Annual Journal of Hydraulic Engineering, JSCE, VOL.44, 2000, February

SIMPLE ANALYTICAL SOLUTION FOR ADVECTION EQUATION IN RECTANGULAR AND COMPOUND CHANNELS

Shiono, K.[1] and Komatsu, T.[2]

1. Ph.D., Professor, Department of Civil and Building Engineering, Loughborough University, UK.
2. Member of JSCE, Dr. of Eng., Professor, Department of Civil Engineering, Kyushu University, Japan

The Gaussian solution for the advection equation in open channel flow was modified with displacements caused by the secondary flow introduced. The solution was compared with experimental data and was in good agreement. There are two concentration peaks for a shallow compound channel flow and the solution reproduces it nicely. A stream function was introduced to predict secondary flow. The solution with the stream function was found to be in good agreement with the experimental data.

Key words: *Tracer concentration, Secondary flow, Analytical solution.*

1. Introduction

Previous work [2,3] show that secondary currents are 1~3 % of the longitudinal component of velocity in a straight prismatic open channel. Dye concentration measurements together with velocity measurements in open channels were carried out [4,5]. The transverse distribution of dye concentration in a region where secondary flow exists has been shown to differ significantly from the Gaussian distribution for both rectangular and compound channels. This paper presents an introduction of an analytical solution, which takes secondary flow into account.

The governing equation of diffusion for uniform flow and isotropic turbulence is:

$$U\frac{\partial C}{\partial x} = K\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) \qquad (1)$$

where C=tracer concentration, x, y, z = longitudinal, lateral and vertical directions, U,V,W = velocity components, K = eddy diffusivity.

An analytical solution for continuously injecting dye as a point source is given by Hinze[1] as:

$$C(x,y,z) = \frac{Q}{4\pi Kr}\exp\left[-\frac{U}{2K}(r-x)\right] \qquad (2)$$

where Q = Material flow rate, m^3/s,

$$r = \left(x^2 + y^2 + z^2\right)^{0.5}$$

The length scale of the turbulent diffusion is normally of one order smaller than that of secondary flow. This means that turbulent diffusion takes place in a Gaussian distribution manner while dye

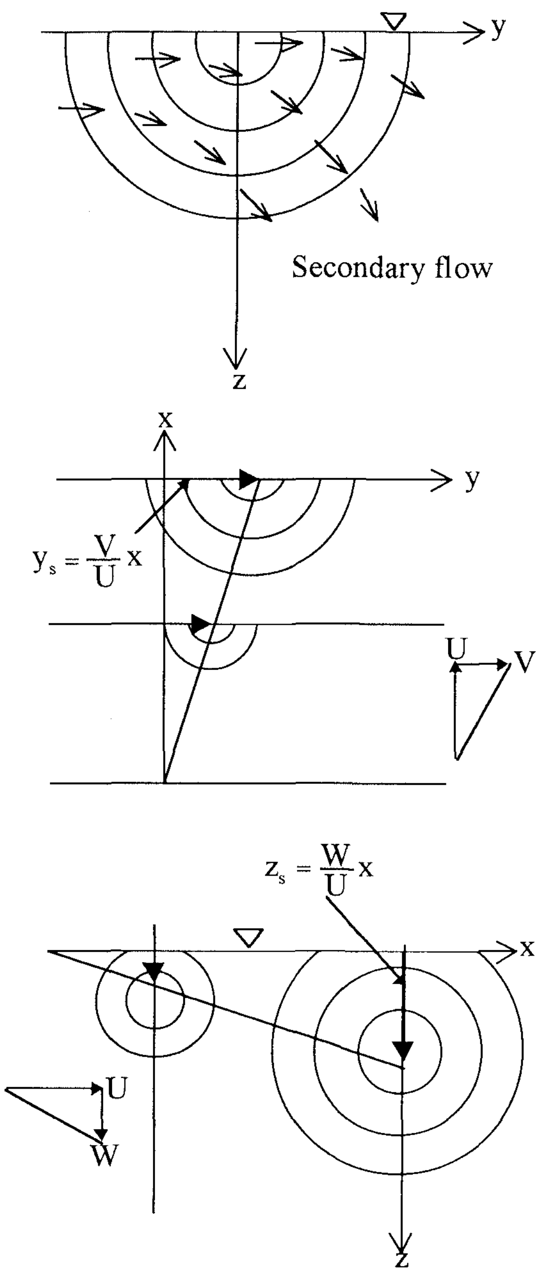

Fig.1 Illustration of displacement caused by secondary flow.

