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THERMAL FLEXURE OF A THIN PLATE HEATED ON ONE SURFACE, EXTENSIONAL STRESSES TAKEN INTO ACCOUNT.

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Synopsis

This paper is the continuation of my former paper on the thermal flexure of a thin plate, one surface of which is heated uniformly. In this I have taken into account the extensional stresses, which were not considered in the previous discussion. The result was that the fundamental equations are no longer linear, and moreover they are susceptible of being solved only with certain limitations.

In the following, I have taken up cases of an infinitely extended strip and a circular plate as being approximatively soluble. The results obtained show slight variations from those in the former paper, the difference naturally becoming negligible when the supports are not strictly rigid.

(1) Fundamental Equations.

A year ago I tried to compute the thermal flexure of a thin plate, one surface of which was uniformly heated. At that time I assumed the supports were not so rigid as to give any extensional (or contractional) stresses to the Neutral Plane.

If the supports, however, are not sufficiently yielding, the above mentioned stresses must be taken into account as I remarked in the foot notes of my former paper.[1] Now we deduce the fundamental equations of such a case.

Assume the temperature $t=t(x,y,z)$

(1) On the Thermal Bending of a Plane Wall Heated on One Surface. (Jour. Civil Eng. Soc. Japan., Vol. XIII, No. 4.)

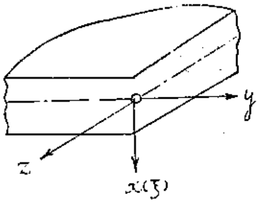


Fig. 1.

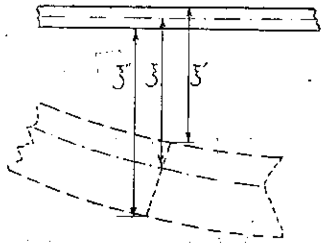


Fig. 2.

A point (x,y,z) in the plate is subjected to the total displacements ξ, η, ζ. Each of them consists of the displacements due to the strains and to the temperature. But it may be assumable that the displacement ξ is independent of x (i. e. it is sensibly of the same value along the normal to the plate), as the thickness of the plate is assumed to be sufficiently thin. (**Fig. 2**) On the Neutral Plane; we have $t_0=t(o,y,z)$.

The strain solely due to the stresses are;

$$\left.\begin{aligned}\varepsilon_y&=\frac{\partial\eta'}{\partial y}+\frac{1}{2}\left(\frac{\partial\xi}{\partial y}\right)^2\\ \varepsilon_z&=\frac{\partial\zeta'}{\partial z}+\frac{1}{2}\left(\frac{\partial\xi}{\partial z}\right)^2\\ \gamma_{yz}&=\frac{\partial\eta'}{\partial z}+\frac{\partial\zeta'}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\xi}{\partial z}\end{aligned}\right\}\cdots\cdots(1)$$

where η' and ζ' are the parts of displacements due to the stresses.

And we have

$$\left.\begin{aligned}\frac{\partial\eta}{\partial y}&=\frac{\partial\eta'}{\partial y}+\alpha t_0\\ \frac{\partial\zeta}{\partial z}&=\frac{\partial\zeta'}{\partial z}+\alpha t_0\\ \frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}&=\frac{\partial\eta'}{\partial z}+\frac{\partial\zeta'}{\partial y}\end{aligned}\right\}\cdots\cdots(2)$$

where η and ζ are the *total* displacements due to stress & temperature and α is the linear expansion coefficient of the material.

Therefore, we have

$$\left.\begin{aligned}\varepsilon_y&=\frac{\partial\eta}{\partial y}-\alpha t_0+\frac{1}{2}\left(\frac{\partial\xi}{\partial y}\right)^2\\ \varepsilon_z&=\frac{\partial\zeta}{\partial z}-\alpha t_0+\frac{1}{2}\left(\frac{\partial\xi}{\partial z}\right)^2\end{aligned}\right\}\cdots\cdots(3)$$

$$\gamma_{yz}=\frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\xi}{\partial z}$$

If we take away for the time being the strains due to the curvature by bending action, the above values may be considered as the mean strains across the thickness of the plate, and we may treat them as the case of the generalized plane strains or stresses.

We have

$$\left.\begin{aligned}\varepsilon_y&=\frac{1}{E}\left(\sigma_y-\frac{1}{m}\sigma_z\right)\\ \varepsilon_z&=\frac{1}{E}\left(\sigma_z-\frac{1}{m}\sigma_y\right)\\ \gamma_{yz}&=\frac{1}{G}\tau_{yz}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(4)$$

or

$$\left.\begin{aligned}\sigma_y&=\frac{E}{1-\frac{1}{m^2}}\left(\varepsilon_y+\frac{1}{m}\varepsilon_z\right)=\frac{E}{1-\frac{1}{m^2}}\left\{\frac{\partial\eta}{\partial y}+\frac{1}{m}\frac{\partial\zeta}{\partial z}-\left(1+\frac{1}{m}\right)\alpha t_0+\frac{1}{2}\left(\frac{\partial\xi}{\partial y}\right)^2+\frac{1}{2m}\left(\frac{\partial\xi}{\partial z}\right)^2\right\}\\ \sigma_z&=\frac{E}{1-\frac{1}{m^2}}\left(\varepsilon_z+\frac{1}{m}\varepsilon_y\right)=\frac{E}{1-\frac{1}{m^2}}\left\{\frac{\partial\zeta}{\partial z}+\frac{1}{m}\frac{\partial\zeta}{\partial y}-\left(1+\frac{1}{m}\right)\alpha t_0+\frac{1}{2}\left(\frac{\partial\xi}{\partial z}\right)^2+\frac{1}{2m}\left(\frac{\partial\xi}{\partial y}\right)^2\right\}\\ \tau&=G\gamma_{yz}=G\left\{\frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\xi}{\partial z}\right\}\end{aligned}\right\}\ (5)$$

where E, G and m are Young's Modulus, Rigidity Modulus and Poisson's Number, resp.

The identical relation (the condition of compatibility) between the total strains $\frac{\partial\eta}{\partial y}$, $\frac{\partial\zeta}{\partial z}$ and $\frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}$ are

$$\frac{\partial^2}{\partial z^2}\left(\frac{\partial\eta}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial\zeta}{\partial z}\right)-\frac{\partial^2}{\partial y\partial z}\left(\frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}\right)=0\quad\ldots\ldots\ldots(6)$$

Putting the values of total strains by the equations (3) in the above equation, we have

$$\frac{\partial^2 \varepsilon_y}{\partial z^2}+\frac{\partial^2 \varepsilon_z}{\partial y^2}-\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}=-\alpha\left(\frac{\partial^2 t_0}{\partial y^2}+\frac{\partial^2 t_0}{\partial z^2}\right)+\frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\frac{\partial \xi}{\partial y}\right)^2+\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\frac{\partial \xi}{\partial z}\right)^2$$
$$-\frac{\partial^2}{\partial y \partial z}\left(\frac{\partial \xi}{\partial y}\frac{\partial \xi}{\partial z}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Putting the values of ε_y, ε_z and γ_{yz} by the equations (4) in the above equation and using the following Airy's relations to the plane stresses which are derived from the equation of equilibrium,

$$\sigma_y=\frac{\partial^2 F}{\partial z^2} \quad \sigma_z=\frac{\partial^2 F}{\partial y^2} \text{ and } \tau_{yz}=-\frac{\partial^2 F}{\partial y \partial z} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

we have

$$\nabla^2\nabla^2 F=-\alpha E \nabla^2 t_0+E\left\{\left(\frac{\partial^2 \xi}{\partial y \partial z}\right)^2-\frac{\partial^2 \xi}{\partial y^2}\frac{\partial^2 \xi}{\partial z^2}\right\} \quad \ldots\ldots\ldots\ldots\ldots (9)$$

where $\nabla^2=\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

Now we consider the strains due to the flexure only.

If the plate is sufficiently thin, the deflection occurs in such way as the normal lines to the Neutral Plane remain straight and normal to the Elastic Surface, (**Fig. 3**)

whence

$$\left.\begin{aligned}\varepsilon_y+\alpha(t-t_0)&=-x\frac{\partial^2 \xi}{\partial y^2}\\ \varepsilon_z+\alpha(t-t_0)&=-x\frac{\partial^2 \xi}{\partial z^2}\\ \gamma_{yz}&=-2x\frac{\partial^2 \xi}{\partial y \partial z}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (10)$$

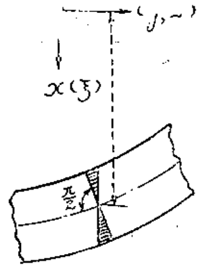


Fig. 3.

and we have the stresses,

$$\left.\begin{aligned}\sigma_y&=\frac{E}{1-\frac{1}{m^2}}\left(\varepsilon_y+\frac{1}{m}\varepsilon_z\right)=-\frac{E}{1-\frac{1}{m^2}}\left\{x\frac{\partial^2 \xi}{\partial y^2}+\frac{x}{m}\frac{\partial^2 \xi}{\partial z^2}+\alpha\left(1+\frac{1}{m}\right)(t-t_0)\right\}\\ \sigma_z&=\frac{E}{1-\frac{1}{m^2}}\left(\varepsilon_z+\frac{1}{m}\varepsilon_y\right)=-\frac{E}{1-\frac{1}{m^2}}\left\{x\frac{\partial^2 \xi}{\partial z^2}+\frac{x}{m}\frac{\partial^2 \xi}{\partial y^2}+\alpha\left(1+\frac{1}{m}\right)(t-t_0)\right\}\\ \tau_{yz}&=G\gamma_{yz}=-2Gx\frac{\partial^2 \xi}{\partial y \partial z}\end{aligned}\right\} (11)$$

