A Study on Diffusion Wave Runoff Model By Equivalent Frequency Transfer Function

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An application of Equivalent Frequency Transfer Function (EFTF) was applied in this study. This EFTF has originally been used among control engineers to analyze non-linear elements; for example threshold and saturation elements. Previous study by the authors had successfully derived the EFTF between rainfall input and discharge for real catchment area based on kinematic wave model. This study developed application of EFTF by using diffusion wave model with zero gradient of depth downstream boundary condition. Gain, time lag characteristics and vector locus are proposed in order to investigate efficiency and effectiveness of EFTF. It is also able to extend this technique to multi element basin.

Key Words: *Equivalent frequency transfer function (EFTF), Diffusion wave model, Gain, Time lag characteristics, Vector locus*

1. INTRODUCTION

The relationship between rainfall and runoff has been a significant theme of hydrological research for several years. The calculation of effective rainfall and derivation of the unit hydrograph are the main objectives of such studies. Many models have been proposed for modeling overland flow on a plane subject to a lateral inflow or rainfall. Typical models are the St.Venant model and its related models, for example, kinematic, diffusion and gravity wave which are obtained from the St.Venant equations. In 1970s Fujita M.[1), 2)] had pointed out the helpfulness of the frequency transfer function to analyze runoff phenomena. Hereafter, Saga[3)] had proposed an application of the frequency transfer function to storage function runoff model including time delay element for linearity system. Fujita M. and Tanaka G.[4), 5)] showed an application of the equivalent frequency response method (EFRM) to non-linear runoff models such as storage function model and kinematic wave model. But they did not lead the equivalent frequency transfer function. Luai H. and Fujita M.[6)] proposed a general method for calculating the equivalent frequency transfer function (EFTF) between rainfall input and discharge using distributed runoff model over slope of the basin. Extension of EFTF was carried out by Wanphen S. and Fujita M.[7]. They successfully applied EFTF to St.Venant equation and kinematic wave model with obtained data from the observation. This study proposed application of EFTF to diffusion wave model with zero depth gradient lower boundary condition. The obtained EFTF is a lumped function of distributed parameter runoff models over frequency domain. Gain, time lag characteristics and vector locus can be obtained by EFTF. Furthermore we have developed the application of EFTF to diffusion wave model from a slope to a general basin.

2. SINGLE SLOPE ANALYSIS

Beginning from the simplest one, single slope analysis was considered first. Its continuity equation and momentum equation are expressed as following.

$$\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=r(t) \quad ; \quad 0 \le x \le l \tag{1}$$

$$q=ch^{3/2}\sqrt{i-\frac{\partial h}{\partial x}} \tag{2}$$

Where h : water depth r : effective rainfall

q : discharge per unit width

l : slope length t : time

i : slope gradient x : distance along slope

c : Chezy roughness

To determine the solution, the initial and upper boundary conditions are required.

$$q(0,x)=0 \quad , \quad q(t,0)=0 \tag{3}$$

In order to obtain the equivalent frequency transfer function, we assume that ;

$$r(t)=\bar{r}+Ae^{j\omega t} \tag{4}$$

$$h(t,x)=\bar{h}(x)+B(x)e^{j\omega t} \tag{5}$$

$$q(t,x)=\bar{q}(x)+C(x)e^{j\omega t} \tag{6}$$

j : imaginary unit ω : frequency

$B(x)$ and $C(x)$ are complex functions.

In this study, we employ zero-depth-gradient as lower boundary condition. This condition defined as

$$\left.\frac{\partial h}{\partial x}\right|_{x=l}=0 \tag{7}$$

From eq. (5) and (7),we obtain

$$\left.\frac{d\bar{h}}{dx}\right|_{x=l}=0 \quad , \quad \left.\frac{dB}{dx}\right|_{x=l}=0 \tag{8}$$

Consider at steady-state condition, therefore

$$\bar{h}(0)=0 \quad , \quad \bar{q}(0)=0 \tag{9}$$

From eq. (1),(4),(5) and (6),we obtain

$$\bar{q}(x)=\bar{r}x=c\bar{h}^{3/2}\sqrt{i-\frac{d\bar{h}}{dx}} \tag{10}$$

$$\frac{dC(x)}{dx}+j\omega B=A \tag{11}$$

Substitution eq.(5) into eq.(2) leads to eq. (12).

