LUMPING PROCESS BASED ON UNSATURATED INFILTRATION THEORY

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Abstract

The Richards equation is adopted as a fundamental equation for the solution of unsaturated infiltration flow. At the first stage, a new boundary condition has been introduced and its accuracy has been cross checked with experiments. At the second stage, on the lumping process of non−dimensional form of Richards equation, a compensation factor has been proposed to equalize the semi−lumped equations. Finally, the relation between storage and discharge has been achieved by fully lumped equations.

Key words: lumping of unsaturated flow equations, semi-lumped equations, storage function model.

1.Introduction:

Basically the infiltration flow is dependent on the physical properties of the soil such as water retention, permeability, slope, slope length and slope depth. Also there are many other parameters in the field which are difficult to obtain. On the other hand, the computation base on Richards equation takes much time.

Therefore, to decrease the computation time, a storage function model needs to be derived. Fujita,M.(1981) and Takasao,T.(1985) derived a storage function model using kinematic wave theory and Matsubayashi,U.(1994) proposed a storage function model using Richards equation. Matsubayashi,U. (1994) didn't explain the lumping process in detail. One of the aims of this paper is a detail elaboration of lumping process using Richards equation.
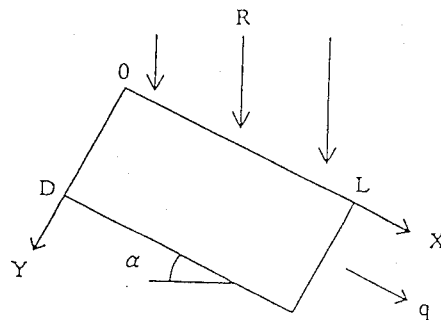

Fig.1:Profile of soil column

2. Two Dimensional Unsaturated Flow Equation:

The infiltration of rain into the soil can be expressed by Richards equation (equation (1)) which is applied to a soil column. The schematic conditions of soil column are shown in figure (1) where (L) is slope length, (D) is slope depth and (α) is slope.

$$\frac{\partial \theta}{\partial t} = -\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} \tag{1}$$

(θ)is moisture content in the soil,(t) is time,(v_x) and (v_y) are fluxes in(x) and (y) directions. These fluxes can be derived by Darcy's equation.

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$$v_x = -K \frac{\partial \phi}{\partial x} \quad (2) \qquad\qquad v_y = -K \frac{\partial \phi}{\partial y} \quad (3)$$

(ϕ) is water potential in the soil and (K) is conductivity of soil which can be expressed as:

$$K = K_s \left(\frac{\theta - \theta_r}{\theta_s - \theta_r}\right)^\beta \tag{4}$$

Where (K_s) is saturated conductivity of the soil, (β) is dependent on soil property, (θ_s) and (θ_r) are maximum (saturated) and minimum water content of the soil. The equation of water content can be derived by Haverkamp equation.

$$\theta = \frac{a^2}{a^2 + \psi^2}(\theta_s - \theta_r) + \theta_r \tag{5}$$

(ψ) is suction in the soil and (a) is dependent on soil property. Equations (2) and (3) can be rewritten as following:

$$v_x = K_s \left(\frac{a^2}{a^2 + \psi^2}\right)^\beta \left(\sin\alpha - \frac{\partial \psi}{\partial x}\right) \quad (6) \qquad v_y = K_s \left(\frac{a^2}{a^2 + \psi^2}\right)^\beta \left(\cos\alpha - \frac{\partial \psi}{\partial y}\right) \quad (7)$$

The solution of Richards equation has been considered by following boundary conditions.

$$y=0 \quad v_y = r\cos\alpha \quad (8) \; , \qquad y=D \quad v_y = 0 \quad (9) \; , \qquad x=0 \quad v_x = 0 \quad (10)$$

At x=L different boundary conditions have been proposed by researchers for example:

$$\frac{\partial \psi}{\partial x} = 0 \tag{11} \qquad\qquad \frac{\partial^2 \psi}{\partial x^2} = 0 \tag{12}$$

These equations are physically explained that the suction (ψ) doesn't change abruptly near the boundary and therefore they have an effect to depress the discharge. Matsubayashi,U.(1994) increased the horizontal hydraulic conductivity ten times greater than vertical hydraulic conductivity to avoid such a phenomena. In the present study, we introduce a new boundary condition (equation (13)) and its validity has been cross checked with experiments. The experimental conditions are shown in table (1). The soil properties such as ($\theta_s, \theta_r, k_s, \beta$ and a) are also obtained by other experiments.

