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HIGHER ORDER FDS SCHEME FOR RAPIDLY VARIED 2-D FLOW SIMULATIONS

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Numerical models based on flux-difference splitting (FDS) technique are developed for simulating rapidly varied flows in two-dimensional space co-ordinates. A first-order accurate model using Roe's numerical flux and a second-order accurate scheme using Lax-Wendroff numerical flux are constructed. Roe's averaging for velocity and celerity ensures conservation and consistency while entropy satisfying solution is guaranteed by theoretically sound treatment. Flux limiters used in second-order accurate model yields oscillation-free results while maintaining high shock-resolution. The models' validity and applicability are demonstrated by comparing computed results with analytical and experimental results for some exacting hydraulic problems.

Key Words : *shock-capturing method, flux-difference splitting, rapidly varied flow, 2-D flow*

1. INTRODUCTION

Rapidly varied flows test the limits of numerical models' applicability because maintaining conservation across a discontinuity and handling differing features of signal propagation in regions of sub- and supercritical flows are requirements that most conventional models fail to fulfil. Schemes such as MacCormack[1,2)], although conservative, treat the problem in a lump-sum way and thus cannot work well where such detail as direction of signal propagation becomes a dominant feature. On the other hand, schemes such as Gabutti[3)], although correctly handle signal propagation, are non-conservative. It has been demonstrated in case of Beam and Warming scheme[4)], that handles signal propagation through automatic switching of difference operators, that ensuring conservation does yield significant enhancement in accuracy.

Lately, several shock-capturing schemes have been found to give very accurate results with rather ease[5,6)]. These schemes essentially apply upwind differencing to a linearized Riemann problem[7]. While that takes care of directionality of signal propagation, the approximate Jacobian developed by Roe[7)] enables conservative splitting of flux differences. The second-order accurate versions of this class of schemes suitably limit[8)] contribution of higher-order terms for oscillation-free results. Although the rigorous theoretical development for FDS schemes have been confined to 1-D flows, their logical extensions for solving 2-D problems recently appearing in the literatures[9,10,11)] show significant promise for further development. Mostly, these schemes use finite-volume methods and rely on MUSCL technique for higher order of accuracy.

This paper follows FDS technique for developing 2-D models. Both first- and second-order accurate schemes on the basis of Roe and Lax-Wendroff numerical fluxes, respectively, are presented and their relative merits in simulating rapidly varied 2-D flows are examined. Roe's[7)] conservative and consistent averaging for velocity and celerity is supported by Harten and Hymen's[12)] treatment that avoids unphysical solutions. The enhanced shock-resolution by the second-order scheme is kept free from any dispersion error by limiting the second-order flux through suitable flux limiters. The validity and applicability of the models are demonstrated by comparing numerical results with experimental and analytical solutions. Results for some test problem are also compared with previously reported solutions by other schemes.

2. GOVERNING EQUATIONS

The governing equations for two-dimensional free-surface flows can be written as

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{E}}{\partial x}+\frac{\partial \mathbf{F}}{\partial y}+\mathbf{S}=0 \qquad (1)$$

Where **U** is the vector of unknowns, **E** and **F** are components of flux vectors and **S** is the vector containing source and sink terms. These vectors are given by

$$\mathbf{U}=\begin{pmatrix}h & uh & vh\end{pmatrix}^T \qquad (2a)$$

$$\mathbf{E}=\begin{pmatrix}uh & u^2h+0.5gh^2 & uvh\end{pmatrix}^T \qquad (2b)$$

$$\mathbf{F}=\begin{pmatrix}vh & uvh & v^2h+0.5gh^2\end{pmatrix}^T \qquad (2c)$$

$$\mathbf{S}=\begin{pmatrix}0 & -gh(S_{ox}-S_{fx}) & -gh(S_{oy}-S_{fy})\end{pmatrix}^T \qquad (2d)$$

where h = flow depth, u = flow velocity along x-direction, v = flow velocity along y-direction, g = acceleration due to gravity, S_o and S_f are bed and energy slopes, respectively. Once the technique for solving homogeneous part of Eq.(1) is established, the source term S can be incorporated separately without affecting the overall formulation. Therefore, the source term is dropped from the following considerations for the development of numerical technique.