Therefore we have

$$
\left.
\begin{aligned}
M_y &= \int_{(h)} \sigma_y x\,dx = -D\left\{\frac{\partial^2 \xi}{\partial y^2} + \frac{1}{m}\frac{\partial^2 \xi}{\partial z^2} + \left(1+\frac{1}{m}\right)\frac{\Phi}{I}\right\} \\
M_z &= \int_{(h)} \sigma_z x\,dx = -D\left\{\frac{\partial^2 \xi}{\partial z^2} + \frac{1}{m}\frac{\partial^2 \xi}{\partial y^2} + \left(1+\frac{1}{m}\right)\frac{\Phi}{I}\right\} \\
M_{yz} &= \int_{(h)} \tau_{yz} x\,dx = -D\left(1-\frac{1}{m}\right)\frac{\partial^2 \xi}{\partial y \partial z}
\end{aligned}
\right\} \qquad \ldots\ldots\ldots\ldots (12)
$$

where $\Phi = \alpha \int_{(h)} (t-t_0)x\,dx$, $I = \int_{(h)} x^2 dx$ and $D = \dfrac{EI}{1-\dfrac{1}{m^2}}$

The shears are

$$
\left.
\begin{aligned}
V_y &= \frac{\partial M_y}{\partial y} + \frac{\partial M_{yz}}{\partial z} \\
V_z &= \frac{\partial M_z}{\partial z} + \frac{\partial M_{yz}}{\partial y}
\end{aligned}
\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)
$$

The equation of equilibrium is

$$
\frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} + p + h\sigma_y \frac{\partial^2 \xi}{\partial y^2} + h\sigma_z \frac{\partial^2 \xi}{\partial z^2} + 2\tau_{yz} h \frac{\partial^2 \xi}{\partial y \partial z} = 0 \qquad \ldots\ldots\ldots\ldots (14)
$$

where p is the distributed load.

Or we have

$$
-D\left\{\nabla^2\nabla^2\xi + \frac{1+\dfrac{1}{m}}{I}\nabla^2\Phi\right\} + p + h\left\{\frac{\partial^2 F}{\partial z^2}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 \xi}{\partial z^2} - 2\frac{\partial^2 F}{\partial y \partial z}\frac{\partial^2 \xi}{\partial y \partial z}\right\} = 0 \ldots (15)
$$

When the one surface of the plate is *heated uniformly all over*, as we are going to deal with here, we have $\nabla^2 t_0 = 0$ and $\nabla^2\Phi = 0$ in the equations (9) & (15) and they simplify themselves in the following forms;

$$
\left.
\begin{aligned}
\nabla^2\nabla^2 F &= E\left\{\left(\frac{\partial^2 \xi}{\partial y \partial z}\right)^2 - \frac{\partial^2 \xi}{\partial y^2}\frac{\partial^2 \xi}{\partial z^2}\right\} \\
D\,\nabla^2\nabla^2 \xi &= p + h\left\{\frac{\partial^2 F}{\partial z^2}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 \xi}{\partial z^2} - 2\frac{\partial^2 F}{\partial y \partial z}\frac{\partial^2 \xi}{\partial y \partial z}\right\}
\end{aligned}
\right\} \qquad \ldots\ldots\ldots\ldots (16)
$$

These are the same as Föppl-Kármán's equations of the flexure of the thin plate with large deflection by lateral loading.[1] In our present problems, we must, of course, put the temperature effect in F.

(2) Infinitely Extended Strip.

In this case it is easy to solve the equations (1) (16).

If $p=0$ and ξ is independent of z, we have

$$\left.\begin{aligned} &\nabla^2\nabla^2 F=0 \\ &D\frac{\partial^4 \xi}{dy^4}-h\frac{\partial^2 F}{\partial z^2}\frac{d^2\xi}{dy^2}=0 \end{aligned}\right\} \quad \cdots\cdots(1)$$

The Moments and Shears are,

$$\left.\begin{aligned} &M_y=-D\left\{\frac{d^2\xi}{dy^2}+\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\right\},\ M_z=-D\left\{\frac{1}{m}\frac{d^2\xi}{dy^2}+\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\right\},\ M_{yz}=0 \\ &V_y=-D\left\{\frac{d^3y}{dy^3}\right\},\quad V_z=0 \end{aligned}\right\}$$

$$\cdots\cdots(2)$$

(I) If $t_0>0$, we have $\sigma_y=\dfrac{\partial^2 F}{\partial z^2}=c<0$ for the inextensional supports.

Therefore

$$\frac{d^4\xi}{dy^4}+n^2\frac{d^2\xi}{dy^2}=0, \qquad n^2=-\frac{hC}{D} \quad \cdots\cdots(3)$$

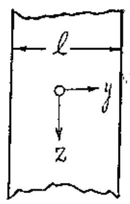


Fig. 3.

The general solution of this equation;

$$\xi=C_1+C_2y+D_1\cos ny+D_2\sin ny$$

If we take the origin at the center of the strip (**Fig. 3.**), ξ will be an even function.

$$\xi=C_1+D_1\cos ny$$

C_1 & D_1 are determined by the following boundary conditions;

$\xi=0$ and $M_y=0$ at $y=\pm\dfrac{l}{2}$ for the *supported edges.*

---

(1) A. Nádai (Elastische Platten S. 264) deduced similar equations to (9) & (15), assuming the plate will remain perfectly in a plane with respect to the extensional stresses. But the present author ventures to think that they are to be modified a little, as such a plate will be subjected very soon to a finite amount of deflection.

$$\left.\begin{aligned} &C_1+D_1\cos\frac{nl}{2}=0 \\ &-D_1n^2\cos\frac{nl}{2}+\left(1+\frac{1}{m}\right)\frac{\Phi}{I}=0 \end{aligned}\right\}$$

whence we have

$$C_1=-\left(1+\frac{1}{m}\right)\frac{\Phi}{n^2I} \text{ and } D_1=\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\frac{1}{n^2\cos\frac{nl}{2}}$$

Therefore we have

$$\xi=-\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\frac{\cos\frac{nl}{2}-\cos ny}{n^2\cos\frac{nl}{2}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

If C is very small i.e. n^2 is very small;

$$\xi=-\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\frac{1-\frac{\left(\frac{nl}{2}\right)^2}{2}+\ldots\ldots-\left\{1-\frac{(ny)^2}{2}+\ldots\right\}}{n^2\left\{1-\frac{\left(\frac{nl}{2}\right)^2}{2}+\ldots\right\}}$$

$$\doteqdot\frac{1}{2}\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\left\{\left(\frac{l}{2}\right)^2-y^2\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

This coincides with the equation of my former paper (loc. cit.) p. 11, where the extensional strains are not taken into account.

The value of C is to be determined by the condition that $\eta=0$ at $y=\pm\frac{l}{2}$ for *the inextensional supports.*

$$C=\sigma_y=\frac{E}{1-\frac{1}{m^2}}\left\{\frac{d\eta}{dy}-\left(1+\frac{1}{m}\right)\alpha t_0+\frac{1}{2}\left(\frac{d\xi}{dy}\right)^2\right\}$$

$$\int_0^{\frac{l}{2}}Cdy=\frac{E}{1-\frac{1}{m^2}}\left\{\int_0^{\frac{l}{2}}\frac{d\eta}{dy}\,dy-\left(1+\frac{1}{m}\right)\alpha t_0\int_0^{\frac{l}{2}}dy+\frac{1}{2}\int_0^{\frac{l}{2}}\left(\frac{d\xi}{dy}\right)^2dy\right\}$$

$$\frac{Cl}{2}=\frac{E}{1-\frac{1}{m^2}}\left\{-\left(1+\frac{1}{m}\right)\alpha t_0\frac{l}{2}+\frac{1}{2}\int_0^{\frac{l}{2}}\left(\frac{d\xi}{dy}\right)^2dy\right\}$$

But $$C=-\frac{n^2D}{h}=-\frac{n^2Eh^2}{12\left(1-\frac{1}{m^2}\right)}$$

$$\therefore\quad n^2=\frac{12}{h^2 l}\left\{\left(1+\frac{1}{m}\right)\alpha\, t_0 l-\int_0^{\frac{l}{2}}\left(\frac{d\xi}{dy}\right)^2 dy\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

From (4) we have

$$\left(\frac{d\xi}{dy}\right)^2=\left(1+\frac{1}{m}\right)^2\left(\frac{\Phi}{I}\right)^2\frac{\sin^2 ny}{n^2\cos^2\frac{nl}{2}}$$

$$\int_0^{\frac{l}{2}}\left(\frac{d\xi}{dy}\right)^2 dy=\left(1+\frac{1}{m}\right)^2\left(\frac{\Phi}{I}\right)^2\frac{1}{n^2\cos^2\frac{nl}{2}}\left\{\frac{l}{4}-\frac{\sin nl}{4n}\right\}$$

Therefore, we have

$$n^2=\frac{12}{h^2}\left\{\left(1+\frac{1}{m}\right)\alpha\, t_0-\left(1+\frac{1}{m}\right)^2\left(\frac{\Phi}{I}\right)^2\frac{1}{2n^2}\left[\frac{1}{2\cos^2\frac{nl}{2}}-\frac{1}{nl}\tan\frac{nl}{2}\right]\right\}$$

or

$$n^4-\frac{12}{h^2}\left\{\left(1+\frac{1}{m}\right)\alpha t_0 n^2-\left(1+\frac{1}{m}\right)^2\left(\frac{\Phi}{I}\right)^2\frac{1}{2}\left[\frac{1}{2\cos^2\frac{nl}{2}}-\frac{1}{nl}\tan\frac{nl}{2}\right]\right\}=0.\quad\ldots(7)$$

This is a transcendental equation to determine n[1].