$$x=L \qquad\qquad \frac{\partial v_x}{\partial x} = 0 \tag{13}$$

By using experimental condition and new boundary condition (equation (13)) the Richards equation has been solved and the result is shown in figure (2). The symbol (o o o) shows the discharge, resulted from calculation. The proposed boundary condition possess a good fitness to the experimental discharge. Further, we computed the ($\tilde{s}-\tilde{q}$) relation and the storage is calculated in two ways. The relation ($\tilde{s}_1-\tilde{q}$) is derived according to equation (14) and the result is shown in figure (3). Besides the relation ($\tilde{s}_2-\tilde{q}$) is derived by using equation (15) and the result is shown

Slope Length	500 (cm)
Slope Depth	40 (cm)
Slope Angle	0.1-0.2 (rdn)
θ_s	0.4
θ_r	0.05
K_s	0.009 (cm/sec)
β	2
a	65 (cm)

Table 1:Range of parameters used in experiment

in figure (4). Both relations show good accuracy. In the present study, we used a new numerical method which due to the lack of space we can not elaborate.

$$\tilde{s}_1 = \int_0^t r\, dt - \int_0^t \tilde{q}\, dt \tag{14}$$

$$\tilde{s}_2 = \int_0^L \int_0^D \theta(x,y,t)\, dy\, dx - \tilde{s}_0 \qquad (15) \quad \text{where } \tilde{s}_0 = \text{initial storage}$$

3. Non Dimensional Form of The Basic Equation:

The physical parameters of basic equations can be reduced by converting it into the non-dimensional form. For this purpose the following relationships have been considered.

$$t = t_* T, \qquad \psi = \psi_* \Phi, \qquad x = x_* X, \qquad y = y_* Y, \qquad v_x = v_* \nabla_x,$$
$$v_y = v_* \nabla_y, \qquad r = r_* R, \qquad \theta = \theta_* \Theta, \qquad \tilde{q} = q_* Q, \qquad \tilde{s} = s_* S \tag{16}$$

Following assumptions have been made for non-dimensionalizing process.

$$x_* = 1, \qquad\qquad y_* = d, \qquad v_* = K_s, \qquad r_* = K_s,$$
$$\theta_* = \theta_s - \theta_r, \qquad t_* = \frac{d\,\theta_*}{k_s}, \qquad \psi_* = d, \qquad q_* = k_s\, d, \qquad s_* = \theta_*\, d\, l \tag{17}$$

The non-dimensional form of equations (1),(6),(7),(4) and (5) can be written as following:

$$\frac{\partial \Theta}{\partial T} = -\left(\frac{d}{l}\right)\frac{\partial \nabla_x}{\partial X} - \frac{\partial \nabla_y}{\partial Y} \quad (18) \qquad\qquad \frac{a}{d} = A \tag{19}$$

$$\nabla_x = \left\{\frac{A^2}{A^2 + \Phi^2}\right\}^\beta \left(\sin\alpha - \frac{d}{l}\frac{\partial \Phi}{\partial X}\right) \quad (20) \qquad \nabla_y = \left\{\frac{A^2}{A^2 + \Phi^2}\right\}^\beta \left(\cos\alpha - \frac{\partial \Phi}{\partial Y}\right) \tag{21}$$

$$K = \left\{\frac{A^2}{A^2 + \Phi^2}\right\}^\beta \qquad (22) \qquad\qquad \Theta = \frac{A^2}{A^2 + \Phi^2} + \frac{\theta_r}{\theta_s - \theta_r} \tag{23}$$

The non-dimensional initial and boundary conditions can be written as:

$$\Phi = \frac{l}{d}(X-1)\sin\alpha + (Y-1)\cos\alpha + \frac{c}{d} \tag{24}$$

$$X=0 \quad \nabla_x = 0 \qquad (25) \qquad\qquad X=1 \quad \frac{\partial \nabla_x}{\partial X} = 0 \tag{26}$$

$$Y=0 \quad \nabla_y = R\cos\alpha \qquad (27) \qquad\qquad Y=1 \quad \nabla_y = 0 \tag{28}$$

The initial condition in equation (24) means initially, water is available in the soil but there is no discharge. This initial condition has been satisfied the experiment's initial condition.