In order to effectively and justifiably apply the Riemann solver and flux difference splitting technique developed for 1-D problem to the 2-D problem defined by Eq.(1), help is sought from the operator splitting technique[13]. That in effect means that instead of solving Eq.(1), we solve the following two equations successively.

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{E}}{\partial x}=0 \qquad (3a)$$

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}}{\partial y}=0 \qquad (3b)$$

These equations can now be viewed as two one-dimensional problems. Therefore, the techniques developed for 1-D problems can be directly applied to the above pair of equations. The equivalence of Eq.(1) and (3) is approximate, not exact. The solution of Eq.(3a) is described in the following. The flux vector **E** is related to **U** through it's Jacobian **A** as

$$\mathbf{A}=\frac{\partial \mathbf{E}}{\partial \mathbf{U}}=\begin{pmatrix}0 & 1 & 0\\ c^2-u^2 & 2u & 0\\ -uv & v & u\end{pmatrix} \qquad (4)$$

Where c is celerity defined as $c=(gh)^{1/2}$. The hyperbolic nature of the governing equations means that the Jacobian **A** has a complete set of independent and real eigenvectors expressed as

$$\begin{pmatrix}\mathbf{e}^1 & \mathbf{e}^2 & \mathbf{e}^3\end{pmatrix}=\begin{pmatrix}1 & 0 & 1\\ u+c & 0 & u-c\\ v & c & v\end{pmatrix} \qquad (5)$$

The corresponding eigenvalues are

$$\lambda^1=u+c;\quad \lambda^2=u;\quad \lambda^3=u-c \qquad (6a,b,c)$$

Roe[7] constructed an approximate Jacobian by using average values for velocity and celerity. The details of these averages in case of 1-D shallow water equation may be referred to Jha et al[5]. Following similar search, the average velocities and celerity can be obtained as

$$\tilde{u}=\left(u_R\sqrt{h_R}+u_L\sqrt{h_L}\right)/\left(h_R+h_L\right) \qquad (7a)$$

$$\tilde{v}=\left(v_R\sqrt{h_R}+v_L\sqrt{h_L}\right)/\left(h_R+h_L\right) \qquad (7b)$$

$$\tilde{c}=\sqrt{0.5g(h_R+h_L)} \qquad (7c)$$

wherein the subscripts R and L refer to the right and left states (i.e. for i+1/2,j the left state is i,j and the right state is i+1,j).

3. FIRST-ORDER SCHEME

The first-order scheme for Eq.(3a) can be written as

$$\mathbf{U}_{i,j}^{t+1}=\mathbf{U}_{i,j}^{t}-\gamma\,\Delta\mathbf{E}_{i,j}^{t} \qquad (8)$$

where i,j = space indices along x- and y-directions; t = time index; γ = Δt/Δx; Δt, Δx = time and space increments,. The flux difference ΔE is written in split form as

$$\Delta\mathbf{E}=\sum_{k=1}^{3}\alpha^k\left(\lambda^+ + \lambda^-\right)^k\mathbf{e}^k \qquad (9)$$

where k = wave number and α = wave strength expressed as

$$\alpha^1=0.5\Delta h+0.5\left(\Delta(uh)-\tilde{u}\,\Delta h\right)/\tilde{c} \qquad (10)$$

$$\alpha^2 = \left(\Delta(vh) - \tilde{v}\,\Delta h\right)/\tilde{c} \qquad (11)$$

$$\alpha^3 = \Delta h - \alpha^1 \qquad (12)$$

The splitting of flux-difference allows introduction of upwinding into the scheme. Insertion of Eq.(9) into Eq.(8) and subsequent mathematical manipulation yields the following first-order upwind finite-difference scheme

$$\mathbf{U}^{t+1}_{I,J} = \mathbf{U}^{t}_{I,J} - \gamma\left(\mathbf{E}^{N}_{I+1/2,J} - \mathbf{E}^{N}_{I-1/2,J}\right) \qquad (13)$$

where the numerical flux $\mathbf{E}^N$ is expressed as

$$\mathbf{E}^{N} = 0.5\left(\mathbf{E}_R + \mathbf{E}_L\right) - 0.5\sum_{k=1}^{3}\tilde{\alpha}^k\left|\tilde{\lambda}^k\right|\,\tilde{\mathbf{e}}^k \qquad (14)$$