We can obtain an approximate value by developing cos & tan into the series and taking the first terms.

$$\frac{1}{2\cos^2\frac{nl}{2}}=\frac{1}{1+\cos nl}=\frac{1}{1+1-\frac{(nl)^2}{2}+\ldots\ldots}\doteqdot\frac{1}{2}$$

$$\frac{1}{nl}\tan\frac{nl}{2}=\frac{1}{nl}\left\{\frac{nl}{2}+\frac{1}{3}\left(\frac{nl}{2}\right)^3+\ldots\ldots\right\}\doteqdot\frac{1}{2}$$

Therefore, dropping the term concerning the thermal bending, we have the equation to determine the first approximate value of n;

$$n^2=\frac{12}{h^2}\left(1+\frac{1}{m}\right)\alpha\, t_0$$

or

$$C=-\frac{n^2D}{h}=-\frac{12}{h^3}D\left(1+\frac{1}{m}\right)\alpha\, t_0=-\frac{E}{1-\frac{1}{m}}\alpha t_0\quad\ldots\ldots\ldots(8)$$

(1) We have many solutions of this equation which correspond to various modes of ξ. But we take only the lowest value of them, i. e. the gravest mode in terms of vibration theory.

This value of C is that of the plane thermal stress.

Multiplying both sides of the first equation of (8) by l^2, we have

$$n^2 l^2 = 12\left(\frac{l}{h}\right)^2\left(1+\frac{1}{m}\right)\alpha t_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

The values of nl for $\frac{l}{h}=100$ & $m=4$ or 10 are shown in the following table & graph (**PL. I**)

The Values of nl for $\frac{l}{h}=100$

$t_0(C^\circ)$	$m=4$, $\alpha=.00001$ (Steel)	$m=10$, $\alpha=.00001$ (Concrete)
1	1.22	1.15
5	2.73	2.58
10	3.86	3.64
15	4.725	4.45
20	5.45	5.15
25	6.10	5.75
30	6.68	6.30
40	7.72	7.27
50	8.63	8.13
81	10.98	10.35
100	12.20	11.50

If we put the above value of n^2 shown by (8) as the first approximate value into the first equation of (7), we have

$$n^2 = \frac{12}{h^2}\left\{\left(1+\frac{1}{m}\right)\alpha t_0 - \left(1+\frac{1}{m}\right)\left(\frac{\Phi}{I}\right)^2 \frac{Kh^2}{24\,\alpha t_0}\right\}$$

where

$$K = \frac{1}{2\cos^2\frac{nl}{2}} - \frac{\tan\frac{nl}{2}}{nl} = \frac{nl-\sin nl}{2\,nl\cos^2\frac{nl}{2}}$$

If t is linear w. r. t. x, i. e. $\Phi=\alpha\int_{(h)}(t-t_0)\,x\,dx=\alpha\Theta I$ [Θ is the temperature gradient.], we have

$$\left.\begin{aligned} n^2 &= \left(1+\frac{1}{m}\right)\frac{12}{h^2}\alpha t_0\left\{1-K\frac{\Theta^2}{24}\left(\frac{h}{t_0}\right)^2\right\} \\ \text{or} \quad n^2 l^2 &= 12\left(1+\frac{1}{m}\right)\alpha t_0\left\{1-K\frac{\Theta^2}{24}\left(\frac{h}{t_0}\right)^2\right\}\left(\frac{l}{h}\right)^2 \end{aligned}\right\} \quad \ldots\ldots (10)$$

$$C=-\frac{E\alpha t_0}{1-\frac{1}{m}}\left\{1-K\frac{\Theta^2}{24}\left(\frac{h}{t_0}\right)^2\right\} \qquad \dots\dots\dots(11)$$

These solutions fail if the value of K becomes as large as $K\frac{\Theta^2}{24}\left(\frac{h}{t_0}\right)^2$ be comparable with unity. $K\to\infty$ when $\cos\frac{nl}{2}\to 0$ or $nl=\pi$, then ξ tends to ∞. This corresponds to the lowest buckling thrust of the plate.

The Values of K (PL. II.)

nl	0.5	1.0	1.5	2.0	2.5	3.0	3.14
K	0.022	0.114	0.314	0.933	3.85	95.3	∞

Comparing these values with nl- graph (**PL. I**) we have the limiting value of temperature applicable to our calculations, e.g.

$$\left.\begin{aligned} t_0&=25^\circ \text{ for } \frac{l}{h}=50 \\ t_0&=100^\circ \text{ for } \frac{l}{h}=25 \end{aligned}\right\}$$

For the value less than this temperature, we may have the corresponding values to K and we may or may not have the legitimate values of $K\frac{\Theta^2}{24}\left(\frac{h}{t_0}\right)^2$ (which must be far less than unity).

The bending moments and bending stresses are obtained by

$$\left.\begin{aligned} M_y&=-\frac{E\Phi}{1-\frac{1}{m}}\left\{1-\frac{\cos ny}{\cos\frac{nl}{2}}\right\} \\ \sigma_y&=-\frac{E}{1-\frac{1}{m}}\left\{-x\frac{\Phi}{I}\frac{\cos ny}{\cos\frac{nl}{2}}+\alpha(t-t_0)\right\} \end{aligned}\right\} \qquad \dots\dots(12)$$

If t is linear w. r. t. x, we have

$$\left.\begin{aligned} M_y&=-\frac{E\alpha\Theta I}{1-\frac{1}{m}}\left\{1-\frac{\cos ny}{\cos\frac{nl}{2}}\right\} \\ \sigma_y&=-\frac{Ex\alpha\Theta}{1-\frac{1}{m}}\left\{1-\frac{\cos ny}{\cos\frac{nl}{2}}\right\} \end{aligned}\right\} \qquad \dots\dots(13)$$

We see from this that the bending moment M_y will not vanish except at the supported edges and that its maximum value occurs on the centre line of the strip. We obtain the analogous expressions for M_z and σ_z from the equations (I) (11) & (I) (12). At the extreme fibre we have the fibre stresses

$$\sigma_y^0=\sigma_y+C=-\frac{E\alpha}{1-\frac{1}{m}}\left[\left\{1-K\frac{\Theta^2}{24}\left(\frac{h}{t_0}\right)^2\right\}t_0\mp\frac{\Theta h}{2}\left\{1-\frac{\cos ny}{\cos\frac{nl}{2}}\right\}\right]\dots(14)$$

If $nl\rightarrow\pi$ the above solution fails.

(II) If $t_0<0$, $\sigma_y=C>0$, we have $n^2=-\frac{hC}{D}<0$.

Therefore put in instead of n in the above solutions for $t_0>0$. Seeing that $\cos ix=\cosh x$ and $\tan ix=i\tanh x$, we can convert the above solutions to the case of $t_0<0$.

Or, if we put $n^2=\frac{hC}{D}>0$, we shall have the following equation instead of (4),

$$\xi=\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\frac{\cosh\frac{nl}{2}-\cosh ny}{n^2\cosh\frac{nl}{2}}\quad\dots\dots(15)$$

If C is very small, by developing cosh into series & neglecting higher terms, we have the same value as the eq. (5)

$$\xi=\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\frac{\left\{1+\frac{1}{2}\left(\frac{nl}{2}\right)^2+\frac{1}{24}\left(\frac{nl}{2}\right)^4+\dots\right\}-\left\{1+\frac{1}{2}(ny)^2+\frac{1}{24}(ny)^4+\dots\right\}}{n^2\left\{1+\frac{1}{2}\left(\frac{nl}{2}\right)^2+\frac{1}{24}\left(\frac{nl}{2}\right)^4+\dots\right\}}$$

$$\doteqdot\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\frac{1}{2}\left\{\left(\frac{l}{2}\right)^2-y^2\right\}\quad\dots\dots(16)$$

For the inextensional supports, we have the same equation as (6), and we put this time the following value in it.

$$\int_0^{\frac{l}{2}}\left(\frac{d\xi}{dy}\right)^2dy=\left(1+\frac{1}{m}\right)^2\left(\frac{\Phi}{I}\right)^2\frac{1}{2n^2}\left\{\frac{1}{n}\tanh\frac{nl}{2}-\frac{l}{2}\frac{1}{\cosh^2\frac{nl}{2}}\right\}$$

and we have

$$\left.\begin{aligned} n^2 &= \frac{12}{h^2}\left[\left(1+\frac{1}{m}\right)^2\frac{\Phi^2}{2n^2I^2}\left\{\frac{1}{nl}\tanh\frac{nl}{2}-\frac{1}{2\cosh^2\frac{nl}{2}}\right\}-\left(1+\frac{1}{m}\right)\alpha t_0\right] \\ \text{or } n^4 &- \frac{12}{h^2}\left[\left(1+\frac{1}{m}\right)^2\left(\frac{\Phi}{I}\right)^2\frac{1}{2}\left\{\frac{1}{nl}\tanh\frac{nl}{2}-\frac{1}{2\cosh^2\frac{nl}{2}}\right\}-\left(1+\frac{1}{m}\right)\alpha t_0\right]=0 \end{aligned}\right\} \quad \cdots\cdots(17)$$

This is the transcendental equation which gives n.