$$q=c(\bar{h}+B(x)e^{j\omega t})^{3/2}\sqrt{i-\frac{d\bar{h}}{dx}-\frac{dB}{dx}e^{j\omega t}} \tag{12}$$

$$=c\bar{h}^{3/2}\sqrt{i-\frac{d\bar{h}}{dx}}(1+\frac{3}{2}\frac{B}{\bar{h}}e^{j\omega t})(1-\frac{c^2\bar{h}^3}{2(\bar{r}x)^2}\frac{dB}{dx}e^{j\omega t}) \tag{13}$$

We neglect high-order term and use approximation method. Then eq. (14) is derived.

$$q \approx \bar{r}x\left\{1+\left(\frac{3B}{2\bar{h}}-\frac{c^2\bar{h}^3}{2(\bar{r}x)^2}\frac{dB}{dx}\right)e^{j\omega t}\right\} \tag{14}$$

We can derive eq.(15) from eq.(6) and (14).

$$C(x)=\bar{r}x\left(\frac{3B}{2\bar{h}}-\frac{c^2\bar{h}^3}{2(\bar{r}x)^2}\frac{dB}{dx}\right) \tag{15}$$

From eq.(11),we obtained

$$B=\frac{1}{j\omega}(A-\frac{dC(x)}{dx}), \quad \frac{dB}{dx}=\frac{-1}{j\omega}\frac{d^2C(x)}{dx} \tag{16}$$

Combination of eq.(15) and (16) finally yield

$$\frac{d^2C(x)}{dx^2}+g_{s1}\frac{dC(x)}{dx}+g_{s2}C(x)=g_{s1}A \tag{17}$$

$$C(0)=0 \quad \text{and} \quad \left.\frac{d^2C(x)}{dx}\right|_{x=l}=0 \tag{18}$$

And eq.(7) and (16) give,

$$g_{s1}=-\frac{3(\bar{r}x)^2}{c^2\bar{h}^4} \quad , g_{s2}=-\frac{2\bar{r}x}{c^2\bar{h}^3}j\omega \tag{19}$$

Eq.(17) is a boundary value problem with these conditions below.

$$\left.\frac{d\bar{h}}{dx}\right|_{x=0} = \frac{ic^2\bar{h}^3 - (\bar{r}x)^2}{c^2\bar{h}^3} = 0 \qquad (20)$$

$$\bar{h}(l) = \left(\frac{\bar{r}l}{ic}\right)^{2/3} \qquad (21)$$

Definition of the EFTF between rainfall $r(t)$ and discharge $q(t,l)$ is expressed by

$$Z(j\omega) = \frac{C(l)}{Al} \qquad (22)$$

The gain and time lag characteristics are

$$G(\omega) = |Z(j\omega)|,\ T_L(\omega) = -\frac{\angle Z(j\omega)}{\omega} \qquad (23)$$

We need numerical calculation in order to check the accuracy of theoretical method by assuming the following rainfall.

$$r(t) = \bar{r} + A\sin(\omega t) \qquad (24)$$

Where A means the constant amplitude. **Figure 1** shows the schematic relationship between a sinusoidal input and its output. The gain function is calculated numerically as

$$G(\omega) = \frac{B}{A} \qquad (25)$$

The time lag function, $T_L(\omega)$ is calculated by the time interval between both sinusoidal peaks as shown in **Figure 1**.

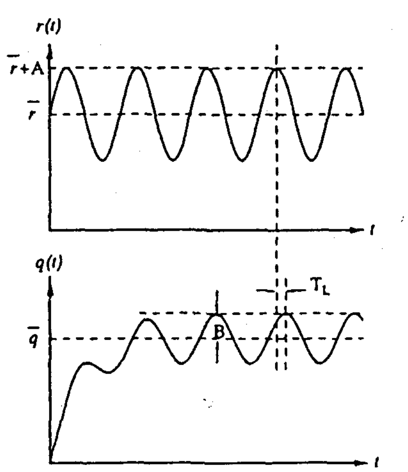

Fig.1 Schematic relationship between sinusoidal input and its output.