The average velocities and celerity given by Eq.(7) are conservative and consistent with the governing equation. However, it converges to an unphysical solution by violating entropy inequality condition in case of rarefaction waves (dam-break case, for example). The problem can be overcome by replacing the modulus of λ in Eq.(14) by a suitable function of λ and a small quantity δ whenever modulus of λ is less than δ[12]. We avoid the trial procedure for finding value of δ and use the following expression instead[12].

$$\delta^k = \max\left(0, \lambda_{LR} - \lambda(\mathbf{U}_L), \lambda(\mathbf{U}_R) - \lambda_{LR}\right) \qquad (15)$$

$$\lambda_{LR} = \lambda\left(\mathbf{U}_L, \mathbf{U}_R\right) \qquad (16)$$

4. SECOND-ORDER SCHEME

The second-order accurate scheme is obtained by using Lax-Wendroff numerical flux in Eq.(13). Using the approximate Jacobian of Roe[7], the Lax-Wendroff numerical flux can be written as

$$\mathbf{E}^{N(LW)} = 0.5\left(\mathbf{E}_R + \mathbf{E}_L\right) - 0.5\sum_{k=1}^{3}\tilde{\alpha}^k\left|\tilde{\lambda}^k\right|\,\tilde{\mathbf{e}}^k$$

$$+ 0.5\sum_{k=1}^{3}\phi\,\tilde{\alpha}^k\left|\tilde{\lambda}^k\right|\left(1 - \gamma\left|\tilde{\lambda}^k\right|\right)\tilde{\mathbf{e}}^k \qquad (17)$$

where ϕ is the flux limiter designed to prevent oscillations due second order of accuracy. The flux limiter is a non-linear function of

$$r^k_{I+1/2,J} = \left(\alpha^k_{I+1/2-\mathrm{sign}(\tilde{\lambda}^k_{I+1/2,J}),J} \,/\, \alpha^k_{I+1/2,J}\right) \qquad (18)$$

We use Van Albada limiter[8] which is expressed as

$$\phi = (r + r^2)/(1 + r^2) \qquad (19)$$

Eqs.(15) and (16) are implemented for satisfying entropy inequality condition just as in the case of first-order scheme.

5. NUMERICAL STABILITY

The schemes presented herein are explicit and, therefore, require strict observance of stability criteria for successful execution. The following has been found to give stable results.

$$\Delta t \le C_n \frac{\min(\Delta x, \Delta y)}{\max\left(c + \sqrt{u^2 + v^2}\right)} \qquad (20)$$

where C_n=the Courant number

6. NUMERICAL RESULTS

The models are first verified against Stoker solution[14] for 1-D dam-break problem.

The dam-break problem is considered in a (200m x 200m) horizontal and frictionless area (Fig.1) which is divided into 5m x 5m cells, 40 along each direction. The dam is placed parallel to y-direction at 100m from either end of the flood that separates reservoir and flood-plain. The initial water depths in reservoir, h_r and flood-plain, h_f are 10m and 5m, respectively. The whole dam is taken off at the initiation of computation that simulates total collapse of the dam, for which 1-D analytical solution is applicable. The computed results at time