We can obtain an approx. solution by developing tanh & cosh into the series and taking the first few terms when nl is small.

$$\frac{1}{nl}\tanh\frac{nl}{2}=\frac{1}{2}-\frac{(nl)^2}{24}+\ldots\ldots \doteqdot \frac{1}{2}$$

$$\frac{1}{2\cosh^2\frac{nl}{2}}=\frac{1}{1+\cosh nl}=\frac{1}{1+1+\left(\frac{nl}{2}\right)^2+\ldots} \doteqdot \frac{1}{2}$$

Therefore neglecting the term concerning the thermal bending, we have the equation to determine n,

$$\left.\begin{aligned} n^2 &= -\frac{12}{h^2}\left(1+\frac{1}{m}\right)\alpha t_0 \\ C &= \frac{n^2D}{h} = -\frac{E}{1-\frac{1}{m}}\alpha t_0 \end{aligned}\right\} \quad \cdots\cdots(18)$$

This is the same as the plane thermal stresses. As $t_0<0$, $C>0$ i.e. the stresses C is in tension.

If we put the above value of n^2 as the first approximate solution into the first term of the right hand member of the equation (17), we have

$$\left.\begin{aligned} n^2 &= \frac{12}{h^2}\left\{-\left(1+\frac{1}{m}\right)\left(\frac{\Phi}{I}\right)^2\frac{h^2}{24\alpha t_0}N-\left(1+\frac{1}{m}\right)\alpha t_0\right\} \\ \text{where} \quad N &= \frac{1}{nl}\tanh\frac{nl}{2}-\frac{1}{2\cosh^2\frac{nl}{2}} \end{aligned}\right\} \quad \cdots\cdots(19)$$

If t is linear w.r.t. x, $\Phi=\alpha\Theta I$

$$\left.\begin{aligned} n^2 &= -\left(1+\frac{1}{m}\right)\frac{12}{h^2}\alpha t_0\left\{1+\frac{\Theta^2h^2}{24\,t_0}N\right\} \\ C &= -\frac{E\alpha t_0}{1-\frac{1}{m}}\left\{1+\frac{\Theta^2h^2}{24\,t_0}N\right\} \end{aligned}\right\} \quad \cdots\cdots(20)$$

The Values of N (PL. III.)

nl	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5
N	.021	.070	.125	.171	.199	.211	.212	.206	.195	.184	.172	.161	.150	.140	.132	.124	.117

If C is large or n is large, $\tanh\dfrac{nl}{2}\to 1$ and

$$\frac{1}{\cos^2\dfrac{nl}{2}}=\frac{1}{\left(\dfrac{e^{\frac{nl}{2}}+e^{\frac{-nl}{2}}}{2}\right)^2}\doteqdot\frac{4}{e^{nl}}$$

$$\left.\begin{aligned}\therefore\quad n^2&=-\frac{12}{h^2}\left(1+\frac{1}{m}\right)\alpha t_0\left\{1-\left(1+\frac{1}{m}\right)\frac{\Phi^2 N'}{2\alpha t_0 n^2 I^2}\right\}\\ \text{where}\quad N'&=\frac{1}{nl}-\frac{2}{e^{nl}}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(21)$$

The first approximation is

$$n^2=-12\left(1+\frac{1}{m}\right)\frac{\alpha t_0}{h^2}\quad\text{or}\quad C=-\frac{E}{1-\dfrac{1}{m}}\alpha t_0\quad\ldots\ldots\ldots\ldots(22)$$

This is the same case as n is very small.

Putting the above value of n^2 into the first term of the right hand member of the equation (21), we have

$$\left.\begin{aligned}n^2&=-\frac{12}{h^2}\alpha t_0\left(1+\frac{1}{m}\right)\left\{1+\frac{\Theta^2 h^2}{24\,t_0}N'\right\}\\ \text{or}\qquad C&=-\frac{E\alpha t_0}{1-\dfrac{1}{m}}\left\{1+\frac{\Theta^2 h^2}{24\,t_0}N'\right\}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

for the linear distribution of temperature.

The Values of N′ (PL. IV)

nl	1	2	5	8	10	11	12	13	14	15	16	17	18	19	20	30	40	50	100
N'	.264	.230	.186	.124	.100	.091	.083	.077	.071	.067	.062	.058	.055	.052	.050	.033	.025	.020	.010

The Bending Moment and Bending Stress along y axis are;

$$M_y=-\frac{E\Phi}{1-\dfrac{1}{m}}\left\{1-\frac{\cosh ny}{\cosh\dfrac{nl}{2}}\right\}=-\frac{E\alpha\Theta I}{1-\dfrac{1}{m}}\left\{1-\frac{\cosh ny}{\cosh\dfrac{nl}{2}}\right\}$$

$$\sigma_y = -\frac{E}{1-\frac{1}{m}}\left\{-x\frac{\Phi}{I}\frac{\cosh ny}{\cosh\frac{nl}{2}} + \alpha\,(t-t_0)\right\} = -\frac{Ex}{1-\frac{1}{m}}\alpha\Theta\left\{1-\frac{\cosh ny}{\cosh\frac{nl}{2}}\right\} \Bigg\} \quad (24)$$

The last members are for the linear distribution of temperature. The analogous expressions for M_z and σ_z are easily obtained from the equations (I) (11) and (I) (12). These solutions will not become indefinitely large as cosh never crosses the zero line.

We see from this that the Bending Moment will not vanish except at the supported edges, and that its maximum value occurs on the center line of the strip.

At the extreme fibre we have the fibre stresses

$$\sigma_y^0 = C + \sigma_y = -\frac{E\alpha}{1-\frac{1}{m}}\left[t_0 + \frac{\Theta^2 h^2}{24 t_0^2} N''' N' \pm \frac{\Theta h}{2}\left\{1-\frac{\cosh ny}{\cosh\frac{nl}{2}}\right\}\right] \quad ..(25)$$

Remarks: for the fixed edges (encastré) we have the boundary conditions

$\xi = 0$ and $\frac{d\xi}{dy} = 0$ at $y = \pm\frac{l}{2}$, which give

$$\left.\begin{aligned} C_1 + D_1 \cos\frac{nl}{2} &= 0 \\ \mp D_1 n \sin\frac{nl}{2} &= 0 \end{aligned}\right\} \text{ for the case } t_0 > 0$$

From this we have $C_1 = D_1 = 0$, that is, the strip will not bend at all in the wider domain than that of the preceeding case. It will buckle, if the values of $\frac{nl}{2} \to \pi$.

(3) Circular plate.

Taking the origin of the coordinates at the center of plate and treating only the case in which F and ξ have the axial symmetry, from the eq. (I) (16), we have

$$\left.\begin{aligned} &\nabla^2\nabla^2 F = -\frac{E}{r}\frac{d\xi}{dr}\frac{d^2\xi}{dr^2} \\ &D\nabla^2\nabla^2\xi = p + h\left\{\frac{1}{r}\frac{dF}{dr}\frac{d^2\xi}{dr^2} + \frac{1}{r}\frac{d^2F}{dr^2}\frac{d\xi}{dr}\right\} = p + \frac{h}{r}\frac{d}{dr}\left(\frac{dF}{dr}\frac{d\xi}{dr}\right) \\ &\text{where } \nabla^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) \end{aligned}\right\} \quad ....(1)$$

The moments and shears are

$$\left.\begin{aligned} M_r &= -D\left\{\frac{d^2\xi}{dr^2}+\frac{1}{m}\frac{1}{r}\frac{d\xi}{dr}+\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\right\} \\ M_t &= -D\left\{\frac{1}{r}\frac{d\xi}{dr}+\frac{1}{m}\frac{d^2\xi}{dr^2}+\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\right\} \\ M_{rt} &= 0 \end{aligned}\right\} \quad \cdots\cdots(2)$$

$$\left.\begin{aligned} V_r &= -D\frac{d}{dr}\nabla^2\xi \\ V_t &= 0 \end{aligned}\right\} \quad \cdots\cdots(3)$$

The stresses due to the flexure are

$$\left.\begin{aligned} \sigma_r &= -\frac{E}{1-\frac{1}{m^2}}\left\{x\frac{d^2\xi}{dr^2}+\frac{1}{m}\frac{x}{r}\frac{d\xi}{dr}+\alpha\left(1+\frac{1}{m}\right)(t-t_0)\right\} \\ \sigma_t &= -\frac{E}{1-\frac{1}{m^2}}\left\{\frac{x}{r}\frac{d\xi}{dr}+\frac{x}{m}\frac{d^2\xi}{dr^2}+\alpha\left(1+\frac{1}{m}\right)(t-t_0)\right\} \\ \tau_{rt} &= 0 \end{aligned}\right\} \quad \cdots\cdots(4)$$

The extensional stresses in the Neutral Surface are

$$\sigma_r=\frac{1}{r}\frac{dF}{dr}, \quad \sigma_t=\frac{d^2F}{dr^2}, \quad \tau_{rt}=0 \quad \cdots\cdots(5)$$

The extensional strains of the Neutral Plane are obtained as follows. The strains solely due to the stresses are;

$$\left.\begin{aligned} \varepsilon_r &= \frac{d\Delta r'}{dr}+\frac{1}{2}\left(\frac{d\xi}{dr}\right)^2 \\ \varepsilon_t &= \frac{\Delta r'}{r} \\ \gamma_{rt} &= 0 \end{aligned}\right\} \quad \cdots\cdots(6)$$

where $\Delta r'$ is the radial displacement due to the stresses. And we have

$$\left.\begin{aligned} \frac{d\Delta r}{dr} &= \frac{d\Delta r'}{dr}+\alpha t_0 \\ \frac{\Delta r}{r} &= \frac{\Delta r'}{r}+\alpha t_0 \end{aligned}\right\} \quad \cdots\cdots(7)$$

where Δr is the total radial displacement due to stress and temperature.