Solid lines in **Figure 2** (A), (B) and (C) show the vector locus, gain and time lag characteristics of slope of basin from theoretical solutions. The circles show numerical results while the dash line show the one obtained from kinematic wave model.

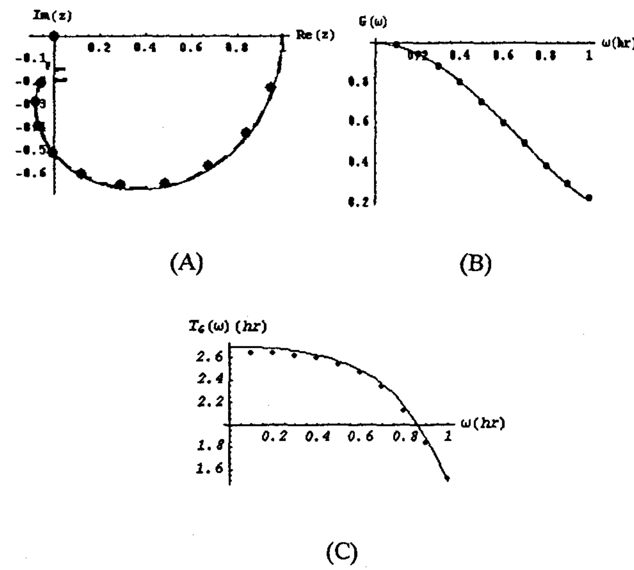

Fig.2 (A),(B) and (C) Vector locus, gain and Time lag of diffusion wave model.

The results obtained from application of EFTF are comparable to the one from numerical calculation. The concerned condition are $l = 1000m, \theta = 5^\circ$, $\bar{r} = 1mm/hr$ and $a = 0.1mm/hr$

3.UNIT BASIN ANALYSIS

Unit basin consists of 2 single slopes and 1 channel as illustrated in **Figure 3**.

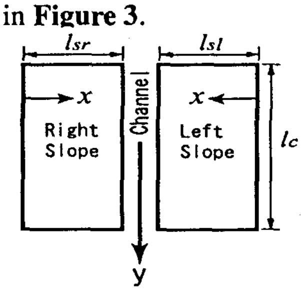

Fig.3 Top view of unit basin

In unit basin, we adopt kinematic wave for both sides for slopes. Because it is reasonable between kinematic wave and diffusion wave model as shown in **Figure 2**.

Continuity and momentum equations are

$$\frac{\partial a}{\partial t}+\frac{\partial q_c}{\partial y}=q_{sr}(t,l_{sr})+q_{sl}(t,l_{sl}) \tag{26}$$

$$q_c=\frac{ca^{3/2}}{\sqrt{W}}\sqrt{i-\frac{1}{W}\frac{\partial a}{\partial y}} \tag{27}$$

where a,y and W are cross-sectional area of channel, distance along channel and width of rectangular channel respectively. Subscript c represent channel of unit basin. Initial and upper boundary conditions are

$$q(0,y)=0 \quad , \quad q(t,0)=0 \tag{28}$$

Similar to slope analysis, we assume that

$$a(t,y)=\bar{a}(y)+B(y)e^{j\omega t}, \quad q_c(t,y)=\bar{q_c}(y)+C_c(y)e^{j\omega t} \tag{29}$$

The right and left slopes assumptions are the same as the ones expressed in slope analysis. Also, the zero depth gradient lower boundary condition is utilized.

$$\left.\frac{\partial h}{\partial y}\right|_{y=l_c}=\frac{1}{W}\left(\frac{\partial a}{\partial y}\right)_{y=l_c}=0 \tag{30}$$

From eq. (6),(29) and (30),we derived

$$\left(\frac{\partial \bar{a}}{\partial y}\right)_{y=l_c}=0 \quad , \quad \left(\frac{\partial B}{\partial y}\right)_{y=l_c}=0 \tag{31}$$