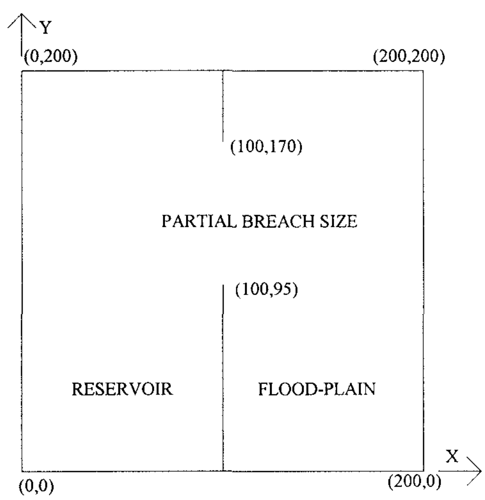

Fig.1 Definition Sketch for Dam-Break Problem

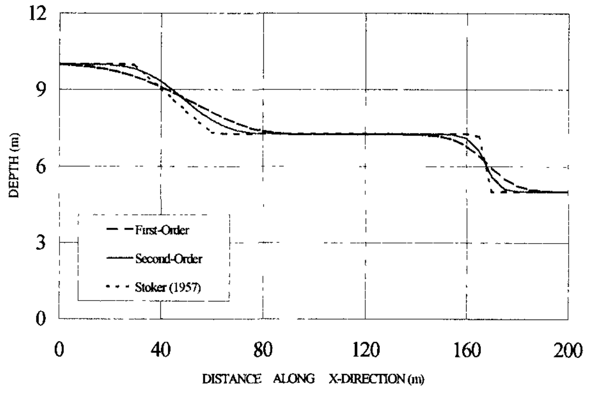

Fig.2 Comparison with Analytical Solution for the Case of Total Dam Collapse

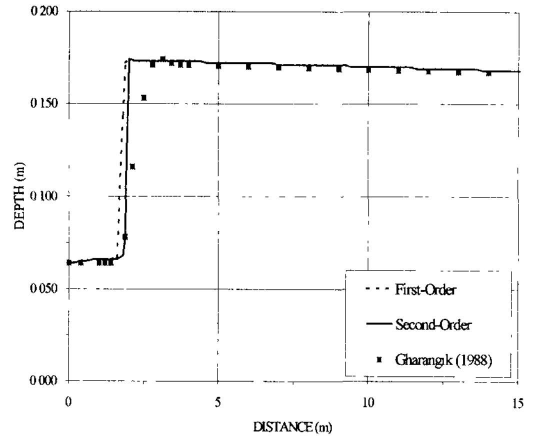

Fig.3 Hydraulic Jump in Rectangular Channel

7+Δt seconds is compared against analytical solution in Fig.2. Both first and second-order models are found to compare excellently with analytical solution. The negative wave as well as the positive wave front are correctly captured by the models and, as expected, the higher-order of accuracy results in slightly better shock-resolution. It can also be seen that there is no trace of any entropy violation due to the remedial treatment.

The models are next verified against experimental data of a hydraulic jump from experiments conducted by Gharangik[15] in a 13.9m long and 0.45 m wide straight, horizontal, rectangular channel. The Manning's n for experimental conditions were reported between 0.008 and 0.011. The constant discharge was 0.053 m^3/s. The upstream flow depth was 0.064m (velocity =1.82 m/s,Fr=2.3) and the conjugate depth was 0.17m (velocity=0.69m/s, Fr=0.53). The grid size for this problem is 0.05m x 0.05m. At the upstream end, both depth and velocity is specified and at the downstream end, a hypothetical rating curve is specified with a point corresponding to downstream measured conditions. We obtained good results with Manning's value equal to 0.008. The steady state results are compared with experimental data in Fig.3. The location of jump at about 1.6 m agrees well with the experimental data and so does the jump height. It may be noted, however, that the experimental jump profile is slightly diffused when compared with sharply resolved discontinuity in the numerical results.

Having verified the models, we apply them to 2-D dam-break problems. The layout is

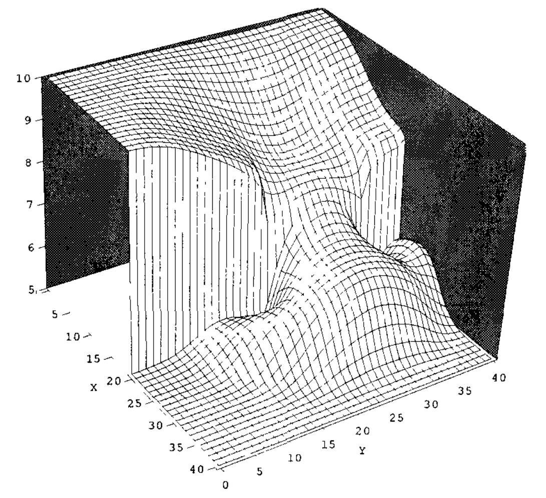

Fig.4 Water Surface Profile 7 sec. After Partial Dam-Break Computed by First-Order Scheme (h_r/h_t=10/5).