convects with the secondary flow illustrated as in **Fig. 1**. Therefore a simple Gaussian model is suggested, with displacements, y_s & z_s caused by secondary flow introduced into the Gaussian solution. Introducing the displacements, $y_s = \frac{V}{U}x$ and $z_s = \frac{W}{U}x$, in the 2 directions, y and z, respectively, (see Fig. 1), the new co-ordinate system can be written as:

$$y' = y + \frac{V}{U}x \text{ and } z' = z + \frac{W}{U}x \tag{3}$$

When the following conditions are satisfied:

$$1 >> 2\frac{\partial V}{\partial y}\frac{x}{U}, \quad 1 >> 2\frac{\partial W}{\partial z}\frac{x}{U}, \quad 1 >> 2\frac{\partial W}{\partial y}\frac{x}{U}$$

and $1 >> 2\frac{\partial V}{\partial z}\frac{x}{U}$, and ignoring higher order terms, then Equation (1) becomes

$$U\frac{\partial C}{\partial x} = K\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y'^2} + \frac{\partial^2 C}{\partial z'^2}\right). \tag{4}$$

Therefore the solution becomes

$$C(x, y', z') = \frac{Q}{4\pi K r'}\exp\left[-\frac{U}{2K}(r' - x)\right] \tag{5}$$

where

$$r' = \left\{x^2 + (y + y_s)^2 + (z + z_s)^2\right\}^{0.5}.$$

This solution will be used to demonstrate the effect of secondary flow on concentration distribution using the measured data.

2. Experimental data

Experiments were carried out in the 1.2m wide, 0.085m deep, rectangular channel and the 0.2m wide, 0.11m deep, asymmetric compound channel. Three components of velocity and tracer concentration were obtained using a combination of Laser Doppler Anemometer (LDA) and Laser Induced Fluorescence (LIF) in rectangular [4] and compound channels [5]. Brief descriptions of the data with **Figs. 2,3 & 4** in both channels are as follows:

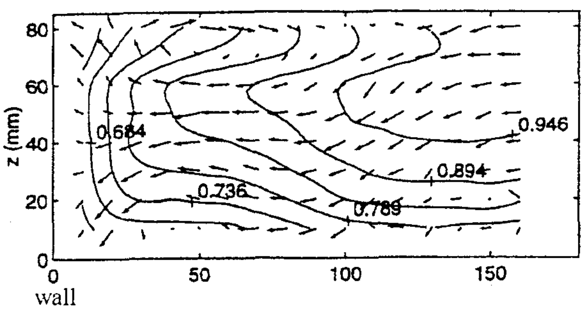

Fig.2 Velocity distribution in rectangular channel.

(1) Velocity in rectangular channel

An area of the velocity measurement for the rectangular open channel was restricted to 180 mm from the wall of the channel owing to the limited focal length of the two probes. **Fig. 2** shows contour lines of the mean primary velocity U normalised by the section mean velocity, Uo, together with secondary current vectors. It is noticed that the contour lines bulge towards the wall at the mid water depth and gradually bend over near the free surface approaching the centre part of the channel. The contour lines also bulge towards the bottom corner. These features correspond well to the behaviour of the secondary flow and agree with the visualisation result near the wall in a rectangular channel obtained by Ishigaki [2].

(2) Velocity in compound channel

Contour lines of velocity together with secondary vectors for the relative depths, Dr, ($Dr = (H - h)/H$, H=main channel water depth, h=floodplain height) of 0.27 and 0.5 are plotted in **Figs. 3 & 4**. A typical mean flow structure for a compound channel can be observed from the figures in that bulging contour lines appear from the edge of the flood plain and coincide well with distinct inclined secondary currents. The inclined secondary currents generate the twin vortices in the main channel and flood plain. The mean velocity structures show very similar trends to those obtained

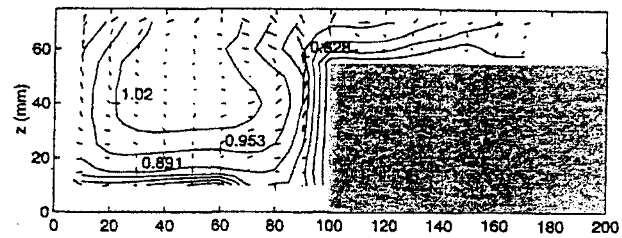

Fig.3 Velocity distribution in compound channel for Dr=0.27.

