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ON THE STRESSES AROUND A HORIZONTAL CIRCULAR HOLE IN GRAVITATING ELASTIC SOLID

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Synopsis

The stresses around a horizontal circular hole in gravitating elastic solid with the horizontal top surface were calculated by the method of stress-function.

Firstly, the stresses vanishing at infinity and having the same values with opposite signs on a circular boundary with those in undisturbed gravitating elastic solid were obtained. Superposing these with the undisturbed ones we obtained the sufficiently approximate values of the required stresses in the case the hole is not situated very near the top surface.

The results worth noticing are as follows:

1. Just as the plate with a circular hole pressed at both ends the circumferential tension takes place at the top and bottom of the hole if we assume the Poisson's ratio $\sigma=\frac{1}{5}$, and the maximum circumferential compression takes place on its both sides.

2. When the hole lies sufficiently deep from the top surface the above mentioned stresses are symmetrical with respect to the horizontal diameter of the hole. When it approaches to the top surface the bottom tension becomes greater than the crown tension.

These mathematical results were ascertained by the model experiments with agar-agar, and they might be used as an elementary theory of a circular tunnel without lining.

1. Introduction.

The problem of elasticity of the plate with a circular hole considered in connection with the rivet hole has become almost classical. Several investigators treated the problem both theoretically and experimentally, amongst whom Profs. Suyehiro and Yokota of Tokyo Imperial University and Profs. Kirsch, Leon, Föppl and Morley may be noted for their theoretical investigations. Recently W. G. Bickley took up the same problem and treated it elaborately by the method of stress-function.[1] This has stimulated the present writer to investigate the stress problem around a horizontal circular hole in

(1 Phil. Trans. Roy. Soc. A Vol. 227 (1928).

gravitating elastic solid by the same method. The problem is usually treated as the *generalized plane stresses*, when it is applied to a thin plate. But here we treat it as the *plane strains* with the aim of applying the results to the stability of a circular tunnel without lining.[1]

2. The Stress Distribution in Undisturbed Gravitating Elastic Solid with Horizontal Top Surface.

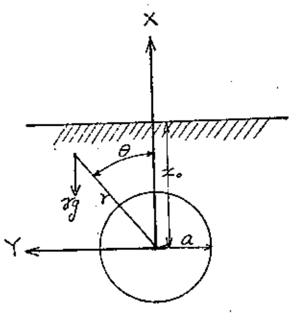

Fig. 1

Taking the co-ordinate axes as shown in **Fig. 1**, we have the equation of equilibrium

$$\left.\begin{aligned}\frac{\partial \widehat{xx}}{\partial x}+\frac{\partial \widehat{xy}}{\partial y}&=\rho g\\ \frac{\partial \widehat{yy}}{\partial y}+\frac{\partial \widehat{xy}}{\partial x}&=0\end{aligned}\right\}\qquad\qquad (1)$$

where ρ is the density of mass, i.e., ρg is the unit weight of mass. From this we obtain the stress-function F defined by

$$\nabla^4 F=0 \qquad\qquad (2)$$

just the same as in the case of mass with no body force. The stresses are given by

$$\widehat{xx}=\frac{\partial^2 F}{\partial y^2}+\int \rho g dx, \quad \widehat{yy}=\frac{\partial^2 F}{\partial x^2}, \quad \widehat{xy}=-\frac{\partial^2 F}{\partial x \partial y} \qquad\qquad (3)$$

Assuming that ρg is independent of the depth and that the strains will not be produced laterally, we easily obtain the following values of the stresses in undisturbed gravitating elastic solid.

$$\widehat{xx}=-\rho g(x_0-x), \quad \widehat{yy}=\widehat{zz}=-\frac{\sigma}{1-\sigma}\rho g(x_0-x), \quad \widehat{xy}=\widehat{zx}=\widehat{yz}=0 \qquad (4)$$

where σ is the Poisson's ratio.

If we write these in the polar co-ordinates, we have

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(1) One of my colleagues Prof. Y. Tanaka has kindly noticed me that Dr. H. Schmid treated the same problem in his treatise, "Statische Probleme des Tunnel- und Druckstollenbaues" (1926). He treated it rather technically and the result is more complicated than mine. I wonder if it may be regarded as the same with mine.

$$\left.\begin{aligned}
\widehat{rr}&=-\frac{W}{2}\left(1+\frac{\sigma}{1-\sigma}\right)+\frac{V}{4}\left(3+\frac{\sigma}{1-\sigma}\right)\frac{r}{a}\cos\theta-\frac{W}{2}\left(1-\frac{\sigma}{1-\sigma}\right)\cos 2\theta\\
&\quad+\frac{V}{4}\left(1-\frac{\sigma}{1-\sigma}\right)\frac{r}{a}\cos 3\theta\\
\widehat{\theta\theta}&=-\frac{W}{2}\left(1+\frac{\sigma}{1-\sigma}\right)+\frac{V}{4}\left(1+\frac{3\sigma}{1-\sigma}\right)\frac{r}{a}\cos\theta+\frac{W}{2}\left(1-\frac{\sigma}{1-\sigma}\right)\cos 2\theta\\
&\quad-\frac{V}{4}\left(1-\frac{\sigma}{1-\sigma}\right)\frac{r}{a}\cos 3\theta\\
\widehat{r\theta}&=-\frac{V}{4}\left(1-\frac{\sigma}{1-\sigma}\right)\frac{r}{a}\sin\theta+\frac{W}{2}\left(1-\frac{\sigma}{1-\sigma}\right)\sin 2\theta\\
&\quad-\frac{V}{4}\left(1-\frac{\sigma}{1-\sigma}\right)\frac{r}{a}\sin 3\theta\\
\widehat{zz}&=\sigma(\widehat{rr}+\widehat{\theta\theta})
\end{aligned}\right\}\qquad(5)$$

Here $W=\rho g x_0$ and $V=\rho g a$, where x_0 is the distance of the center of the hole from the top surface and a is the radius of the hole.

3. The Stresses with Any Possible Values on the Circumference of a Circular Hole and Vanishing at Infinite Distance from it.