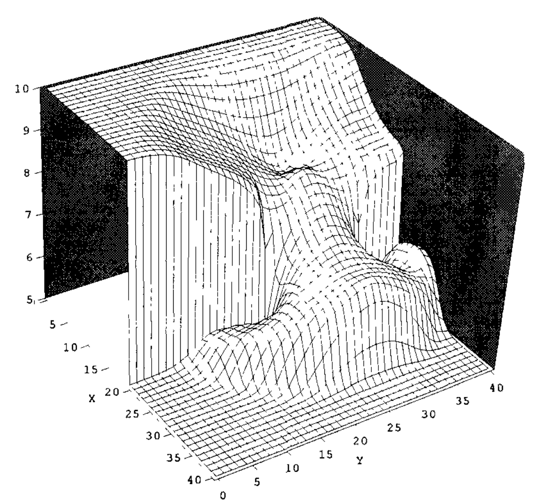

Fig.5 Water Surface Profile 7 sec. After Partial Dam-Break Computed by Second-Order Scheme (h_r/h_t=10/5).

taken from Fennema[16] for which results computed by other approaches have been previously reported[9,11,14]. The problem is schematically shown in Fig.1 with the partial breach section. The simulated results are shown in Figs.4 and 5. The first-order model yields remarkably good results but the shock –resolution is less sharp than that given by second-order accurate scheme. The flow depth contours corresponding to Figs.4 and 5 are shown in Figs.6 and 7. These results compare very well with previously reported results[9,11,14].

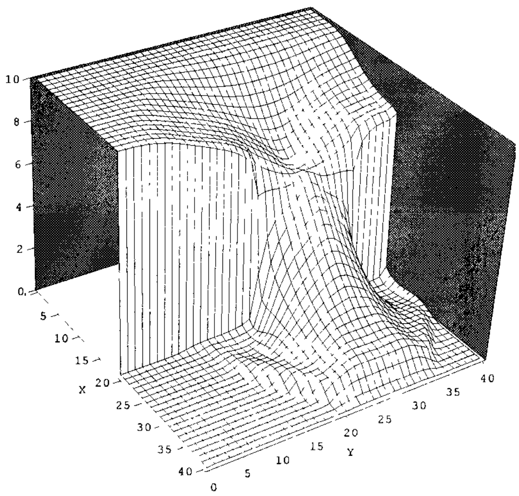

Fig.8 Water Surface Profile 7 sec. After Partial Dam-Break Computed by Second-Order Scheme (h_r/h_t=10/.05).

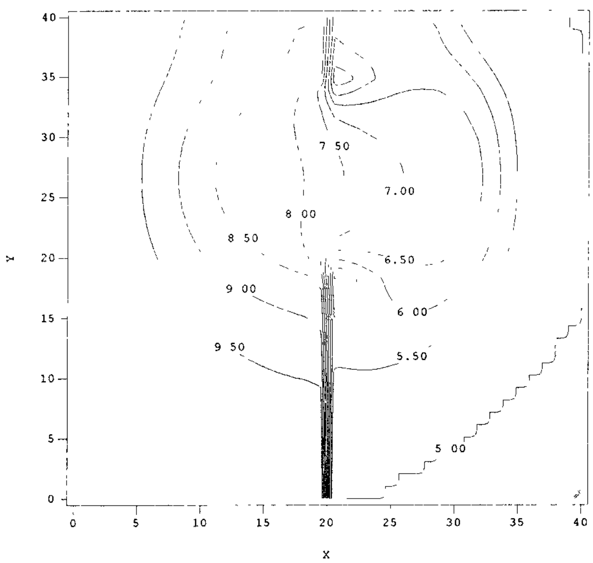

Fig.6 Water Depth Contour 7 sec. After Partial Dam-Break By First-Order Scheme (for Fig.4)