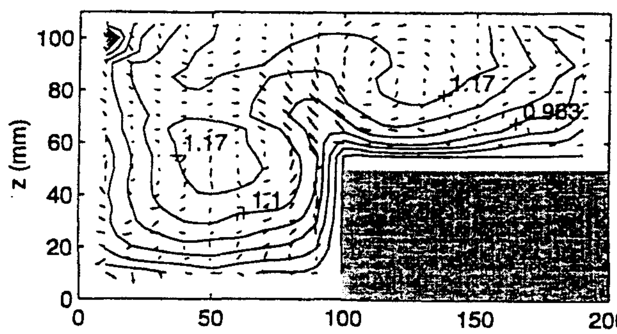

Fig.4 Velocity distribution in compound channel for Dr=0.5.

by Ishigaki[2], Shiono[6], Tominaga[7] with the "2D-2D" measurements.

(3) Concentration

The dye, Rhodamine6G, was injected continuously at the water surface, z=82mm, in the rectangular channel and the measurement section was 1m downstream from the injection point. Dye concentration measurements were carried out at three levels, z= 80mm, 70mm and 60mm for 3 injection locations, y=3mm, 85mm and 170mm. The transverse dye concentration distributions for the 170mm and 85mm injection cases are shown in **Figs. 5 & 6**. From the measurement result, the concentration distributions show more or less Gaussian distribution with the peaks shifted from the injection location.

3. Solution

The isotropic eddy diffusivity was calculated using the measured mean velocity and peak concentration for the 170mm injection case since the vertical

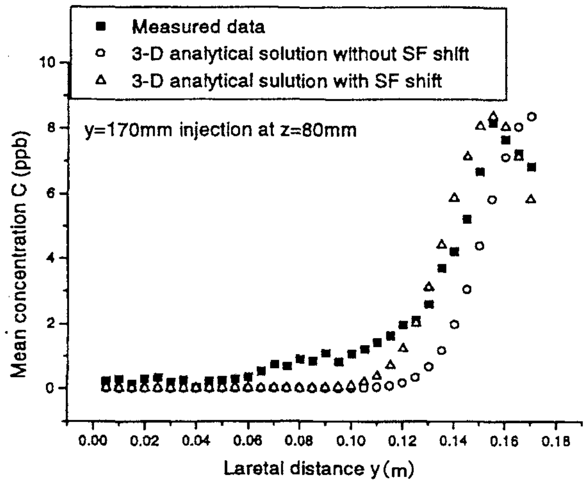

Fig.5 Dye concentration distribution at Z=80mm for 170mm injection case.

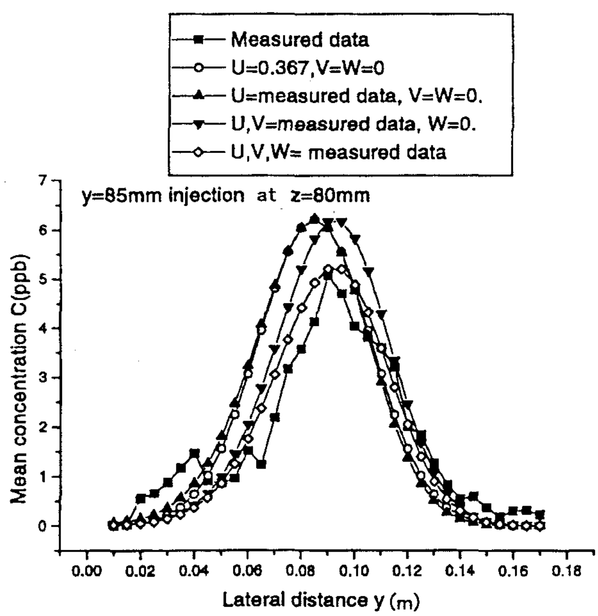

Fig.6 Dye concentration distribution at Z=80mm for 85mm injection case

component of secondary flow is almost zero in the peak concentration region. The value of K was thus calculated as 0.0000650 m^2/s and the non-dimensional value, K/u_*H , was 0.040. This value is smaller than the depth averaged value, 0.067, for a 2-dimensional uniform flow in open channel and smaller than the usual values quoted in the literature. The concentration was then calculated using equation (5), with the non-dimensional diffusion coefficient of 0.04 and no secondary flow, and the calculated distribution and the measured data plotted together as shown in **Fig. 5**. The calculated distribution is that of exponential decay from the injection location at 170mm, so the peak location is displaced from the measured one, as would be expected. The solution with secondary flow was also determined and is plotted in **Fig. 5** also. The peak location seems to agree well with the experimental data.