Therefore, we have

$$\left.\begin{aligned} \varepsilon_r &= \frac{d\Delta r}{dr}-\alpha t_0+\frac{1}{2}\left(\frac{d\xi}{dr}\right)^2 \\ \varepsilon_t &= \frac{\Delta r}{r}-\alpha t_0 \end{aligned}\right\} \quad \cdots\cdots(8)$$

And we have

$$\left.\begin{aligned}\varepsilon_r&=\frac{1}{E}\left(\sigma_r-\frac{1}{m}\sigma_t\right)\\ \varepsilon_t&=\frac{1}{E}\left(\sigma_t-\frac{1}{m}\sigma_r\right)\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

or

$$\left.\begin{aligned}\frac{d\Delta r}{dr}&=\frac{1}{E}\left(\sigma_r-\frac{1}{m}\sigma_t\right)+\alpha t_0-\frac{1}{2}\left(\frac{d\xi}{dr}\right)^2\\ \frac{\Delta r}{r}&=\frac{1}{E}\left(\sigma_t-\frac{1}{m}\sigma_r\right)+\alpha t_0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(10)$$

We have the stresses in terms of strains as follows:

$$\left.\begin{aligned}\sigma_r&=\frac{E}{1-\frac{1}{m^2}}\left(\varepsilon_r+\frac{1}{m}\varepsilon_t\right)=\frac{E}{1-\frac{1}{m^2}}\left\{\frac{d\Delta r}{dr}+\frac{1}{m}\frac{\Delta r}{r}-\alpha t_0\left(1+\frac{1}{m}\right)+\frac{1}{2}\left(\frac{d\xi}{dr}\right)^2\right.\\ \sigma_t&=\frac{E}{1-\frac{1}{m^2}}\left(\varepsilon_t+\frac{1}{m}\varepsilon_r\right)=\frac{E}{1-\frac{1}{m^2}}\left\{\frac{\Delta r}{r}+\frac{1}{m}\frac{d\Delta r}{dr}-\alpha t_0\left(1+\frac{1}{m}\right)+\frac{1}{2m}\left(\frac{d\xi}{dr}\right)^2\right.\end{aligned}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Assuming the extensional stress $\sigma_r=C$, const, as the first approximation, we have $\frac{dF}{dr}=Cr$ and $\frac{d^2F}{dr^2}=\sigma_t=C$, i. e. this is the case of the hydrostatic pressure.

The second equation of (1) is

$$D\nabla^2\nabla^2\xi=\frac{h}{r}\frac{d}{dr}\left(Cr\frac{d\xi}{dr}\right)\quad\text{if}\quad p=0\quad\ldots\ldots\ldots\ldots\ldots(12)$$

And the equations (10) lead to

$$\left.\begin{aligned}\frac{d\Delta r}{dr}&=\frac{1}{E}\left(1-\frac{1}{m}\right)C+\alpha t_0-\frac{1}{2}\left(\frac{d\xi}{dr}\right)^2\\ \frac{\Delta r}{r}&=\frac{1}{E}\left(1-\frac{1}{m}\right)C+\alpha t_0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(13)$$

or

$$\frac{d\Delta r}{dr}=\frac{\Delta r}{r}-\frac{1}{2}\left(\frac{d\xi}{dr}\right)^2\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

And the equations (11) lead to

$$\sigma_r=\sigma_t=\frac{E}{1-\frac{1}{m}}\left(\frac{\Delta r}{r}-\alpha t_0\right)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

This must be equal to a const. C; therefore we have $\frac{\Delta r}{r}=\text{const.}$ On the boundary $r=a$, we have $\Delta r=0$ for *the unyielding support*, or we have

$\left(\frac{\Delta r}{r}\right)_{r=a}=0 \qquad \therefore \quad \frac{\Delta r}{r}=0$ and we have

$$\sigma_r=\sigma_t=-\frac{E}{1-\frac{1}{m}}\alpha t_0=C \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

And from (13) we must have

$$\int_0^a \frac{d\Delta r}{dr}dr=-\frac{1}{2}\int_0^a\left(\frac{d\xi}{dr}\right)^2 dr=0 \quad \ldots\ldots\ldots\ldots (17)$$

Strictly speaking this is not fulfilled unless $\left(\frac{d\xi}{dr}\right)^2$ is zero i. e. the deflection angle is very small, but as it will be seen in the afterwards that it is the same order infinitesimal compared with the approximate value of ξ when C is not large.

From the equation (12) we have, on putting $\varphi=\frac{d\xi}{dr}$ "meridian inclination"

$$D\frac{d}{dr}\left\{r\frac{d}{dr}(r\varphi)\right\}=hC\varphi$$

or

$$\frac{d^2\varphi}{dr^2}+\frac{1}{r}\frac{d\varphi}{dr}+\left(-\frac{hC}{D}-\frac{1}{r^2}\right)\varphi=0$$

(I) If C<O, put $-\frac{hC}{D}=\beta^2$

$$\frac{d^2\varphi}{dr^2}+\frac{1}{r}\frac{d\varphi}{dr}+\left(\beta^2-\frac{1}{r^2}\right)\varphi=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

Its general solution is

$$\varphi=AJ_1(\beta r)+BY_1(\beta r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

We reject Y_1 as it tends to ∞ at the origin.

As the boundary condition of *the supported edge*, we have

$$r=a; \quad M_r=-D\left\{\frac{d\varphi}{dr}+\frac{1}{m}\frac{\varphi}{r}+\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\right\}=0$$

and at the center $r=0$, $\varphi=0$; this is satisfied of itself as $J_1(0)=0$. The above equation determines A.

$$A\beta J_1'(\beta a)+\frac{1}{m}\frac{AJ_1(\beta a)}{a}+\left(1+\frac{1}{m}\right)\frac{\Phi}{I}=0$$

$$\therefore \quad A=-\frac{\left(1+\frac{1}{m}\right)\frac{\Phi}{I}}{\beta J_1'(\beta a)+\frac{1}{ma}J_1(\beta a)}=-\frac{\left(1+\frac{1}{m}\right)\frac{\Phi}{I}}{\beta J_0(\beta a)-\frac{1}{a}\left(1-\frac{1}{m}\right)J_1(\beta a)}^{(1)} \ldots (20)$$

(1) $J_1'(\beta a)=-\frac{1}{\beta a}J_1(\beta a)+J_0(\beta a)$

$$\therefore \quad \xi = -\frac{\left(1+\frac{1}{m}\right)\frac{\Phi}{I}}{\beta J_0(\beta a) - \frac{1}{a}\left(1-\frac{1}{m}\right)J_1(\beta a)} \int J_1(\beta r)\, dr + C'$$

The boundary condition $\xi=0$ at $r=a$ gives;

$$C' = -\frac{\left(1+\frac{1}{m}\right)\frac{\Phi}{I}}{\beta J_0(\beta a) - \frac{1}{a}\left(1-\frac{1}{m}\right)J_1(\beta a)} \frac{J_0(\beta a)}{\beta}{}^{(1)}$$

whence we have

$$\xi = \frac{\left(1+\frac{1}{m}\right)\frac{\Phi}{I}}{J_0(\beta a) - \left(1-\frac{1}{m}\right)\frac{J_1(\beta a)}{\beta a}} \left\{ \frac{J_0(\beta r) - J_0(\beta a)}{\beta^2} \right\} \quad \ldots\ldots\ldots (21)$$

This solution fails when $J_0(\beta a) - \left(1-\frac{1}{m}\right)\frac{J_1(\beta a)}{\beta a} = 0$. With this value of β the plate buckles; for example, if $m=3$, the smallest positive root is $\beta a = 2.05$

$$\therefore \quad C = -\beta^2 \frac{D}{h} = -\frac{4.2025}{a^2}\left(\frac{D}{h}\right) \quad \ldots\ldots\ldots\ldots (22)$$

Our solution is good for the absolute value of C less than the above one. When $|C|$ is small, we have

$$\frac{J_0(\beta r) - J_0(\beta a)}{\beta^2} = \frac{1}{\beta^2}\left\{1 - \frac{(\beta r)^2}{4} + \frac{(\beta r)^4}{64} - \ldots\ldots - \left(1 - \frac{(\beta a)^2}{4} + \frac{(\beta a)^4}{64} - \ldots\ldots\right)\right\}$$

$$= \frac{1}{4}(a^2 - r^2) - \frac{\beta^2}{64}(a^4 - r^4) + \ldots\ldots$$

$$J_0(\beta a) - \left(1-\frac{1}{m}\right)\frac{J_1(\beta a)}{\beta a} = 1 - \frac{(\beta a)^2}{4} + \frac{(\beta a)^4}{64} - \ldots - \left(1-\frac{1}{m}\right)\left\{\frac{1}{2} - \frac{(\beta a)^2}{16} + \frac{(\beta a)^4}{384} - \ldots\right\}$$

$$= \frac{1}{2}\left(1+\frac{1}{m}\right) - \frac{(\beta a)^2}{16}\left(3+\frac{1}{m}\right) + \ldots\ldots$$

$$\therefore \quad \xi = \frac{16(a^2 - r^2) - \beta^2(a^4 - r^4)}{32 - 4\,\dfrac{3+\frac{1}{m}}{1+\frac{1}{m}}(\beta a)^2} \frac{\Phi}{I} \quad \ldots\ldots\ldots\ldots (23)$$

If C or β is very small neglecting 2nd order we have

(1) $\int J_1(\beta r)\, dr = \int \frac{J_1(\beta r)\, d(\beta r)}{\beta} = -\frac{J_0(\beta r)}{\beta}$

$$\xi=\frac{1}{2}(a^2-r^2)\frac{\Phi}{I}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

This is the same as obtained in my previous paper (loc. cit.) Here we will see how far the condition (17) will be fulfilled.

$$\int_0^a\left(\frac{d\xi}{dr}\right)^2dr=\frac{A}{\beta}\int_0^a\{J_1(\beta r)\}^2d(\beta r)$$

$$\{J_1(\beta r)\}^2=\frac{\left(\frac{1}{2}\beta r\right)^2}{(1!)^2}\left\{1-\frac{3(\beta r)^2}{4\cdot 1\cdot 3}+\frac{3\cdot 5}{4\cdot 6}\frac{(\beta r)^4}{1\cdot 2\cdot 3\cdot 4}-\ldots\right\}\overset{(1)}{\doteqdot}\frac{1}{4}(\beta r)^2$$

$$\int_0^a\left(\frac{d\xi}{dr}\right)^2dr=\frac{A}{4\beta}\int_0^a(\beta r)^2d(\beta r)=\frac{A}{4\beta}\frac{\beta^3a^3}{3}$$

This is the same order in β as we have in the equation (23). We may assume this equals to zero if we are contented with the same order of approximation as the value of ξ shown by the eq. (24).