Subsequently eq.(32) and (33) are obtained,

$$\bar{q_c}(y)=\bar{r}(l_{sr}+l_{sl})y=\frac{c\bar{a}^{-3/2}}{\sqrt{W}}\sqrt{i-\frac{1}{\omega}\frac{d\bar{a}}{dy}} \tag{32}$$

$$\frac{dC_c(y)}{dy}+j\omega B=C_{sr}(l_{sr})+C_{sl}(l_{sl}) \tag{33}$$

Definition of q_c is

$$q_c=\frac{c}{\sqrt{W}}(\bar{a}+Be^{j\omega t})^{3/2}\sqrt{i-\frac{1}{\omega}\left(\frac{d\bar{a}}{dy}+\frac{dB}{dy}\right)e^{j\omega t}}$$

$$=\frac{c\bar{a}^{-3/2}}{\sqrt{W}}\sqrt{i-\frac{1}{\omega}\frac{d\bar{a}}{dy}}\left(1+\frac{3}{2}\frac{B}{\bar{a}}e^{j\omega t}\right)\left(1-\frac{c^2\bar{a}^{-3}}{2W^2(\bar{r}(l_{sr}+l_{sl})y)^2}\frac{dB}{dy}e^{j\omega t}\right)$$

$$\approx\bar{r}(l_{sr}+l_{sl})y\left\{1+\left(\frac{3B}{2\bar{a}}-\frac{c^2\bar{a}^{-3}}{2W^2(\bar{r}(l_{sr}+l_{sl})y)^2}\frac{dB}{dy}\right)e^{j\omega t}\right\} \tag{34}$$

Imaginary part of eq.(29) and (34) are separated as

$$C_c(y)=\bar{r}(l_{sr}+l_{sl})y\left(\frac{3B}{2\bar{a}}-\frac{c^2\bar{a}^{-3}}{2W^2(\bar{r}(l_{sr}+l_{sl})y)^2}\frac{dB}{dy}\right) \tag{35}$$

From eq.(33),it is denoted that

$$B=\frac{1}{j\omega}\left(C_{sr}+C_{sl}-\frac{dC_c(y)}{dy}\right), \quad \frac{dB}{dy}=-\frac{1}{j\omega}\frac{d^2C_c(y)}{dy^2} \tag{36}$$

Substitution eq.(35) into eq.(36) provides

$$\frac{d^2C_c(y)}{dy^2}+g_{c1}\frac{dC_c(y)}{dy}+g_{c2}C_c(y)=g_{c1}(C_{sr}+C_{sl})$$

$$=g_{c1}(l_{sr}Z_{sr}(j\omega)+l_{sl}Z_{sl}(j\omega))A \tag{37}$$

$$C_c(0)=0 \quad \text{and} \quad \left.\frac{d^2C_c(y)}{dy}\right|_{x=l_c}=0 \tag{38}$$

$$g_{c1}=-\frac{3W^2\left(\bar{r}(l_{sr}+l_{sc})y\right)^2}{c^2\bar{a}^{-4}} \tag{39}$$

$$g_{c2}=\frac{-2W^2\bar{r}(l_{sr}+l_{sl})y}{c^2\bar{a}^{-3}}j\omega \tag{40}$$

Eq.(37) is a boundary value problem which satisfies eq.(41) and (42).

$$\left.\frac{d\bar{a}}{dy}\right|_{y=l_c}=\frac{ic^2\bar{a}^{-3}W-\left(\bar{r}(l_{sr}+l_{sl})y\right)^2W^2}{c^2\bar{a}^{-3}}=0 \tag{41}$$

$$\bar{a}(l_c)=\left(\frac{W(\bar{r}A_c)^2}{ic^2}\right)^{1/3} \tag{42}$$

The EFTF of unit basin, $Z_c(j\omega)$,is defined by

$$Z_c(j\omega)=\frac{C_c(l_c)}{Al_c(l_{sr}+l_{sl})} \tag{43}$$

4.GENERAL BASIN ANALYSIS

This section focuses on general basin as shown in **Figure 4**. It is possible to employ the results from this kind of basin to further complex one.