At z=80mm for the y=85mm injection case since the velocity is not uniform in the vicinity of the wall, an investigation of the analytical solution was further carried out as to whether the change of velocity over the measured area affected the analytical solution or not. The tests were carried out for 4 conditions which were i) U=constant=0.367 m/s (at y=85mm & z=80mm), V=W=0.0, ii) U= measured (not constant), V=W=0.0, iii) U= measured data, V= constant value used at z=80mm and y=85mm, and W=0.0 and iv) U, V, W= measured data. The isotropic eddy diffusivity of 0.04 used as before was used to calculate the concentration distributions and all the distributions are shown together in **Fig. 6**. The results for i) and ii), give nearly the same distributions, hence the velocity change does not appear to significantly affect the concentration distribution. Introducing the value of V for iii), gives a shift of the peak concentration location. Introducing the values of V and W for iv), gives a significant change in the magnitude and a good prediction. Therefore this test demonstrated clearly the effect of secondary flow on the tracer concentration.

Next this solution was applied to stronger secondary flow in the compound channel. The dye experiments in the compound channel were also carried out for the water depths of 75mm and 110mm equivalent to the relative depths, Dr=0.27 and Dr=0.5 respectively.

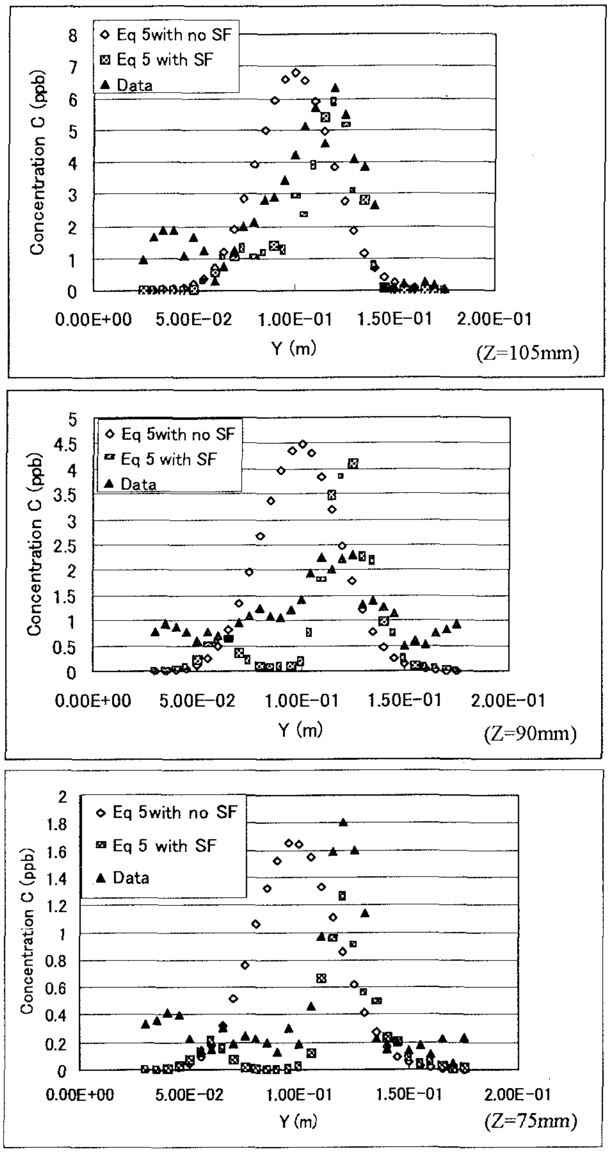

Fig.7 Dye concentration distribution at Z=105mm, Z=90mm and Z=75mm, in compound channel for Dr=0.5.