4. Semi-Lumped Unsaturated Flow Equation:

By integration of the equations (18),(20),(25) and (26) along (Y), we can obtain semi-lumped equations.

$$\frac{\partial \overline{\Theta}}{\partial T} + \left(\frac{d}{l}\right)\frac{\partial Q_x}{\partial X} = R\cos\alpha \qquad (29) \qquad \text{where} \qquad 0 \le X \le 1$$

$$Q_x = \left\{\frac{A^2}{A^2 + \overline{\Phi}^2}\right\}^\beta \left(\sin\alpha - \frac{d}{l}\frac{\partial \overline{\Phi}}{\partial X}\right) \tag{30}$$

$$X=0 \quad Q=0 \qquad (31) \qquad\qquad X=1 \quad \frac{\partial Q}{\partial X} = 0 \tag{32}$$

$(\overline{\mathbf{\Theta}})$ in equation (29) and $(\overline{\mathbf{\Phi}})$ in equation (30) are averaged water content and suction along depth. However,when the relation in equation (32) replaced in equation (29) it has been created disturbance in calculation and it gave defaulted result. To avoid such a condition, following relation has been considered.

$$X=1 \qquad \frac{\partial \overline{\mathbf{\Theta}}}{\partial T}= 0 \qquad or \qquad \overline{\mathbf{\Theta}}= \overline{\mathbf{\Theta}}_{initial}= constant \qquad (33)$$

To equalized the approximation of integration following relation has been considered.

$$f1= \int_0^1 (\frac{A^2}{A^2+\Phi^2})^\beta dY \ / \ (\frac{A^2}{A^2+\overline{\Phi}^2})^\beta \qquad (34)$$

When the relation of equation (34) replaced in equation (30), it has been realized that it destroys the situation of boundary condition at (X=0). Therefore to equalize the relation in equation (30), equation (35) has been introduced.

$$Q_x= \gamma \{\frac{A^2}{A^2+\overline{\Phi}^2}\}^\beta (\sin\alpha-\frac{d}{l}\frac{\partial \overline{\Phi}}{\partial X}) \qquad (35)$$

Where (γ) is obtained through various numerical calculations.

$$\gamma= EXP \{0.1\beta-0.07+(0.09-0.17\beta) \ Ln(A)\} \quad (36) \qquad where \qquad \alpha= 0.2\sim0.6$$

For different ranges of parameters the validity of (γ) is shown in figure (5) briefly. (γ) is obtained under uniform rain therefore to check its accuracy, we applied different type of rain as it is shown in figure (6).

5. Storage Function Model:

The relation between storage and discharge can be derived under the steady state condition.

$$Q_x= \frac{l}{d} R X \cos\alpha \qquad (37)$$

By futher integration of equations (29) and (35) under the condition $(\sin\alpha > \frac{d}{l}\cdot\frac{\partial \overline{\Phi}}{\partial X})$, equations (38) and (39) are obtained.

$$\frac{dS}{dT}+(\frac{d}{l}) Q_{x=1}= R \cos\alpha \qquad (38)$$

$$S= (\frac{\beta}{\beta+1})\{\frac{Q}{\gamma \ \sin\alpha}\}^{\frac{1}{\beta}}+(\frac{\theta_r}{\theta_s-\theta_r}) \qquad (39)$$

Fig .7: Profile of discharge along (X)

Equation (39) is almost similar to Matsubayashi,U.(1994) storage function model. Figure (8) shows the result obtained by semi-lumped equations. In principal, the equations (29) and (35) have to be solved theoretically to find (S~Q) relation at unsteady state. However, it is difficult to solve them theoretically. Therefore, a convenient method has been introduced by authors. Let's assume:

$$\frac{A^2}{A^2+\overline{\Phi}^2}= e^{e\overline{\Phi}} \qquad (40)$$

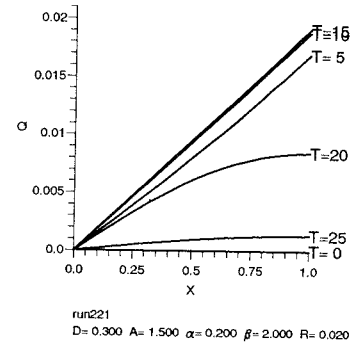By substituting the relation in equation (40) into equation (35), following relation can be achieved.