In the case of a thin plate (i.e. in the state of the generalized plane stresses) the stress-function of the stresses vanishing at infinity and satisfying any possible conditions over a circular boundary was obtained in the general form by Prescott[(1)] and Bickley.[(2)] In adopting their results here we have to reduce them into the state of the *plane strains*. This reduction naturally affects only the terms involving the elastic constants, and the elastic constants enter into the cyclic terms only. The cyclic terms in the stress-function in the polar co-ordinates are as follows:

$$F=A_0 r\theta\sin\theta+B_0 r\theta\cos\theta+A_1 r\log r\cos\theta+B_1 r\log r\sin\theta \quad\ldots\ldots\ldots\ldots\ldots(6)$$

The coefficients A_0, B_0, A_1, B_1 are not independent. They must be determined to give the single-valued displacements.

The displacements u, v along x and y in the state of the *plane strains* are

(1) Prescott: Applied Elasticity p. 381.

(2) Bickley: loc. cit.

$$\left.\begin{aligned} 2\mu u &= \xi - \frac{\partial F}{\partial x} \\ 2\mu v &= \eta - \frac{\partial F}{\partial y} \end{aligned}\right\} \qquad \cdots\cdots (7)^{(1)}$$

where

$$\left.\begin{aligned} \frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} &= (\lambda + 2\mu)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\ -\frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial x} &= \mu\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned}\right\} \qquad \cdots\cdots (8)$$

and λ, μ are Lamé's constants.

Since

$$\widehat{xx} + \widehat{yy} = \nabla^2 F = 2(\lambda + \mu)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} = \frac{\lambda + 2\mu}{2(\lambda + \mu)} \nabla^2 F$$

and as $\lambda = E\sigma/(1+\sigma)(1-2\sigma)$ and $\mu = E/2(1+\sigma)$ (where E is the Young's modulus), we have

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} = (1-\sigma)\nabla^2 F \qquad \cdots\cdots (9)$$

Nabla square is not changed in value if we transform the co-ordinates from x, y to r, θ, i.e. $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.

Therefore we have from (6)

$$\nabla^2 F = \frac{2}{r}\left\{(A_0 + A_1)\cos\theta - (B_0 - B_1)\sin\theta\right\}$$

$$\therefore \quad \left.\begin{aligned} \xi &= (1-\sigma)\{2(A_0 + A_1)\log r + 2(B_0 - B_1)\theta\} \\ \eta &= (1-\sigma)\{2(A_0 + A_1)\theta - 2(B_0 - B_1)\log r\} \end{aligned}\right\} \qquad \cdots\cdots (10)$$

And we have

$$\left.\begin{aligned} 2\mu u &= 2(1-\sigma)(B_0 - B_1)\theta - B_0\theta + \text{acyclic terms} \\ 2\mu v &= 2(1-\sigma)(A_0 + A_1)\theta - A_0\theta + \text{acyclic terms} \end{aligned}\right\} \qquad \cdots\cdots (11)$$

Putting the coefficients of the cyclic terms equal to zero, we have

$$B_1 = \frac{1}{2}\left(1 - \frac{\sigma}{1-\sigma}\right)B_0 \quad \text{and} \quad A_1 = -\frac{1}{2}\left(1 - \frac{\sigma}{1-\sigma}\right)A_0$$

Therefore we have the following stress-function which is sufficiently general for our present purpose.

(1) Love: Elasticity 3rd ed. p. 205.

$$F=a^2\Big[A_0\log\frac{r}{a}+A_1\Big\{\frac{r}{a}\,\theta\sin\theta-\frac{1}{2}\Big(1-\frac{\sigma}{1-\sigma}\Big)\frac{r}{a}\log\frac{r}{a}\cos\theta\Big\}$$
$$+B_1\Big\{\frac{r}{a}\,\theta\cos\theta+\frac{1}{2}\Big(1-\frac{\sigma}{1-\sigma}\Big)\frac{r}{a}\log\frac{r}{a}\sin\theta\Big\}+C_1\frac{a}{r}\cos\theta+D_1\frac{a}{r}\sin\theta$$
$$+\sum_{m=2}^{\infty}\Big\{a^{m-2}r^{-m+2}(A_m\cos m\theta+B_m\sin m\theta)+a^m r^{-m}(C_m\cos m\theta+D_m\sin m\theta)\Big\}\Big] \quad\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

The stresses derived from this are

$$\left.\begin{aligned}
\widehat{rr}&=A_0\frac{a^2}{r^2}+\frac{1}{2}\Big(3+\frac{\sigma}{1-\sigma}\Big)\frac{a}{r}\Big(A_1\cos\theta-B_1\sin\theta\Big)-\frac{2a^3}{r^3}\Big(C_1\cos\theta\\
&\quad+D_1\sin\theta\Big)-\sum_{m=2}^{\infty}\Big\{(m+2)(m-1)a^m r^{-m}(A_m\cos m\theta+B_m\sin m\theta)\\
&\qquad+m(m+1)a^{m+2}r^{-m-2}(C_m\cos m\theta+D_m\sin m\theta)\Big\}\\
\widehat{\theta\theta}&=-A_0\frac{a^2}{r^2}-\frac{1}{2}\Big(1-\frac{\sigma}{1-\sigma}\Big)\frac{a}{r}\Big(A_1\cos\theta-B_1\sin\theta\Big)+\frac{2a^3}{r^3}\Big(C_1\cos\theta\\
&\quad+D_1\sin\theta\Big)+\sum_{m=2}^{\infty}\Big\{(m-2)(m-1)a^m r^{-m}(A_m\cos m\theta+B_m\sin m\theta)\\
&\qquad+m(m+1)a^{m+2}r^{-m-2}(C_m\cos m\theta+D_m\sin m\theta)\Big\}\\
\widehat{r\theta}&=-\frac{1}{2}\Big(1-\frac{\sigma}{1-\sigma}\Big)\frac{a}{r}\Big(A_1\sin\theta+B_1\cos\theta\Big)-\frac{2a^3}{r^3}\Big(C_1\sin\theta-D_1\cos\theta\Big)\\
&\quad+\sum_{m=2}^{\infty}\Big\{m(m-1)a^m r^{-m}(-A_m\sin m\theta+B_m\cos m\theta)\\
&\qquad+m(m+1)a^{m+2}r^{-m-2}(-C_m\sin m\theta+D_m\cos m\theta)\Big\}
\end{aligned}\right\}\ \ldots (13)$$