The Bending Moments are

$$\left.\begin{aligned}
M_r&=-\left(1+\frac{1}{m}\right)\frac{\Phi}{I}D\left\{1-\frac{\beta J_0(\beta r)-\frac{1}{r}\left(1-\frac{1}{m}\right)J_1(\beta r)}{BJ_0(\beta a)-\frac{1}{a}\left(1-\frac{1}{m}\right)J_1(\beta a)}\right\}\\
M_t&=-\left(1+\frac{1}{m}\right)\frac{\Phi}{I}D\left\{1-\frac{\frac{\beta}{m}J_0(\beta r)+\frac{1}{r}\left(1-\frac{1}{m}\right)J_1(\beta r)}{BJ_0(\beta a)-\frac{1}{a}\left(1-\frac{1}{m}\right)J_1(\beta a)}\right\}\\
M_{rt}&=0\\
&\text{where}\quad \beta^2=-\frac{hC}{D}\quad[C<0]
\end{aligned}\right\}\quad\ldots(25)$$

M_r is not exactly zero except at $r=a$.

The stresses due to the flexure

$$\left.\begin{aligned}
\sigma_r&=-\frac{E}{1-\frac{1}{m}}\left\{\alpha(t-t_0)-\frac{\beta J_0(\beta r)-\frac{1}{r}\left(1-\frac{1}{m}\right)J_1(\beta r)}{\beta J_0(\beta a)-\frac{1}{a}\left(1-\frac{1}{m}\right)J_1(\beta a)}\frac{x\Phi}{I}\right\}\\
\sigma_t&=-\frac{E}{1-\frac{1}{m}}\left\{\alpha(t-t_0)-\frac{\frac{\beta}{m}J_0(\beta r)+\frac{1}{r}\left(1-\frac{1}{m}\right)J_1(\beta r)}{\beta J_0(\beta a)-\frac{1}{a}\left(1-\frac{1}{m}\right)J_1(\beta a)}\frac{x\Phi}{I}\right\}\\
\tau_{rt}&=0
\end{aligned}\right\}\quad\ldots(26)$$

(1) Watson: Bessel Function; p. 32-33.

σ_r is zero only when $r=a$ and $\Phi=\alpha\Theta I$ $\left(\text{where } \Theta=\dfrac{t-t_0}{x}\right)$.

(II) If C>0 or $\dfrac{hC}{D}>0$, we have

$$\frac{d^2\varphi}{dr^2}+\frac{1}{r}\frac{d\varphi}{dr}-\left(\beta^2+\frac{1}{r^2}\right)\varphi=0 \quad\ldots\ldots(27)$$

on putting $\dfrac{hC}{D}=\beta^2$

Its general solution is

$$\varphi=AI_1(\beta r)+BK_1(\beta r) \quad\ldots\ldots(28)$$

where $I_1(\beta r)$ and $K_1(\beta r)$ are the Modified Bessel Function of the First and Second kind respectively.

We reject K_1 as it tends to ∞ at the origin.

At the center $r=0$, we have $\varphi=0$ as $I_1(0)=0$.

As the boundary condition of *the supported edge*, we have

$$M_r=0 \quad \text{or} \quad \frac{d\varphi}{dr}+\frac{1}{m}\frac{\varphi}{r}+\left(1+\frac{1}{m}\right)\frac{\Phi}{I}=0 \quad \text{at} \quad r=a.$$

This determines A;

$$A\beta I_1'(\beta a)+\frac{1}{m}\frac{AI_1(\beta a)}{a}+\left(1+\frac{1}{m}\right)\frac{\Phi}{I}=0$$

$$\therefore \quad A=-\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\frac{1}{\beta I_0(\beta a)-\dfrac{1}{a}\left(1-\dfrac{1}{m}\right)I_1(\beta a)}^{(1)} \quad\ldots\ldots(29)$$

$$\therefore \quad \xi=-\frac{\left(1+\dfrac{1}{m}\right)\dfrac{\Phi}{I}}{\beta I_0(\beta a)-\dfrac{1}{a}\left(1-\dfrac{1}{m}\right)I_1(\beta a)}\int I_1(\beta r)\,dr^{(2)}+C'$$

The boundary condition $\xi=0$ at $r=a$ gives;

$$C'=\frac{\left(1+\dfrac{1}{m}\right)\dfrac{\Phi}{I}}{I_0(\beta a)-\left(1-\dfrac{1}{m}\right)\dfrac{I_1(\beta a)}{\beta a}}\frac{I_0(\beta a)}{\beta^2}$$

$$\therefore \quad \xi=\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\frac{1}{I_0(\beta a)-\left(1-\dfrac{1}{m}\right)\dfrac{I_1(\beta a)}{\beta a}}\left\{\frac{I_0(\beta a)-I_0(\beta r)}{\beta^2}\right\} \quad\ldots(30)$$

(1) $(\beta a)\,I_1'(\beta a)=-I_1(\beta a)+\beta a\,I_0(\beta a)$

(2) $\int I_1(\beta r)\,dr=\dfrac{\int I_1(\beta r)\,d(\beta r)}{\beta}=\dfrac{I_0(\beta r)}{\beta}$

For the very small value of C or β;

$$\left.\begin{aligned}
&I_0(\beta a)=1+\frac{(\beta a)^2}{4}+\frac{(\beta a)^4}{64}+\dots\\
&I_1(\beta a)=\frac{1}{2}\beta a+\frac{1}{16}(\beta a)^3+\dots\\
&\xi=\frac{\Phi}{I}\frac{16(a^2-r^2)+\beta^2(a^4-r^4)}{32+4\dfrac{3+\frac{1}{m}}{1+\frac{1}{m}}\beta^2a^2}\\
\text{or}\quad &\xi=\frac{\Phi}{I}\frac{1}{2}(a^2-r^2)
\end{aligned}\right\}\dots\dots(31)$$

This is the same with (23) and (24).

The denominator $I_0(\beta a)-\left(1-\frac{1}{m}\right)\frac{I_1(\beta a)}{\beta a}$ (1) has no real root, hence we will not encounter with the failing case. For large value of C or β, from the asymptotic expansion of I_n, we have $\frac{I_0}{I_1}\doteqdot 1$.

$$\therefore\quad \xi=\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\frac{1}{1-\left(1-\frac{1}{m}\right)\frac{1}{\beta a}}\left\{\frac{1}{\beta^2}-\frac{I_0(\beta r)}{\beta^2 I_0(\beta a)}\right\}\dots\dots(32)$$

The center deflection ξ_0 is as follows;

$$\xi_0=\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\frac{1}{1-\left(1-\frac{1}{m}\right)\frac{1}{\beta a}}\left\{\frac{1}{\beta^2}-\frac{1}{\beta^2 I_0(\beta a)}\right\}\dots\dots(33)$$

Here if β is very large, ξ_0 tends to zero.

The Bending Moments are

$$\left.\begin{aligned}
M_r&=-\left(1+\frac{1}{m}\right)\frac{\Phi}{I}D\left\{1-\frac{\beta I_0(\beta r)-\frac{1}{r}\left(1-\frac{1}{m}\right)I_1(\beta r)}{\beta I_0(\beta a)-\frac{1}{a}\left(1-\frac{1}{m}\right)I_1(\beta a)}\right\}\\
M_t&=-\left(1+\frac{1}{m}\right)\frac{\Phi}{I}D\left\{1-\frac{\frac{\beta}{m}I_0(\beta r)+\frac{1}{r}\left(1-\frac{1}{m}\right)I_1(\beta r)}{\beta I_0(\beta a)-\frac{1}{a}\left(1-\frac{1}{m}\right)I_1(\beta a)}\right\}
\end{aligned}\right\}\dots(34)$$

(1) If $m=3$; $I_0(\beta a)-\left(1-\frac{1}{m}\right)\frac{I_1(\beta a)}{\beta a}=\left(1-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{24}\right)(\beta a)^2+\dots$

$$=\frac{2}{3}+\frac{5}{24}(\beta a)^2+\dots$$

$$M_{rt}=0$$

where $\beta^2=\dfrac{hC}{D}\quad [C>0]$

M_r is not exactly zero except at $r=a$.

