - \beta > 2$

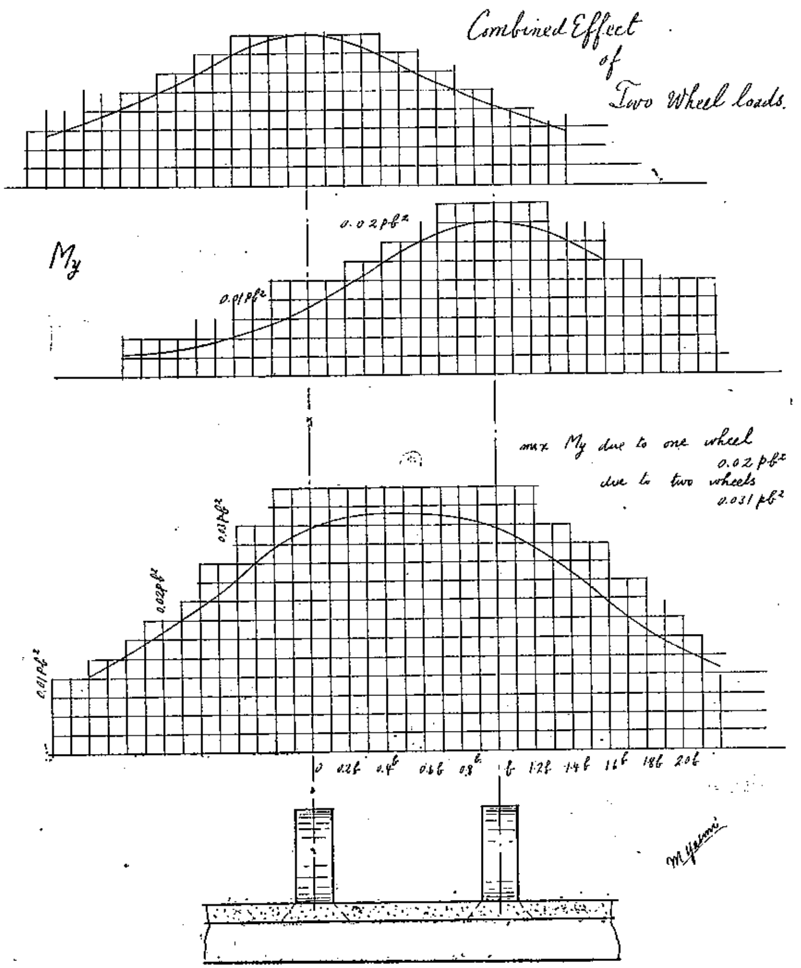
Moment Ratio for Extreme Loading.

(due to first term calc.)

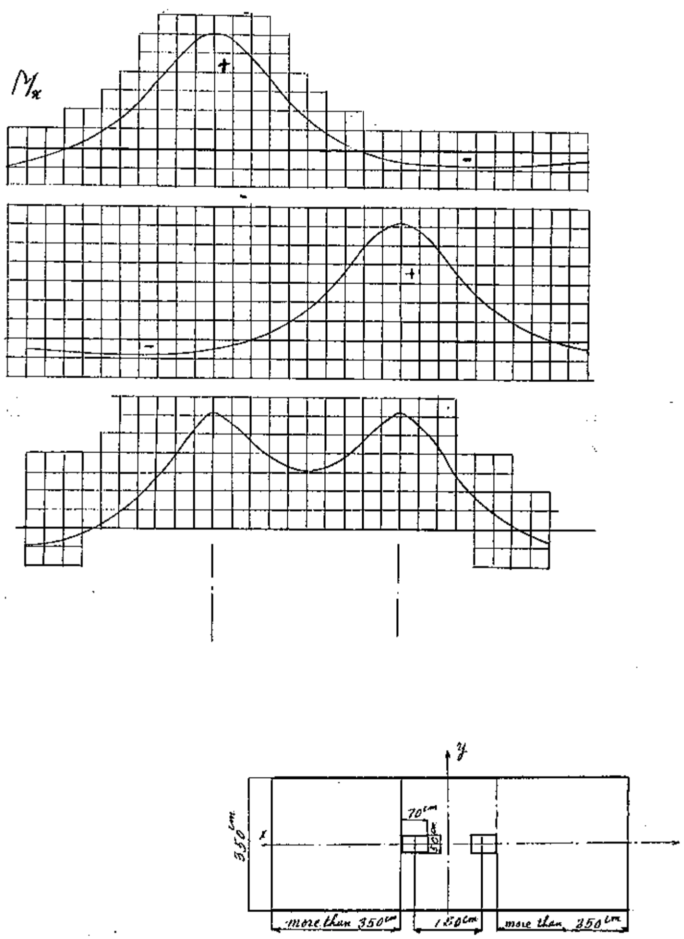
Moment ratio $= \dfrac{(M_y)_{x=0,\,y=0}}{\text{Simple Beam Mt}}$