4. The Stress Distribution Near the Hole in Gravitating Elastic Solid with Horizontal Top Surface.

The radial and shearing stresses in undisturbed elastic solid on the circumference of a circle is

$$\left.\begin{aligned}
(\widehat{rr})_a&=-\frac{W}{2}\Big(1+\frac{\sigma}{1-\sigma}\Big)+\frac{V}{4}\Big(3+\frac{\sigma}{1-\sigma}\Big)\cos\theta-\frac{W}{2}\Big(1-\frac{\sigma}{1-\sigma}\Big)\cos2\theta\\
&\quad+\frac{V}{4}\Big(1-\frac{\sigma}{1-\sigma}\Big)\cos3\theta\\
(\widehat{r\theta})_a&=-\frac{V}{4}\Big(1-\frac{\sigma}{1-\sigma}\Big)\sin\theta+\frac{W}{2}\Big(1-\frac{\sigma}{1-\sigma}\Big)\sin2\theta-\frac{V}{4}\Big(1-\frac{\sigma}{1-\sigma}\Big)\sin3\theta
\end{aligned}\right\}(14)$$

Picking up the relevant terms from (13) on putting $r=a$, and comparing with $-(\widehat{rr})_a$ and $-(\widehat{r\theta})_a$ of the above expression, we have

$$\left.\begin{aligned} A_0=\frac{W}{2}\left(1+\frac{\sigma}{1-\sigma}\right),\quad A_1=-\frac{V}{2},\quad A_2=-\frac{W}{2}\left(1-\frac{\sigma}{1-\sigma}\right),\quad A_3=\frac{V}{8}\left(1-\frac{\sigma}{1-\sigma}\right) \\ C_1=0,\quad C_2=\frac{W}{4}\left(1-\frac{\sigma}{1-\sigma}\right),\quad C_3=-\frac{V}{12}\left(1-\frac{\sigma}{1-\sigma}\right) \end{aligned}\right\}\quad(15)$$

Therefore we have

$$\left.\begin{aligned} \widehat{rr}=&\frac{W}{2}\left(1+\frac{\sigma}{1-\sigma}\right)\left(\frac{a}{r}\right)^2-\frac{V}{4}\left(3+\frac{\sigma}{1-\sigma}\right)\left(\frac{a}{r}\right)\cos\theta+\frac{W}{2}\left(1-\frac{\sigma}{1-\sigma}\right)\left\{4\left(\frac{a}{r}\right)^2\right. \\ &\left.-3\left(\frac{a}{r}\right)^4\right\}\cos 2\theta-\frac{V}{4}\left(1-\frac{\sigma}{1-\sigma}\right)\left\{5\left(\frac{a}{r}\right)^3-4\left(\frac{a}{r}\right)^5\right\}\cos 3\theta \\ \widehat{\theta\theta}=&-\frac{W}{2}\left(1+\frac{\sigma}{1-\sigma}\right)\left(\frac{a}{r}\right)^2-\frac{V}{4}\left(1-\frac{\sigma}{1-\sigma}\right)\left(\frac{a}{r}\right)\cos\theta \\ &+\frac{W}{2}\left(1-\frac{\sigma}{1-\sigma}\right)3\left(\frac{a}{r}\right)^4\cos 2\theta-\frac{V}{4}\left(1-\frac{\sigma}{1-\sigma}\right)\left\{4\left(\frac{a}{r}\right)^5-\left(\frac{a}{r}\right)^3\right\}\cos 3\theta \\ \widehat{r\theta}=&\frac{V}{4}\left(1-\frac{\sigma}{1-\sigma}\right)\left(\frac{a}{r}\right)\sin\theta-\frac{W}{2}\left(1-\frac{\sigma}{1-\sigma}\right)\left\{3\left(\frac{a}{r}\right)^4-2\left(\frac{a}{r}\right)^2\right\}\sin 2\theta \\ &+\frac{V}{4}\left(1-\frac{\sigma}{1-\sigma}\right)\left\{4\left(\frac{a}{r}\right)^5-3\left(\frac{a}{r}\right)^3\right\}\sin 3\theta \end{aligned}\right\}\quad(16)$$

These are the stresses vanishing at infinity and giving $-(\widehat{rr})_a$ and $-(\widehat{r\theta})_a$ on the circular hole. Practically we may assume that they vanish readily at a certain distance from the hole. Therefore, combining these with the undisturbed stresses (5) we obtain the required stresses, if the center of the hole is not situated *very near* the top surface.

$$\left.\begin{aligned} \widehat{rr}=&-\frac{W}{2}\left(1+\frac{\sigma}{1-\sigma}\right)\left\{1-\left(\frac{a}{r}\right)^2\right\}+\frac{V}{4}\left(3+\frac{\sigma}{1-\sigma}\right)\left\{\frac{r}{a}-\frac{a}{r}\right\}\cos\theta \\ &-\frac{W}{2}\left(1-\frac{\sigma}{1-\sigma}\right)\left\{1-4\left(\frac{a}{r}\right)^2+3\left(\frac{a}{r}\right)^4\right\}\cos 2\theta \\ &+\frac{V}{4}\left(1-\frac{\sigma}{1-\sigma}\right)\left\{\frac{r}{a}-5\left(\frac{a}{r}\right)^3+4\left(\frac{a}{r}\right)^5\right\}\cos 3\theta \\ \widehat{\theta\theta}=&-\frac{W}{2}\left(1+\frac{\sigma}{1-\sigma}\right)\left\{1+\left(\frac{a}{r}\right)^2\right\}+\frac{V}{4}\left\{\left(1+\frac{3\sigma}{1-\sigma}\right)\frac{r}{a}-\left(1-\frac{\sigma}{1-\sigma}\right)\frac{a}{r}\right\}\cos\theta \\ &+\frac{W}{2}\left(1-\frac{\sigma}{1-\sigma}\right)\left\{1+3\left(\frac{a}{r}\right)^4\right\}\cos 2\theta \end{aligned}\right\}\quad(17)$$

$$
\left.
\begin{aligned}
&-\frac{V}{4}\left(1-\frac{\sigma}{1-\sigma}\right)\left\{\frac{r}{a}-\left(\frac{a}{r}\right)^3+4\left(\frac{a}{r}\right)^5\right\}\cos 3\theta \\
\widehat{r\theta}=&-\frac{V}{4}\left(1-\frac{\sigma}{1-\sigma}\right)\left\{\frac{r}{a}-\frac{a}{r}\right\}\sin\theta \\
&+\frac{W}{2}\left(1-\frac{\sigma}{1-\sigma}\right)\left\{1+2\left(\frac{a}{r}\right)^2-3\left(\frac{a}{r}\right)^4\right\}\sin 2\theta \\
&-\frac{V}{4}\left(1-\frac{\sigma}{1-\sigma}\right)\left\{\frac{r}{a}+3\left(\frac{a}{r}\right)^3-4\left(\frac{a}{r}\right)^5\right\}\sin 3\theta
\end{aligned}
\right\}
$$