$$Q = \gamma \sin\alpha \ e^{\beta\varepsilon\overline{\Phi}} - \frac{\gamma d \cdot}{l\beta\varepsilon} \frac{\partial e^{\beta\varepsilon\overline{\Phi}}}{\partial X} \tag{41}$$

With the respect to figure(7) following relation can be assumed. The relation in equation (42) means the steady state condition of discharge along (X) where the relation in equation (43) means the unsteady state condition.

$$Q(X) = Q_{X=1} \ X \qquad (42) \qquad\qquad Q(X) = -X(X-2) \ Q_{X=1} \tag{43}$$

Let's assume a new coordinate system $\acute{X} = 1-X$ where according to boundary condition at $\qquad \acute{X} = 0 \qquad\qquad e^{\beta\varepsilon\overline{\Phi}} = e^{\beta\varepsilon\overline{\Phi}_0}$ (44)

By considering the new coordinate system and substituting the relation in equation (42) in equation (35) following relation can be achieved.

$$Z_1 = e^{\beta\varepsilon\overline{\Phi}} = (\frac{Q_{\acute{X}=0}}{\gamma \sin\alpha}) \{1 - e^{-C_1\acute{X}} - \frac{1}{C_1}(C_1\acute{X}-1) - \frac{1}{C_1}e^{-C_1\acute{X}}\} \quad (45) \text{ where } \quad C_1 = \frac{l \ \beta \ \varepsilon \ \sin\alpha}{d} \quad (46)$$

Similarly, by replacing the equation (43) in equation (35) we can derive.

$$Z_2 = e^{\beta\varepsilon\overline{\Phi}} = (\frac{Q_{\acute{X}=0}}{C_1}) \left\{ (1 - e^{-C_1\acute{X}}) - (\acute{X}^2 - \frac{2\acute{X}}{C_1} + \frac{2}{C_1^2}) + \frac{2}{C_1^2}e^{-C_1\acute{X}} \right\} \tag{47}$$

The $(S\sim Q)$ relation can be obtained by following equations.

$$S_1 = \int_0^1 Z_1^{\frac{1}{\beta}} \ dX = \{\frac{Q_{\acute{X}=0}}{\gamma\sin\alpha}\}^{\frac{1}{\beta}} \ G_1(C_1,\beta) \qquad (48) \qquad S_2 = \int_0^1 Z_2^{\frac{1}{\beta}} \ dX = \{\frac{Q_{\acute{X}=0}}{\gamma\sin\alpha}\}^{\frac{1}{\beta}} \ G_2(C_1,\beta) \quad (49)$$

Where $G_1(C_1,\beta)$ and $G_2(C_1,\beta)$ are obtained by numerical calculations.

$$G_1(C_1,\beta) = 0.151 \ \log C_1 + 0.331 \ \log\beta + 0.241 \tag{50}$$

$$G_2(C_1,\beta) = (-0.064 \ \log\beta + 0.21) \ \log C_1 + 0.31 \ \log\beta + 0.31 \tag{51}$$

Due to the lack of space, we can not show the results of equations (51) and (52).

6.Conclusion:

At the first stage, we introduced a new boundary condition at the outlet of soil column and its accuracy is examined with experimental results. At the second stage, we proposed a compensation factor (γ) to equalize the semi lumped equation to the Richards equation. Finally, we found out the storage function model can not satisfy the actual (S-Q) relation and it may satisfy in case of long duration of rainfall.

7.References:

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Matsubayashi,U. Takagi,F. and Furuta,S. (1994) "On The Lumping of Runoff Model Based On The Unsaturated Flow Theory", Journal of JSCE No.497/II-28, 21-29

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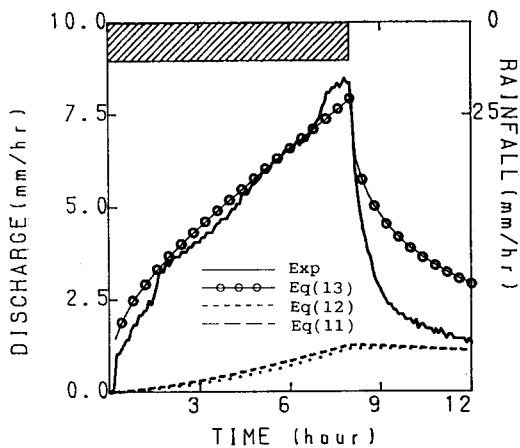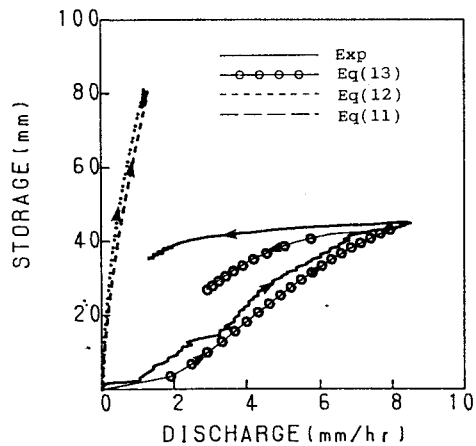
Fig.2:Profile of discharge $\alpha = 0.1$