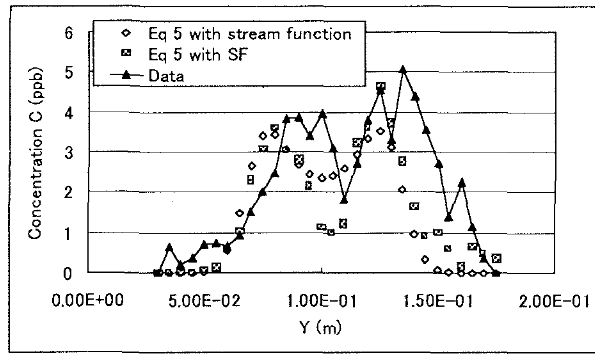

Fig.8 Dye concentration distribution at z=70mm for compound channel, Dr=0.27

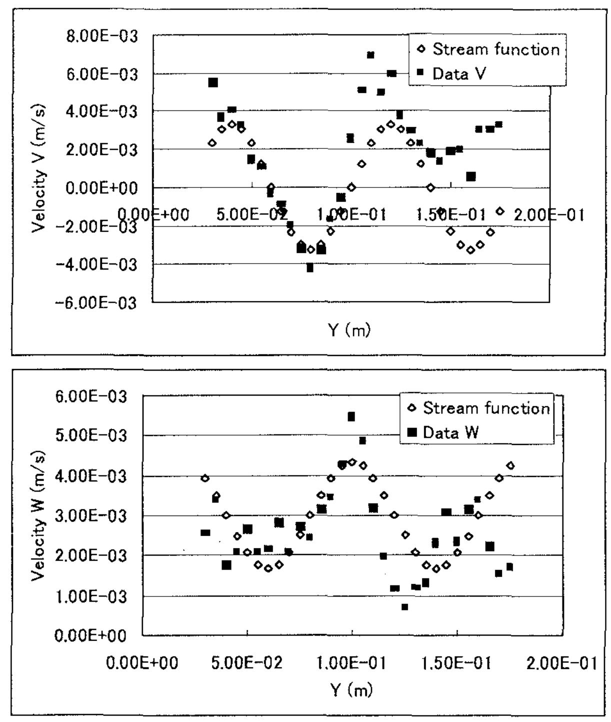

Fig.9 Secondary flow V and W.

Three injection locations for Dr=0.5 were at y=50mm, y=100mm, and y=150mm near the water surface with z=108mm. For Dr=0.5, the K value was calibrated and was $0.035u_*H$. The solution for the injection location, y=100mm at z=108mm is shown in **Fig. 7** and is seen to agree well with the experimental data, except at z=90mm. The peak concentration location and magnitude are in good agreement at z=105mm and z=75mm. For Dr=0.27 for the injection location, y=100mm at z=73mm, there are two dye concentration peaks in the main channel and the flood plain, so the distribution is clearly not Gaussian. The non-dimensional eddy diffusivity was calibrated and found to be $0.05u_*H$. It should be noted that the water depth in the main channel was used. The solution reproduces the double peaks well as shown in **Fig. 8.** There are clearly some discrepancies in the solution in that the whole concentration distribution is slightly shifted to the left. However it should satisfy the conditions given in the previous section (also not including the full advection terms and an-isotropic eddy diffusivity), so some discrepancies would be expected in this complex flow. However, despite this the predictions of concentration using this solution are surprisingly good. This result indicates that secondary flow considerably affects the dye concentration distribution.

In order to use this solution, secondary flow data are required. We attempted to express secondary flow as a stream function, such as $\Psi(y,z) = A\sin(2\pi y/2L)\sin(2\pi z/2H)$, L=width of secondary flow cell and H=depth of secondary flow cell. A value of A should be determined from the data. The vertical and lateral components of velocity are: $V = \partial\Psi/\partial z$ and $W = -\partial\Psi/\partial y$.

For Dr=0.27, L, H and A were determined from the data, and the lateral and vertical components of velocity are shown in **Fig. 9**, together with measured data. The stream function gives reasonable result at this test location and the solution also reproduces two peaks and reasonable agreement with the experimental data shown in **Fig. 8.**

4. Conclusions

The modified Gaussian solution was introduced to predict solution concentration distributions in the rectangular and compound channel flows. The solution surprisingly gives a good prediction for the experimental data although the flow conditions are not perfectly satisfied. The eddy diffusivity was found to be smaller than the typical values quoted in the literature. When the secondary flow was expressed as a stream function at one location in the complex flow of a compound channel for Dr=0.27, the solution gave a good prediction. Therefore when a general formulation for secondary flow was found, this solution could be used to predict a solute concentration distribution in straight channel flow.

References

1) Hinze, J. (1976), Turbulence, McGraw-Hill Book Company.

2) Ishigaki, T. (1993), 3-D flow in compound cannels, Doctoral thesis, Kyoto Unversity, Japan.

3) Nezu and Nakagawa (1993), Turbulence in open channel flow, IAHR monogragh.

4) Shiono, K., Feng, T. and Muto,Y. (1998), Reynolds stress and flux measurements in open channel flow, Conference on Refined model and turbulent measurements, Tainan, Taiwan.

5) Shiono, K. and Feng, T, (1999), Turbulence measurements of solute concentration in a compound channel, submitted for publication.

6) Shiono, K. and Knight, D.W. (1991), Turbulent open channel flows with variable depth across the channel, JFM. Vol. 222, pp617-647.

7) Tominaga, A. and Nezu, I. (1991), Turbulent structure in compound channel flows, JHE, ASCE, Vol. 117, No. 1, pp21-41.

(Received September 30,1999)