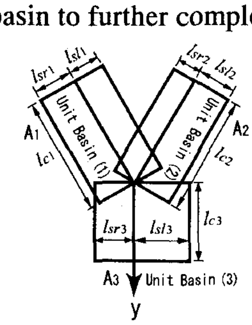

Fig.4 General basin

Continuity equation and momentum equation are

$$\frac{\partial A_3}{\partial t}+\frac{\partial q_{c3}}{\partial y}=q_{sr3}(t,l_{sr3})+q_{sl3}(t,l_{sl3}) \tag{44}$$

$$q_{c3}=\frac{cA_3^{3/2}}{\sqrt{W}}\sqrt{i-\frac{1}{W}\frac{\partial A_3}{\partial y}} \quad (45)$$

A_3 represents a cross sectional area of the third channel. Upper boundary condition is

$$q_{c3}(t,0)=q_{c1}(t,l_{c1})+q_{c2}(t,l_{c2}) \quad (46)$$

Lower boundary condition is denoted as

$$\left.\frac{\partial h}{\partial y}\right|_{y=l_{c3}}=\frac{1}{W}\left.\frac{\partial A_3}{\partial y}\right|_{y=l_{c3}}=0 \quad (47)$$

Unit basin 1 and 2 have the same definition as we explained in unit basin analysis. The assumptions of A_3 and q_{c3} are

$$A_3(t,y)=\bar{A}_3(y)+B(y)e^{j\omega t} \quad (48)$$

$$q_{c3}(t,y)=\bar{q}_{c3}(y)+C_{c3}(y)e^{j\omega t} \quad (49)$$

From eq.(47) and (48),we obtained

$$\left.\frac{\partial \bar{A}_3}{\partial y}\right|_{y=l_{c3}}=0 \quad , \quad \left.\frac{\partial B}{\partial y}\right|_{y=l_{c3}}=0 \quad (50)$$

The following equations are derived

$$\bar{q}_{c3}(0)=\bar{q}_{c1}(l_{c1})+\bar{q}_{c2}(l_{c2}) \quad (51)$$

$$C_{c3}(0)=C_{c1}(l_{c1})+C_{c2}(l_{c2}) \quad (52)$$

$$\frac{d\bar{q}}{dy}=\bar{q}_{sr}(l_{sr})+\bar{q}_{sl}(l_{sl}) \quad (53)$$

$$\frac{dC_{c3}(y)}{dy}+j\omega B=C_{sr}(l_{sr})+C_{sl}(l_{sl}) \quad (54)$$

From eq.(51) and (52),solution of eq.(53) and (54) are

$$\bar{q}_{c3}(y)=\bar{r}(l_{sr3}+l_{sl3})y+\bar{r}(A_1+A_2)=\frac{c\bar{A}_3}{\sqrt{W}}\sqrt{i-\frac{1}{\omega}\frac{\partial \bar{A}_3}{\partial y}} \quad (55)$$

$$C_{c3}(y)=\frac{c\bar{A}_3\sqrt{i}}{\sqrt{W}}\left(\frac{3B}{2\bar{A}_3}-\frac{3B}{4W\bar{A}_3 i}\frac{d\bar{A}_3}{dy}-\frac{1}{2Wi}\frac{dB}{dy}\right) \quad (56)$$

From eq.(54) and (56),we derived

$$\frac{d^2C_{c3}(y)}{dy^2}+g_{w1}\frac{dC_{c3}(y)}{dy}+g_{w2}C_{c3}(y)=g_{w1}(C_{sr3}+C_{sl3})$$
$$=g_{w1}(l_{sr3}Z_{sr3}(j\omega)+l_{sl3}Z_{sl3}(j\omega))A \quad (57)$$

where $C_{c3}(0)=A\{A_1Z_{c1}(j\omega)+A_2Z_{c2}(j\omega)\}$ (58)

$$\left.\frac{d^2C_{c3}(y)}{dy^2}\right|_{y=l_{c3}}=0 \quad (59)$$

$$g_{w1}=-\frac{3W^2\{\bar{r}(A_1+A_2)+\bar{r}(l_{sr3}+l_{sl3})y\}^2}{c^2\bar{a}^{-4}} \quad (60)$$

$$g_{w2}=-\frac{2W^2\{\bar{r}(A_1+A_2)+\bar{r}(l_{sr}+l_{sl})y\}}{c^2\bar{a}^{-3}}j\omega \quad (61)$$