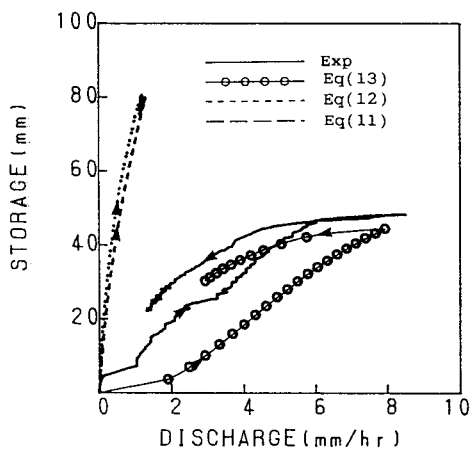Fig.3:Relationship of storage and discharge $(\tilde{S}_1, \tilde{q})$, $\alpha = 0.1$

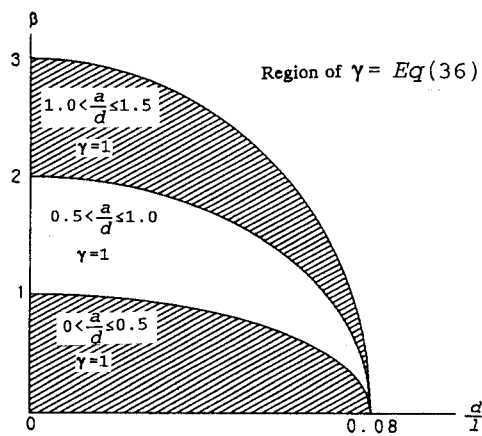Fig.4:Relationship of storage and discharge $(\tilde{S}_2, \tilde{q})$, $\alpha = 0.1$

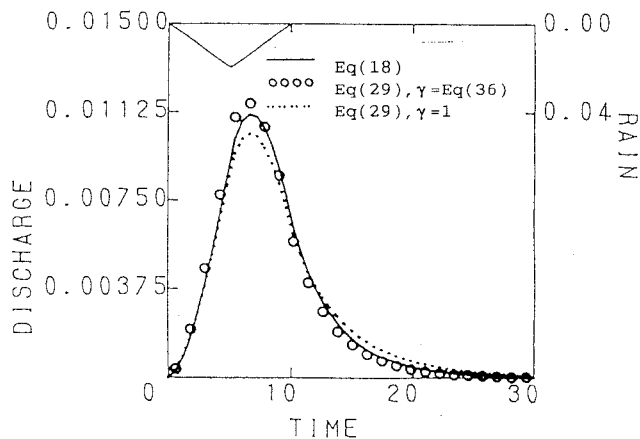Fig.5:Range of parameters for γ

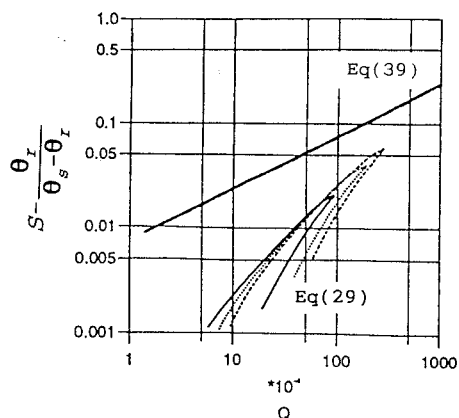Fig.6:Profile of discharge $\frac{d}{l}=0.5, \frac{a}{d}=0.05, \alpha=0.4, \beta=2$

Fig.8:Relationship of storage and discharge $\frac{d}{l}=0.3, a/d=1.5, \alpha=0.2, \beta=2$