And

$$\widehat{zz}=\sigma(\widehat{rr}+\widehat{\theta\theta}),\qquad \widehat{rz}=\widehat{\theta z}=0$$

These will, of course, satisfy the equation of equilibrium in the polar co-ordinates:

$$
\left.
\begin{aligned}
&\frac{\partial\widehat{rr}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{r\theta}}{\partial\theta}+\frac{\widehat{rr}-\widehat{\theta\theta}}{r}=\rho g\ \cos\theta \\
&\frac{1}{r}\frac{\partial\widehat{\theta\theta}}{\partial\theta}+\frac{\partial\widehat{r\theta}}{\partial r}+2\frac{\widehat{r\theta}}{r}=-\rho g\ \sin\theta
\end{aligned}
\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots (18)
$$

(17) may be written in the following form ;

$$
\left.
\begin{aligned}
&\widehat{\quad}=k_1W+k_2V \\
&\widehat{\theta\theta}=l_1W+l_2V \\
&\widehat{r\vartheta}=m_1W+m_2V
\end{aligned}
\right\}\ [r\geqq a]\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)
$$

where

$$
\left.
\begin{aligned}
k_1=&-\frac{1+\frac{\sigma}{1-\sigma}}{2}\left\{1-\left(\frac{a}{r}\right)^2\right\}-\frac{1-\frac{\sigma}{1-\sigma}}{2}\left\{1-4\left(\frac{a}{r}\right)^2+3\left(\frac{a}{r}\right)^4\right\}\cos 2\theta \\
k_2=&\ \frac{3+\frac{\sigma}{1-\sigma}}{4}\left\{\frac{r}{a}-\frac{a}{r}\right\}\cos\theta+\frac{1-\frac{\sigma}{1-\sigma}}{4}\left\{\frac{r}{a}-5\left(\frac{a}{r}\right)^3+4\left(\frac{a}{r}\right)^5\right\}\cos 3\theta \\
l_1=&-\frac{1+\frac{\sigma}{1-\sigma}}{2}\left\{1+\left(\frac{a}{r}\right)^2\right\}+\frac{1-\frac{\sigma}{1-\sigma}}{2}\left\{1+3\left(\frac{a}{r}\right)^4\right\}\cos 2\theta \\
l_2=&\ \frac{1}{4}\left\{\left(1+\frac{3\sigma}{1-\sigma}\right)\left(\frac{r}{a}\right)-\left(1-\frac{\sigma}{1-\sigma}\right)\left(\frac{a}{r}\right)\right\}\cos\theta \\
&-\frac{1-\frac{\sigma}{1-\sigma}}{4}\left\{\frac{r}{a}-\left(\frac{a}{r}\right)^3+4\left(\frac{a}{r}\right)^5\right\}\cos 3\theta
\end{aligned}
\right\}(20)
$$

$$m_1 = \frac{1-\frac{\sigma}{1-\sigma}}{2}\left\{1+2\left(\frac{a}{r}\right)^2-3\left(\frac{a}{r}\right)^4\right\}\sin 2\theta$$

$$m_2 = -\frac{1-\frac{\sigma}{1-\sigma}}{4}\left\{\frac{r}{a}-\frac{a}{r}\right\}\sin\theta - \frac{1-\frac{\sigma}{1-\sigma}}{4}\left\{\frac{r}{a}+3\left(\frac{a}{r}\right)^3-4\left(\frac{a}{r}\right)^5\right\}\sin 3\theta$$

The terms with W have the property similar to the stresses around a circular hole of an infinitely extended plate which is pressed uniformly at $x=\pm\infty$. If we put $\sigma=0$ in the coefficients of W, these coincide exactly with it. The compressive intensity at $x=\pm\infty$ is W. These terms give the stresses which are symmetrical with respect to the y-axis.

The terms with V represent the remaining stresses due to the unbalanced weight coming from the gravity effect. These give the stresses which tend to infinity at the infinite distances upward and downward from the hole.

If we tabulate the values of these coefficients assuming $\sigma=\frac{1}{5}$ (for rock), we have

k_1 (Pl. I)

a/r \ $\theta°$	0 / 180	30 / 150	60 / 120	90
1	0	0	0	0
1/2	$-.539$	$-.504$	$-.434$	$-.398$
1/3	$-.773$	$-.667$	$-.445$	$-.334$
1/4	$-.872$	$-.729$	$-.443$	$-.300$
$1/\infty$	-1.000	$-.815$	$-.437$	$-.250$

k_2 (Pl. II)

a/r \ $\theta°$	0 / 180	30 / 150	60 / 120	90
1	0	0	0	0
1/2	±1.50	±1.055	$\pm.330$	0
1/3	±2.70	±1.876	$\pm.552$	0
1/4	±3.53	±2.639	$\pm.787$	0
$1/\infty$	$\pm\infty$	$\pm\infty$	$\pm\infty$	0