Pl. 9



Pl. 10.



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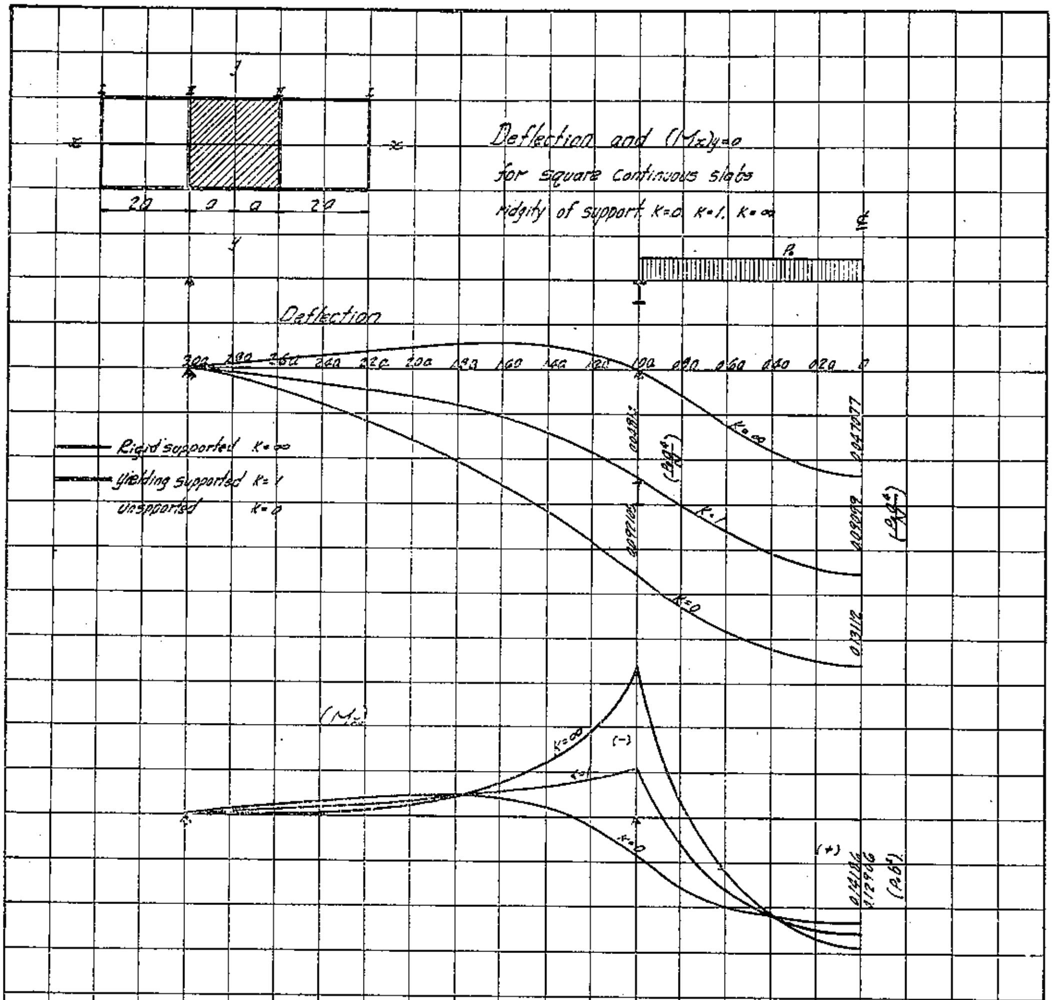
Pl. 11.



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