The stresses due to the flexure are

$$\left.\begin{aligned}
\sigma_r&=-\frac{E}{1-\frac{1}{m}}\left\{\alpha(t-t_0)-\frac{\beta I_0(\beta r)-\frac{1}{r}\left(1-\frac{1}{m}\right)I_1(\beta r)}{\beta I_0(\beta a)-\frac{1}{a}\left(1-\frac{1}{m}\right)I_1(\beta a)}\cdot\frac{x\Phi}{I}\right\}\\
\sigma_t&=-\frac{E}{1-\frac{1}{m}}\left\{\alpha(t-t_0)-\frac{\frac{\beta}{m}I_0(\beta r)+\frac{1}{r}\left(1-\frac{1}{m}\right)I_1(\beta r)}{\beta I_0(\beta a)-\frac{1}{a}\left(1-\frac{1}{m}\right)I_1(\beta a)}\cdot\frac{x\Phi}{I}\right\}\\
\tau_{rt}&=0
\end{aligned}\right\}\qquad(35)$$

σ_r is not zero except for $\Phi=\alpha\Theta I$ and $r=a$.

(III) We have already shown that our assumption about the extensional stress is only fulfilled on condition that $\left(\dfrac{d\xi}{dr}\right)^2$ is small. Our assumption requires, moreover, that $\dfrac{d\xi}{dr}$ or $\dfrac{d^2\xi}{dr^2}$ will be as small as to satisfy the first fundamental equation of (1). We will try how far this condition will be pushed forwards.

Take the approximate value of ξ as is shown by the eq. (24) or (31). And we have $\dfrac{d\xi}{dr}=-r\dfrac{\Phi}{I}$ and $\dfrac{d^2\xi}{dr^2}=-\dfrac{\Phi}{I}$

$\therefore$ the first equation of (1) becomes

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\,r\frac{dF}{dr}\right)\right\}=-\frac{E\Phi^2}{I^2}\quad\ldots\ldots\ldots\ldots\ldots(36)$$

The solution of this equation is

$$F=-\frac{E\Phi^2}{I^2}\frac{r^4}{64}+A+Br^2+C\log r+Dr^2\log r$$

On condition that σ_r and σ_t are of finite value at the origin, we have $C=0$ and $D=0$ and A is not necessary as far as the stresses are concerned. And we have

$$F=-\frac{E\Phi^2}{I^2}\frac{r^4}{64}+Br^2\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(37)$$

Therefore we have

$$\left.\begin{aligned}\sigma_r&=\frac{1}{r}\frac{dF}{dr}=-\frac{E\Phi^2}{I^2}\frac{r^2}{16}+2B\\ \sigma_t&=\frac{d^2F}{dr^2}=-\frac{E\Phi^2}{I^2}\frac{3r^2}{16}+2B\end{aligned}\right\}$$

If we have $\sigma_r=C=-\dfrac{E}{1-\dfrac{1}{m}}\alpha t_0$ at center $r=o$;

$$2B=-E\frac{\alpha t_0}{1-\dfrac{1}{m}}$$

and we have the stresses

$$\left.\begin{aligned}\sigma_r&=-E\left\{\frac{\alpha t_0}{1-\dfrac{1}{m}}+\frac{\Phi^2}{I^2}\frac{1}{16}r^2\right\}\\ \sigma_t&=-E\left\{\frac{\alpha t_0}{1-\dfrac{1}{m}}+\frac{\Phi^2}{I^2}\frac{1}{16}3r^2\right\}\end{aligned}\right\}\quad\cdots\cdots\cdots(38)$$

If the temperature gradient is a straight line, $\Phi=\alpha\Theta I$; here Θ shows the gradient angle. And we have

$$\left.\begin{aligned}\sigma_r&=-E\left\{\frac{\alpha t_0}{1-\dfrac{1}{m}}+\frac{\alpha^2\Theta^2}{16}r^2\right\}\\ \sigma_t&=-E\left\{\frac{\alpha t_0}{1-\dfrac{1}{m}}+\frac{\alpha^2\Theta^2}{16}3r^2\right\}\end{aligned}\right\}\quad\cdots\cdots\cdots(39)$$

If the gradient Θ is not large as compared with t_0, the second terms of the right hand members of the above equations will be negligibly small in comparison with the first, and our assumption will be justified.

To obtain a more accurate expression of ξ, we proceed with successive approximation. Taking the value of σ_r as shown by eq. (39), and putting

$$\frac{dF}{dr}=\left\{C+C'r^2\right\}r\qquad\left[C'=\frac{E\alpha^2\Theta^2}{16}>0\right]$$

in the second equation of (1), we have

$$\frac{d^2\varphi}{dr^2}+\frac{1}{r}\frac{d\varphi}{dr}-\frac{1}{r^2}\varphi-\left\{\frac{hC}{D}-\frac{hC'}{D}r^2\right\}\varphi=0$$

Putting $\frac{hC}{D}=\mp\beta^2$ [take the above sign when $C<0$ & the lower when $C>0$]

and $\frac{hC'}{D}=\gamma^2$

$$\frac{d^2\varphi}{dr^2}+\frac{1}{r}\frac{d\varphi}{dr}-\frac{\varphi}{r^2}-\left\{\mp\beta^2-\gamma^2r^2\right\}\varphi=0$$

or
$$\frac{d^2\varphi}{dr^2}+\frac{1}{r}\frac{d\varphi}{dr}+\left(\mp\beta^2+\gamma^2r^2-\frac{1}{r^2}\right)\varphi=0 \quad\dots\dots\dots\dots\dots(40)$$

Assuming $\varphi=A_m r^m$, and putting it in the above eq. we have

$$(m^2-1)r^{m-2}\pm\beta^2r^m+\gamma^2r^{m+2}=0$$

Putting the first coef. $=0$, we have $m=1$. Therefore we have an odd power series beginning with the term r^1 as a solution of eq. (40).

$$\varphi=A_1r+A_3r^3+A_5r^5+A_7r^7+\dots\dots$$

Substituting this in the above equation, we have

$$\left(3\cdot2\,A_3r+5\cdot4\,A_5r^3+7\cdot6\,A_7r^5+\dots\dots\right)+\left(\frac{A_1}{r}+3A_3r5A_5r^3+7A_7r^5+\dots\dots\right)$$
$$\mp\beta^2\left(A_1r+A_3r^3+A_5r^5+\dots\right)+\gamma^2\left(A_1r^3+A_3r^5+A_5r^7+\dots\right)$$
$$-\left(\frac{A_1}{r}+A_3r+A_5r^3+A_7r^5+\dots\right)=0$$

Putting coefficients of different powers of r equal to zero, we have

$$A_3=\mp\frac{\beta^2}{2\cdot4}A_1,\quad A_5=\frac{1}{4\cdot6}\left(\frac{\beta^4}{2\cdot4}-\gamma^2\right)A_1,\quad A_7=\mp\frac{1}{6\cdot8}\left(\frac{\beta^6}{2\cdot4\cdot4\cdot6}-\frac{\gamma^2\beta^2}{6}\right)A_1\dots\dots$$

$$\therefore\quad \varphi=A_1\left\{r\mp\frac{\beta^2}{2\cdot4}r^3+\frac{1}{4\cdot6}\left(\frac{\beta^4}{2\cdot4}-\gamma^2\right)r^5\mp\frac{1}{6\cdot8}\left(\frac{\beta^6}{2\cdot4\cdot4\cdot6}-\frac{\gamma^2\beta^2}{6}\right)r^7+\dots\right\} \quad (41)$$

Or dividing each term by a & putting $A=aA_1$

$$\varphi=A\left\{\frac{r}{a}\mp\frac{\beta^2a^2}{2\cdot4}\left(\frac{r}{a}\right)^3+\frac{1}{4\cdot6}\left(\frac{\beta^4a^4}{2\cdot4}-\gamma^2a^4\right)\left(\frac{r}{a}\right)^5\mp\frac{1}{6\cdot8}\left(\frac{\beta^6a^6}{2\cdot4\cdot4\cdot6}-\frac{\gamma^2\beta^2a^6}{6}\right)\left(\frac{r}{a}\right)^7\right.$$
$$\left.+\dots\right\}\quad\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(42)$$

We must have one more series to complete the solution.

This latter series has, however, a singular point at the origin.(1) Therefore we take, in the present case, only the solution (41) or (42),

$$\varphi = A\Psi_1(r)$$

(1) To obtain another particular solution in series, put $\varphi = ry$ in the eq. (40)

$$r^2\frac{d^2\varphi}{dr^2} + r\frac{d\varphi}{dr} = \left\{1 \mp \beta^2 r^2 - \gamma^2 r^4\right\}\varphi \quad \ldots\ldots (A)$$

It becomes

$$r^2\frac{d^2y}{dr^2} + 3r\frac{dy}{dr} = (\mp\beta^2 r^2 - \gamma^2 r^4)\, y$$

On putting again $r^2 = x$, we have

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = (\mp a - bx)y \quad \left[a = \left(\frac{\beta}{2}\right)^2,\ b = \left(\frac{\gamma}{2}\right)^2\right] \quad \ldots\ldots (B)$$

Putting $y = x^m + A_1 x^{m+1} + A_2 x^{m+2} \ldots\ldots$ in the above eq. and equating coeffs. of different powers of x, we have

$$y_1 = 1 \mp \frac{a}{2}x + \left(\frac{a^2}{12} - \frac{b}{6}\right)x^2 \mp \frac{1}{12}\left(\frac{a^3}{12} - \frac{2}{3}ab\right)x^3 + \ldots\ldots$$

This is the same solution with (41).