(The upper row of θ takes the upper signs and the lower row of θ takes the lower)

l_1 (Pl. III)

a/r \ $\theta°$	0 / 180	30 / 150	60 / 120	90
1	.250	$-.500$	-2.000	-2.750
1/2	$-.336$	$-.559$	-1.004	-1.227
1/3	$-.306$	$-.500$	$-.839$	-1.083
1/4	$-.284$	$-.477$	$-.851$	-1.043
$1/\infty$	$-.250$	$-.437$	$-.815$	-1.000

l_2 (Pl. IV)

a/r \ $\theta°$	0 / 180	30 / 150	60 / 120	90
1	$\mp.500$	$\pm.217$	$\pm.875$	0
1/2	$\pm.427$	$\pm.677$	$\pm.787$	0
1/3	$\pm.691$	±1.082	±1.184	0
1/4	$\pm.956$	±1.35	±1.602	0
$1/\infty$	$\pm\infty$	$\pm\infty$	$\pm\infty$	0

(The upper row of θ takes the upper signs and the lower row of θ takes the lower)

m_1 (Pl. V)

$\frac{a}{r}$ \ $\theta°$	0 180	30 150	60 120	90
1	0	0	0	0
1/2	0	±.426	±.426	0
1/3	0	±.385	±.385	0
1/4	0	±.362	±.362	0
1/∞	0	±.324	±.324	0

m_2 (Pl. VI)

$\frac{a}{r}$ \ $\theta°$	0 180	30 150	60 120	90
1	0	0	0	0
1/2	0	−.563	−.244	.141
1/3	0	−.830	−.433	.085
1/4	0	−1.117	−.609	.051
1/∞	0	−∞	−∞	0

(The upper row of θ takes the upper signs and the lower row of θ takes the lower)

The stresses $\widehat{\theta\theta}$ are the most important ones concerning the rupture of the hole. If we take the terms with W only we have the maximum compression at $\theta=90°$ and $270°$. This is $(\widehat{\theta\theta})_{90°}=-\left(3-\frac{\sigma}{1-\sigma}\right)W$, and it may be worthy of notice that the *tension* exists at the top and bottom of the hole.(1) Though this amount is not large, the rock usually has the low tensile strength and it might become the weakest points of the tunnel. The effect of W will be applicable to a deep tunnel. In the case of a shallow tunnel we must take the effect of V into account. The terms with V do not affect the maximum compression, but it decreases the crown tension and increases the bottom one. As regards the other stresses, $\widehat{rr}$, $\widehat{r\theta}$, the parts due to W are counteracted by those due to V in the upper half and increased in the lower half. On the top surface all the stresses naturally vanish and with increase of the depth they increase infinitely in magnitude.

5. Numerical Examples.

Example I: To find the stresses in a comparatively *shallow* tunnel. Take the depth of the center of the hole $x_0=20$ m, the radius of the hole $a=5$ m, $\sigma=1/5$, $\gamma g=2.0$ t/m^3(earth). Then we have $W=4$ kg/cm^2 and $V=W/4$.
The stresses are given in the following table, the unit being kgs. per cm^2.

(1) If we take the larger value of the Poisson's ratio, this tension disappears; e. g., we have the hydrostatic pressure for $\sigma=\frac{1}{2}$.

$\widehat{rr}$

$\frac{a}{r}$ \ $\theta°$	0	90	180
1	0	0	0
1/2	—.65	—1.52	—3.65
1/3	—.41	—1.33	—5.81
1/4	—.04*	—1.20	—7.02
1/∞		—1.00	— ∞

$\widehat{\theta\theta}$

$\frac{a}{r}$ \ $\theta°$	0	90	180
1	.50	—11.0	1.50
1/2	—.92	— 4.90	—1.77
1/3	—.53	— 4.33	—1.90
1/4	—.02*	— 4.17	—2.25
1/∞		— 4.00	— ∞

$\widehat{r\theta}$

$\frac{a}{r}$ \ $\theta°$	0	30	60	120	150
1	0	0	0	0	0
1/2	0	1.14	1.46	—1.95	—2.27
1/3	0	.71	1.11	—1.97	—2.37
1/4	0	.33	.84	—2.05	—2.57

* These must be zero rigorously treated, as they are the stresses on the top surface. We see from these that the errors are not important even in the case of such a shallow tunnel.

Terzaghi(1) shows that some clay has the compressive strength as high as 11.6 kg/cm² which is about the same value underlined in the above table of $\widehat{\theta\theta}$.

Example II: To find the largest sustaining depth of a circular hole whose radius is 5 m.

For a *soft sandstone* we have $\gamma g = 2.35$ t/m³, the compressive strength $= 200$ kg/cm² and the tensile strength $= 10$ kg/cm².

We have the maximum compression on both sides;

$$(\widehat{\theta\theta})_{90°} = -6.46\ x_0$$

Equating this to 200 kg/cm² or 2 000 t/m² we obtain the largest sustaining depth for compression $x_0 = 310$ m.

As we have the maximum tension at the bottom

$$(\widehat{\theta\theta})_{180°} = .5875\ (x_0 + 10)$$

Equating this to 10 kg/cm² or 100 t/m², we obtain the largest sustaining depth for tension $x_0 = 160$ m.

For a *hard granite* we have $\gamma g = 2.78$ t/m³, the compressive strength $= 2\ 000$ kg/cm² and the tensile strength $= 77$ kg/cm².

(1) Erdbaumechanik S. 78.

Therefore in the similar way we obtain $x_0=2\,600$ m for compression and $x_0=1\,100$ m for tension.

6. Summary of the Mathematical Investigations.

We, firstly, obtained the stresses vanishing at infinity and having the same values with opposite signs on a circular boundary with those in undisturbed gravitating elastic solid. Superposing these with the undisturbed ones we obtained the stresses near a circular hole in gravitating elastic solid. The results worth noticing are as follows :

(I) Just the same as the plate with a circular hole pressed at both ends the circumferential tension occurs at the top and bottom of the hole if we assume the Poisson's ratio $\sigma=\frac{1}{5}$, and the maximum circumferential compression occurs on the right and left sides.