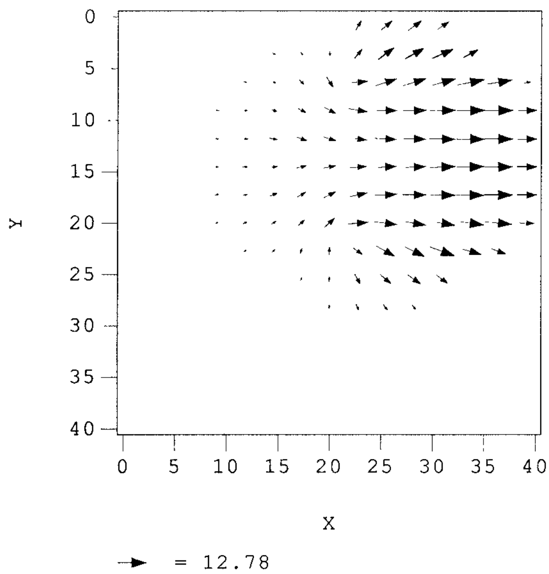

Fig.9 Velocity Vectors Corresponding to Fig.8

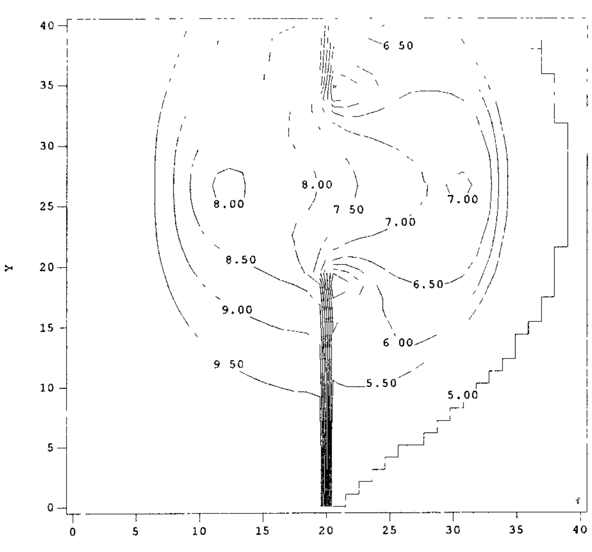

Fig.7 Water Depth Contour 7 sec. After Partial Dam-Break By Second-Order Scheme (for Fig.5)

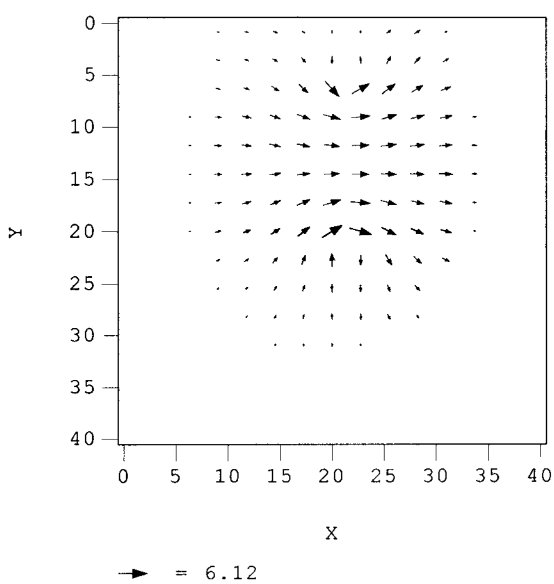

Fig.10 Velocity Vectors Corresponding to Fig.5

In order to demonstrate the models capability to simulate one of the severest problem, the water depth, h_t in the flood-plain is reduced to 0.05m. The water surface profile at 7+Δt seconds computed by second-order scheme is shown in Fig.8.The figure indicates that the bore travels much faster than when the flood-plain had larger water depth. However, the bore height is lower and less pronounced in this case. Fig.9 shows the corresponding velocity vector plot. It clearly shows that the velocity in the main flow direction is much larger than that in the transverse direction. Therefore, the flood-wave spreads comparatively slowly in the direction perpendicular to the main flow. Comparing Fig.9 with the velocity vector plot of Fig.10, corresponding to Fig.5, it is clear that this phenomenon becomes stronger as the difference between reservoir and flood-plain depths increases.

It may be noted that a free-slip boundary condition has been implemented at side walls, which means that there is no side wall friction

7. CONCLUSIONS

A First-order FDS scheme based on Roe's numerical flux and a second-order FDS scheme based on Lax-Wendroff numerical flux are developed for simulating rapidly varied two-dimensional flows. These schemes incorporate Roe's approximate Jacobian for conservative properties and consistency with the governing equations. The inclusion of Harten and Hymens treatment ensures models' compliance with entropy inequality requirements. The models are applied to severe dam-break and hydraulic jump problems. The comparison of numerical results with analytical and experimental results indicates that the models presented in this paper yield accurate results. The shocks are resolved, particularly by the second-order scheme, mostly within one spatial grid. The mass balance was tracked in all cases and the error was always within one percent. It is concluded that these models can be used confidently to simulate 2-D rapidly varied flows.

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