If we know a particular solution y_1 of the linear homogeneous equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

we obtain another solution by

$$y_2 = y_1 \int \frac{dx}{y_1^2} e^{-\int p dx} \quad \text{[Goursat: Cours d'Analyse II. p. 433]}$$

$$\therefore\ y_2 = y_1 \int \frac{dx}{\left\{1 \mp \frac{a}{2}x + \left(\frac{a^2}{12} - \frac{b}{6}\right)x^2 \mp \frac{1}{12}\left(\frac{a^2}{12} - \frac{2}{3}ab\right)x^3 + \ldots\right\}^2} e^{-\int \frac{2dx}{x}}$$

$$= y_1 \int \frac{dx}{x^2}\left\{1 \mp \left[\frac{a}{2}x \mp \left(\frac{a^2}{12} - \frac{b}{6}\right)x^2 + \frac{1}{12}\left(\frac{a^2}{12} - \frac{2}{3}ab\right)x^3 \mp \ldots\right]\right\}^{-2}$$

$$= y_1 \int \frac{dx}{x^2}\left\{1 \pm 2\left[\frac{a}{2}x \mp \left(\frac{a^2}{12} - \frac{b}{6}\right)x^2 + \frac{1}{12}\left(\frac{a^2}{12} - \frac{2}{3}ab\right)x^3 \mp \ldots\right]\right.$$
$$+ 3\left[\frac{a}{2}x \mp \left(\frac{a^2}{12} - \frac{b}{6}\right)x^2 + \frac{1}{12}\left(\frac{a^2}{12} - \frac{2}{3}ab\right)x^3 \mp \ldots\right]^2$$
$$\left. \pm \ldots\ldots\ldots\ldots \right\}$$

$$= y_1 \int \frac{dx}{x^2}\left\{1 \pm ax + \left[\frac{3a^2}{4} - 2\left(\frac{a^2}{12} - \frac{b}{6}\right)\right]x^2 \ldots\right\}$$

$$= y_1 \left\{-\frac{1}{x} \pm a\log x + \left[\frac{3a^2}{4} - 2\left(\frac{a^2}{12} - \frac{b}{6}\right)\right]x \ldots\ldots\right\}$$

The general solution of (B) is

$$y = Ay_1(x) + By_2(x)$$

And that of (A) is

$$\varphi = Ary_1(r^2) + Bry_2(r^2)$$

here $\Psi_1(r)=r\mp\frac{\beta^2}{2\cdot4}r^3+\frac{1}{4\cdot6}\left(\frac{\beta^4}{2\cdot4}-\gamma^2\right)r^5\mp\frac{1}{6\cdot8}\left(\frac{\beta^6}{2\cdot4\cdot4\cdot6}-\frac{\gamma^2\beta^2}{6}\right)r^7+\ \ ..(43)$

As the boundary condition of *the supported edge*, we have

$$M_r=-D\left\{\frac{\partial\varphi}{\partial r}+\frac{1}{m}\frac{\varphi}{r}+\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\right\}=0 \quad \text{at} \quad r=a$$

and at the center $r=0$, $\varphi=0$; the latter condition is satisfied of itself as $\Psi_1(0)=0$.

The former condition determines A;

$$A\Psi_1'(a)+\frac{1}{ma}A\Psi_1(a)+\left(1+\frac{1}{m}\right)\frac{\Phi}{I}=0$$

$$A=-\frac{\left(1+\frac{1}{m}\right)\frac{\Phi}{I}}{\Psi_1'(a)+\frac{1}{ma}\Psi_1(a)}$$

And as $\xi=0$ at $r=a$, we have

$$\xi=\frac{\left(1+\frac{1}{m}\right)\frac{\Phi}{I}}{\Psi_1'(a)+\frac{1}{ma}\Psi_1(a)}\int_r^a\Psi_1(r)dr$$

As $\quad \Psi_1'(r)=1\mp\frac{3\beta^2}{2\cdot4}r^2+\frac{5}{4\cdot6}\left(\frac{\beta^4}{2\cdot4}-\gamma^2\right)r^4\mp\frac{7}{6\cdot8}\left(\frac{\beta^6}{2\cdot4\cdot4\cdot6}-\frac{\beta^2\gamma^2}{6}\right)r^6+\cdots$

$$\int\Psi_1(r)dr=\frac{r^2}{2}\mp\frac{\beta^2r^4}{2\cdot4\cdot4}+\frac{1}{4\cdot6\cdot6}\left(\frac{\beta^4}{2\cdot4}-\gamma^2\right)r^6\mp\frac{1}{6\cdot8\cdot8}\left(\frac{\beta^6}{2\cdot4\cdot4\cdot6}-\frac{\beta^2\gamma^2}{6}\right)r^8+\cdots$$

We have

$$\xi=\frac{\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\left[\frac{r^2}{2}\mp\frac{\beta^2r^4}{2\cdot4\cdot4}+\frac{1}{4\cdot6\cdot6}\left(\frac{\beta^4}{2\cdot4}-\gamma^2\right)r^6\mp\frac{1}{6\cdot8\cdot8}\left(\frac{\beta^6}{2\cdot4\cdot4\cdot6}-\frac{\beta^2\gamma^2}{6}\right)r^8+\dots\right]_r^a}{\left\{1\mp\frac{3\beta^2}{2\cdot4}a^2+5\left(\frac{\beta^4}{2\cdot4\cdot4\cdot6}-\frac{\gamma^2}{4\cdot6\cdot6}\right)a^4+\dots\right\}+\frac{1}{m}\left\{1\mp\frac{\beta^2a^2}{2\cdot4}+\left(\frac{\beta^4}{2\cdot4\cdot4\cdot6}-\frac{\gamma^2}{4\cdot6}\right)a^4+\dots\right\}}$$

$$=\left(1+\frac{1}{m}\right)\frac{\Phi}{I}\,\frac{\frac{1}{2}(a^2-r^2)\mp\frac{\beta^2}{2\cdot4\cdot4}(a^4-r^4)+\left(\frac{\beta^4}{2\cdot4\cdot4\cdot6\cdot6}-\frac{\gamma^2}{4\cdot6\cdot6}\right)(a^6-r^6)+\cdots}{\left(1+\frac{1}{m}\right)\mp\left(3+\frac{1}{m}\right)\frac{\beta^2a^2}{2\cdot4}+\left(5+\frac{1}{m}\right)\left(\frac{\beta^4}{2\cdot4\cdot4\cdot6}-\frac{\gamma^2}{4\cdot6}\right)a^4+\cdots}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(46)$$

Neglecting 4th and higher orders of β and 2nd and higher orders of γ;

$$\xi=\frac{16(a^2-r^2)\mp\beta^2(a^4-r^4)}{32\mp4\dfrac{3+\dfrac{1}{m}}{1+\dfrac{1}{m}}\beta^2a^2}\frac{\Phi}{I}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(44)$$

This is the same as eq. (23) or (31).

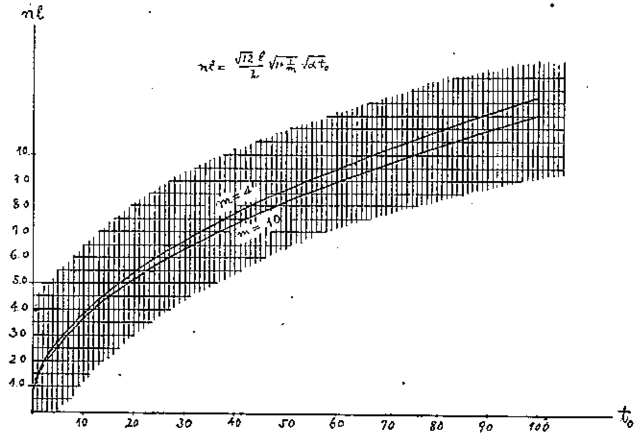
If we neglect, moreover, 2nd and higher orders of β and γ; we shall return to our original equation

$$\xi=\frac{1}{2}\frac{\Phi}{I}(a^2-r^2)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(48)$$

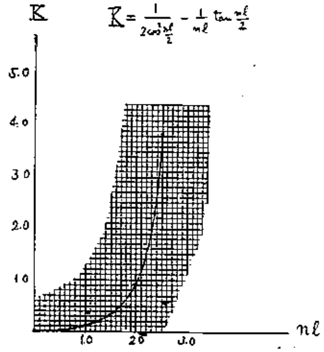
From this we see that the parabolic solution is approximately true either σ_r is assumed to be constant throughout or it may be assumed as eq. (39). In conclusion we see that the solution, which is obtained on condition that σ_r is a constant throughout, is to be applied with sufficient approximation.

Tokyo, Feb. 1928.

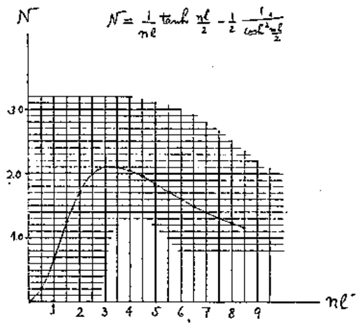
Pl. I.



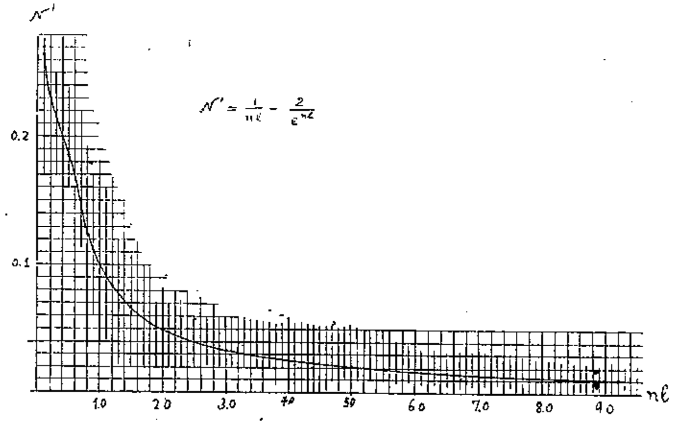
Pl. II.



Pl. III.



Pl. IV.



(土木學會誌第十四卷第三號附圖)