(II) If the hole lies very deep from the top surface the circumferential stresses are symmetrical with respect to the horizontal diameter of the hole. When it approaches to the top surface *the bottom tension becomes greater than the crown tension.*

These results may be used as an elementary theory of a circular tunnel without lining which is not situated *very near* the top surface.[1]

7. Experiments with Agar-agar.

To verify the above obtained mathematical results, some model experiments were made with agar-agar (or Japanese isinglass). Agar-agar with proper content of water has comparatively good elastic property as shown by the load-elongation curve in **Fig. 2.**

Agar-agar was cast in the rectangular box having the size $70\times50\times8$ cm., the least dimension being the depth. When it was cooled a circular hole with the diameter 7.5 cm. was punched after we erected the vessel with its largest side in the vertical position. The initially circular hole deforms into the elliptic one by the weight of the surrounding medium. **Pl. VII** shows the general aspect of the erected state. As the size of the vessel is sufficiently large compared with the diameter of the hole, the boundary effect of the vessel does not appear

(1) If a circular hole lies very near the top surface, we might have to use the bi-polar coordinates which is now under investigation by the present writer.

near the hole. If we prepare beforehand the surface of agar-agar with the cross lines drawn with the hectogragh ink, we can take the prints of it showing the various configurations. **Pl. VIII** is a photograph of such a print taken before the box is erected. **Pl. IX** is a print taken when the box is erected vertically. It shows very clearly the deformation near the circular hole, whose horizontal diameter remaining almost unchanged, while the vertical diameter being shortened. The depth from the top horizontal surface to the center of the hole is 37.5 cm. From this we see that the depth of 37.5 cm. is sufficient to give the symmetrical strains with respect to the horizontal axis of the hole having the diameter of 7.5 cm. Here we may take only the terms with W in the equations (19). If we measure the distances between the cross lines by the comparator, we might estimate the actual stresses taken place at any point on the circumference of the hole by the aid of the load-elongation curve shown in **Fig. 2.** The print shows, to our unexpected interest, the lines of principal stresses which are the traces of the ripples made by the thin crustal film of the surface of agar-agar. **Pl. X** is another print taken, the depth of the center of the hole being 18.75 cm. from the top surface, i. e., the depth is two and half times of the diameter of the hole. When we investigate the print carefully we observe that the top curvature is larger than that of the bottom one, i. e., the top arc is slightly pointed when compared with the bottom one. Here the effect of the terms with V in the equations (19) appears, i. e., the bottom tension is larger than the top one. We might conclude from these experiments that

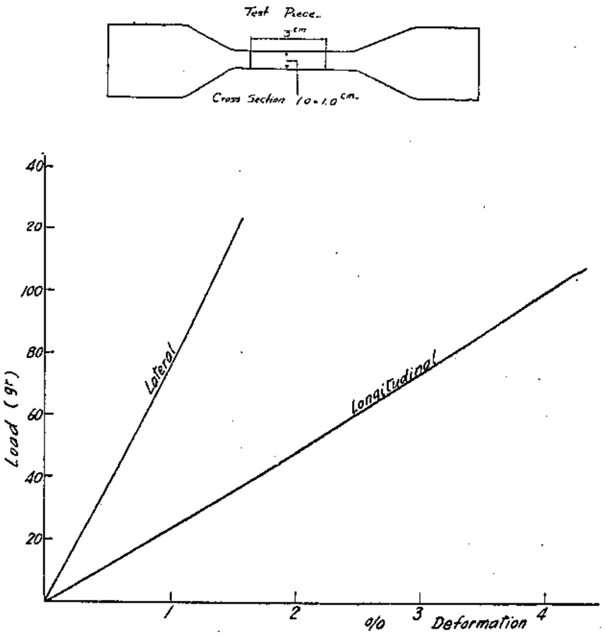
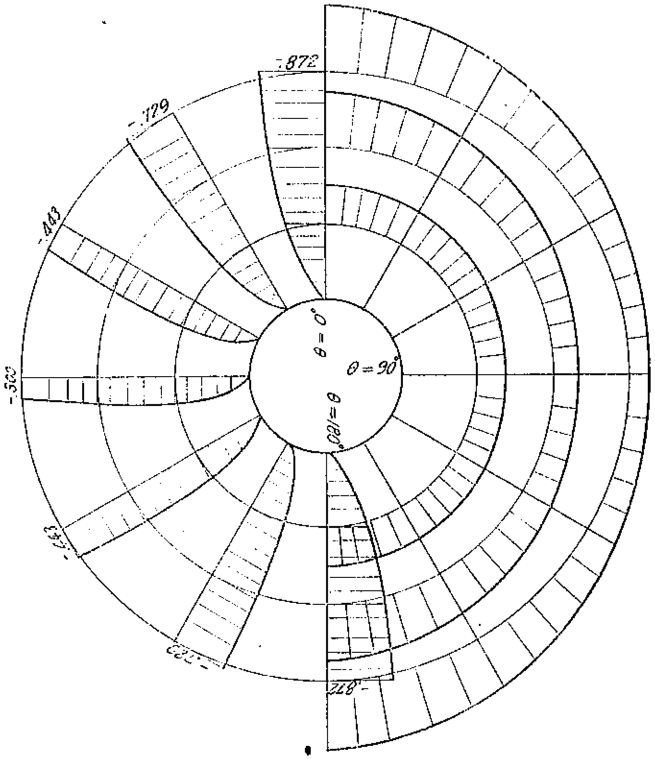


Fig. 2 Lood-Elongation Curve of Agar-agar

the above mathematical results are verified at least qualitatively. Strictly speaking, we had better use harder agar-agar, as it is doubtful that such a large amount of deformation as shown in the present photographs can be treated by the ordinary theory of elasticity. There is one more problem to measure precisely the Poisson's ratio of agar-agar which affects the values of the stresses considerably. All these important experimental investigations which are necessary to obtain the quantitative results are now undertaken in the Research Office of the Japanese Government Railway. The results will be published shortly. In conclusion, the present writer expresses the indebtedness of these experimental performances to his able assistants Messrs. T. Hata and G. Kubota of the Research Office of J. G. R.