Lower boundary conditions are

$$\left.\frac{d\bar{A}_3}{dy}\right|_{y=l_{c3}}=\frac{ic^2\bar{A}_3^{-3}W-\{\bar{r}(A_1+A_2)+\bar{r}(l_{sr}+l_{sl})y\}^2W^2}{c^2\bar{a}^{-3}}=0 \quad (62)$$

$$\bar{A}_3(l_{c3})=\left(\frac{W(\bar{r}A_r)^2}{ic^2}\right)^{1/3} \quad (63)$$

In which $A_r=A_1+A_2+A_3$

The EFTF for general basin is

$$Z_G(j\omega)=\frac{C(l_{c3})}{A.A_r} \quad (64)$$

Gain and Time lag characteristics are defined as

$$G(\omega)=|Z_G(j\omega)| \quad , \quad T_L(\omega)=-\frac{\angle Z_G(j\omega)}{\omega} \quad (65)$$

Figure 5 (A),(B) and (C) show the vector locus, gain and time lag function calculated from EFTF of single slope(slim solid lines), unit basin (dashed lines) and general basin (thick solid lines).

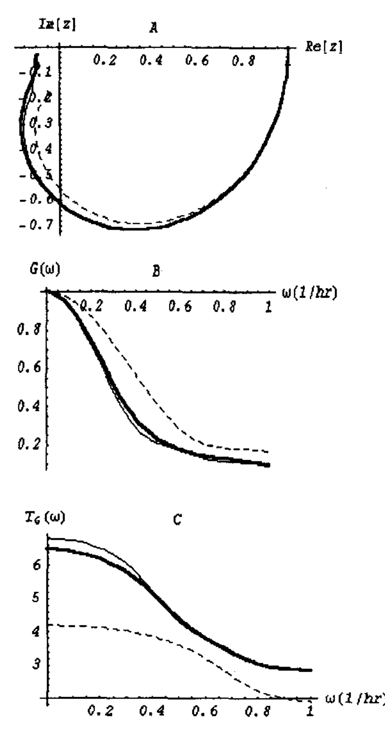

Fig.5 vector locus, gain and time lag function of slope, unit basin and general basin

The achieved EFTF of general basin can be leaded to simpler estimation of discharge. Its vector locus shows that we can apply second order differential equation to describe runoff system as shown in eq.(66)

$$f_1 \frac{d^2q}{dt} + f_2 \frac{dq}{dt} + q = r \qquad (66)$$

$$q(0) = 0, \left.\frac{dq}{dt}\right|_{t=0} = 0 \qquad (67)$$

EFTF of eq.(66) is

$$Z\ (j\omega) = \frac{1}{1 - f_1\omega^2 + jf_2\omega} \qquad (68)$$

$$f_1 = \frac{1}{\omega^2}\left\{1 - \frac{R_e[Z]}{R_e^2[Z] + I_m^2[Z]}\right\} \quad , \quad f_2 = \frac{-I_m[Z]}{\omega(R_e^2[Z] + I_m^2[Z])} \qquad (69)$$

Therefore we suggest the possibility to use parameters f_1 and f_2 to estimate amount of discharge.

DISCUSSION AND CONCLUSION

The diffusion wave, one of distributed parameter runoff models, was analyzed by using the equivalent frequency transfer function (EFTF). The word "EFTF" means relationship between rainfall input and discharge. In this study, we had derived EFTF for the simplest model or slope. The theoretical results agree with the one calculated from numerical method. Therefore the model was extended. Consequently EFTF for general basin was derived. Gain and time lag characteristics show that they decreased as the frequency increased. We suggest that it is feasible to extend application of EFTF to the practical river networks consisting of many slopes and channels. In our further study, we will simplify using EFTF with available data by estimation parameters from second order differential equation.

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