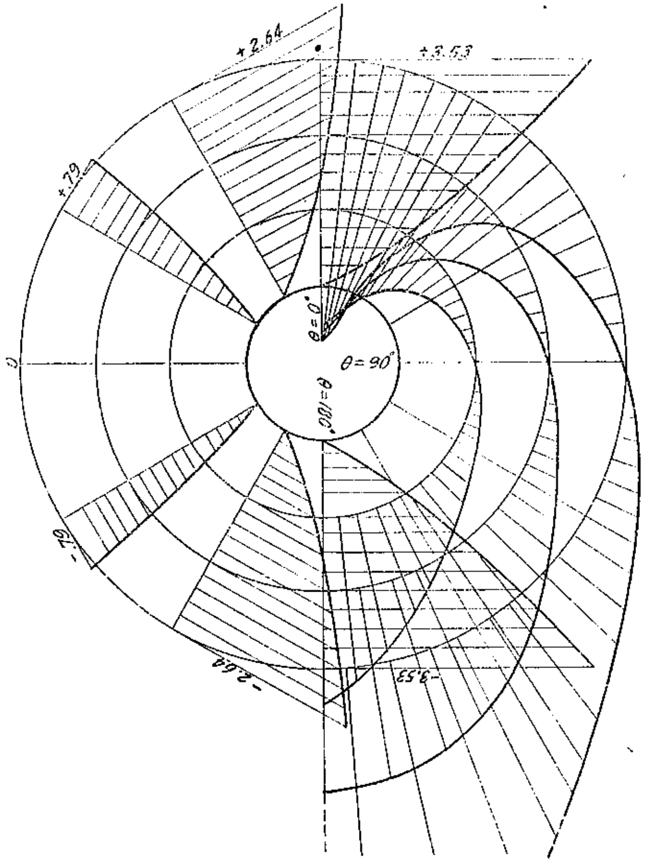
January, 1929,
Tokyo, Japan.

Pl. I



The value of k_1 in $\widehat{rr} = k_1 W$

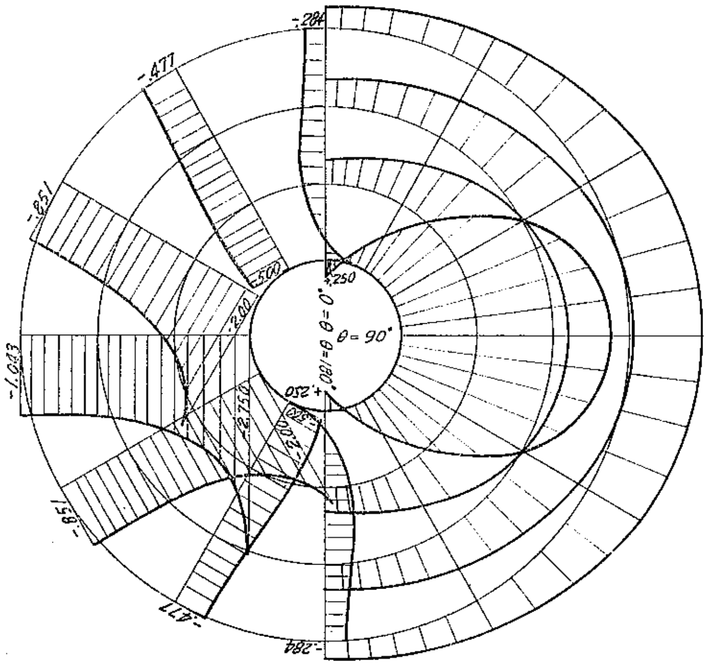
Pl. II



The value of k_2 in $\widehat{rr} = k_2 V$

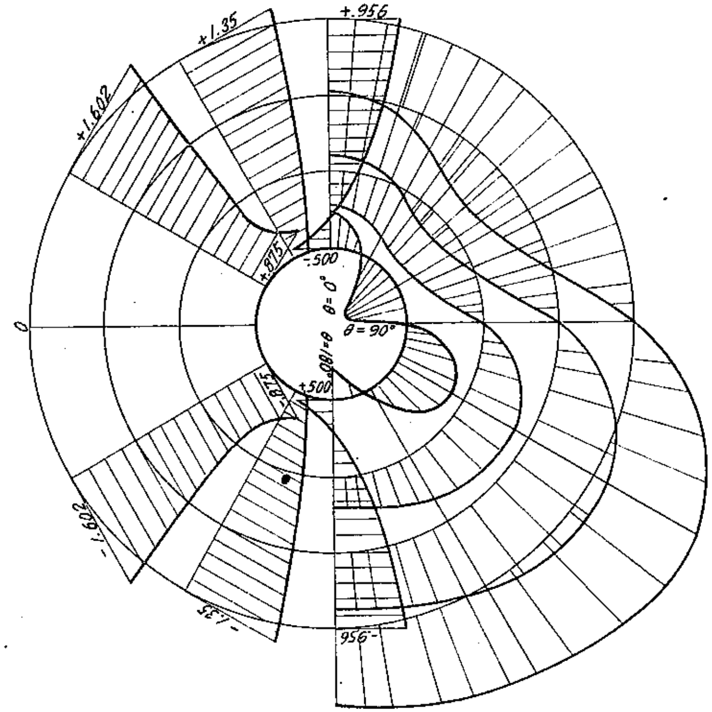
(土木學會誌第十五卷第四號附圖)

Pl. III



The value of l_1 in $\widehat{\theta\theta}=l_1 W$

Pl. IV

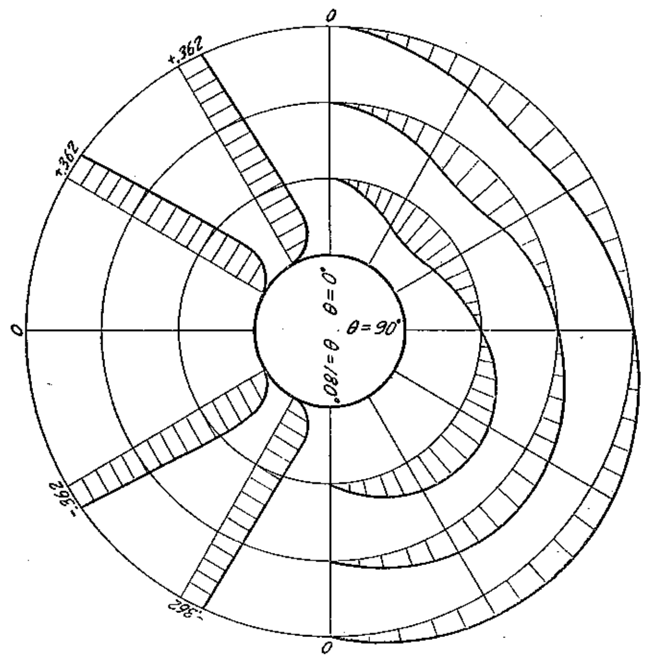


The value of l_2 in $\widehat{\theta\theta}=l_2 V$

(土木學會誌第十五卷第四號附圖)

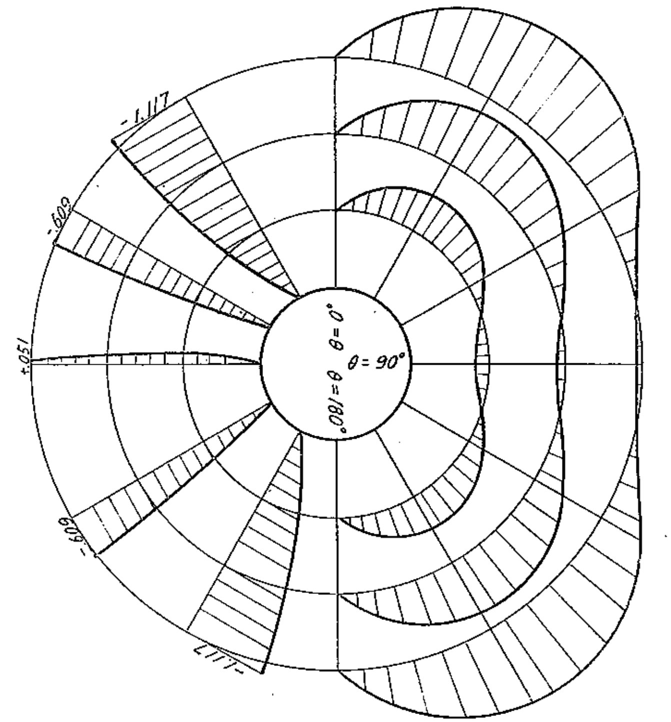
(土木學會誌第十五卷第四號附圖)

Pl. V



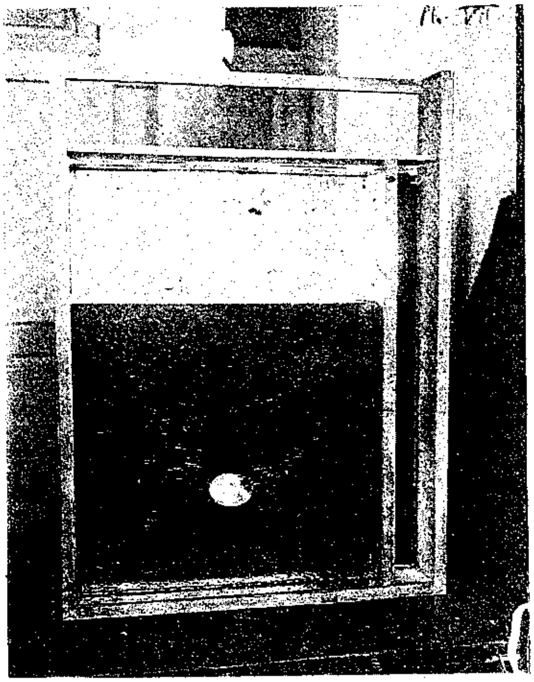
The value of m_1 in $\widehat{r\theta}=m_1 W$

Pl. VI

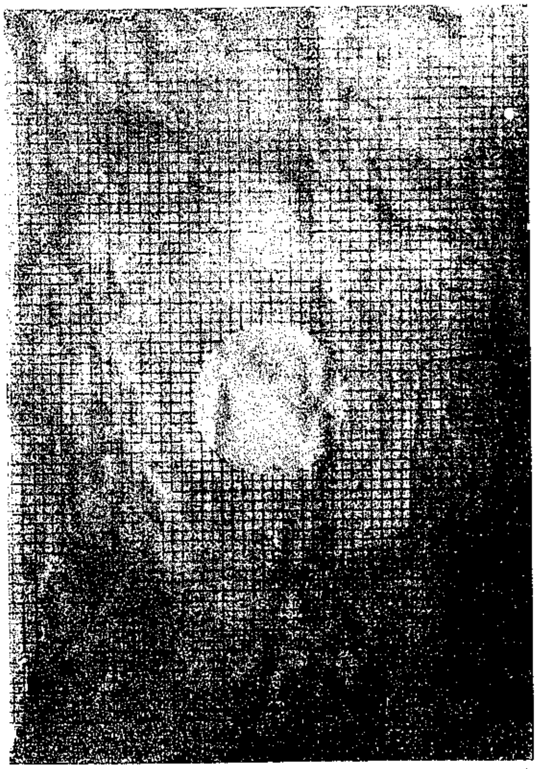


The value of m_2 in $\widehat{r\theta}=m_2 V$

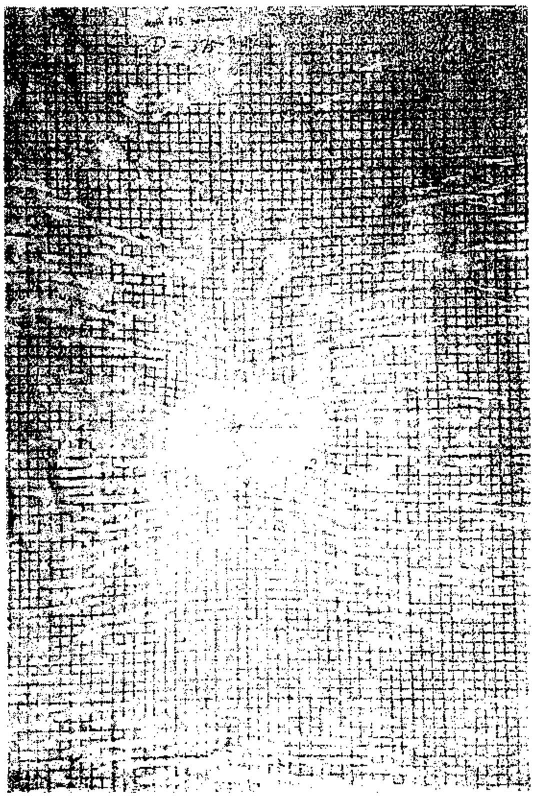
Pl. VII



Pl. VIII



Pl. IX



Pl. X