Diffusion of Suspended Load in Unsteady Open-Channel Flows

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After a brief survey of the major theories on suspended-load diffusion under different flow conditions, we present here an analytical study on the diffusion of suspended load in unsteady open-channel flows. The histograms of suspended load and of flow discharge in rivers are examined, and the concept of sediment-wave celerity is introduced. It is noted that discussed here is but a small and humble step toward understanding the much complex phenomenon of suspended-load diffusion in unsteady flows.

1   Introduction

Sediments transported in natural rivers are usually composed of wash load, suspended load and bed load. Wash load, no matter how much the upstream supplies, is flushed by the flow to the downstream. Suspended load and bed load, however, frequently cause problems. Moreover, in most natural rivers the bulk of the transported materials are carried to the downstream as suspended load. Suspended load, in turn, may create serious problems to waterways, to industrial as well as to agricultural water supply, and to the environment that has attracted more and more attention from different sectors of the society. Needless to say, it is important to treat problems related to suspended-load diffusion in natural rivers. In the present paper, we shall study the diffusion of suspended load in unsteady open-channel flows.

In open-channel flows, a complex diffusion mechanism entrains, mixes, and transports the suspended load through the fluid medium. The intricate structure of turbulence makes it virtually impossible, up to the present state of knowledge at least, to fully describe the properties of the fluid medium and the materials it carries. Graf (1984, p.164) asserted that until the mid-thirties, though observations of the suspension phenomenon had been made and some researchers even suggested predictive models, none of them could present any useful quantitative informations. One of the main obstacles was that, although realized as of considerable importance to the understanding of diffusion processes, turbulence and its associated consequences could be expressed neither physically nor mathematically.

The late thirties witnessed a breakthrough with the now famed Rouse (1938) formula, which predicts the vertical distribution of suspended load in steady uniform flows with equilibrium transport. Kalinske (1940) investigated the factors determining the distance required to reach equilibrium conditions if no sediment is supplied at the entrance section. Dobbins (1943) obtained analytical solutions for a set of problems where the longitudinal dispersion could be neglected, and the results were successfully applied by Camp (1945) to design sedimentation basins. Along the same line, Mei (1969) attacked the problem analytically; and Apmann and Rumer (1970) did it experimentally. Mathematically speaking, all these researchers have used the same basic equation while introducing certain approximations, albeit with different boundary conditions. Again Hjelmfelt and Lenau (1970) treated the same problem with a diffusion coefficient deduced from the well-known log-law for velocity distribution. To be noted also is that in Japan Goda (1953) even attempted to solve a 3-D steady diffusion problem. However, to our knowledge, there has been no investigation on the diffusion of suspended load in unsteady open-channel flows, although it is known to all that during flood periods large amount of suspended-load transportation occurs.

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2   Basic Equation

Considering that within an infinite control volume in the sediment-water mixture, the temporal variation of suspended-load concentration should be equal to the net increase of those entering into, minus those going out, of the control volume, one can write:

$$\frac{\partial c}{\partial t} = -\frac{\partial(uc)}{\partial x} - \frac{\partial(vc)}{\partial y} - \frac{\partial(wc)}{\partial z} + \frac{\partial(\omega c)}{\partial y} + \frac{\partial}{\partial x}\left(\varepsilon_x \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(\varepsilon_z \frac{\partial c}{\partial z}\right) \qquad (1)$$

where: c is the concentration at a point (x, y, z) in space, at time instant, t; x, y and z, the longitudinal, vertical and transverse coordinates, respectively; ω, the settling velocity of sediment particles; ε_x, ε_y and ε_z, sediment diffusion coefficients in the x, y and z directions, respectively.

In a wide open channel the flow can be considered as two dimensional, so all the $\partial/\partial z$ terms in Eq.1 can be dropped out. Further, it is generally believed that in most natural rivers, dispersion in the longitudinal direction is much less important than in the vertical direction, $\frac{\partial}{\partial x}\left(\varepsilon_x \frac{\partial c}{\partial x}\right) << \frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial c}{\partial y}\right)$. So Eq.1 can be reduced to:

$$\frac{\partial c}{\partial t} = -\frac{\partial(uc)}{\partial x} - \frac{\partial(vc)}{\partial y} + \frac{\partial(\omega c)}{\partial y} + \frac{\partial}{\partial y}\left(\varepsilon_y \frac{\partial c}{\partial y}\right) \qquad (2)$$

which is the diffusion equation for suspended load in unsteady, two dimensional open-channel flows. It is too complicated in its present form to be solved analytically. Solutions may be found for some particular cases, and with certain approximations, as has been done by various investigators in the past.

In the following sections, an analytical solution will be derived for the diffusion of suspended load in unsteady open-channel flows. We shall first make some approximations, then define the boundary conditions, and finally obtain the analytical solution.

3   Working Approximations

Before searching for an analytical solution, some approximations are necessary to render the much involved governing equation less difficult to solve.

3.1   Vertical velocity component

The vertical velocity component, v, is negligible. Consequently: $\partial(vc)/\partial y = 0$. Tu and Graf (1992), running a series of hydrographs in a gravel-bedded flume, have justified this approximation by the fact that the vertical velocity component is very small compared with the longitudinal one.

3.2   Longitudinal velocity component

In natural rivers, the longitudinal variation of the velocity component, u, even during flood events, can be considered as negligible when compared with the vertical velocity gradient. In order to reduce the complexity of the much involved governing equation in the present analysis, as has been frequently done by other investigators, we shall assume that all the point velocities are equal to the depth-averaged velocity, V.

3.3   Sediment diffusion coefficient

About the vertical sediment diffusion coefficient, there have been quite different conclusions drawn by various researchers. The coefficient is generally believed to be proportional to the momentum coefficient, ε, such that:

$$\varepsilon_y = \beta\,\varepsilon$$

The question is whether β is equal to, larger or smaller than, one (Graf 1984, pp.176-177). If the suspended load is composed of fine sands, and the concentration is relatively low, good precision can be expected even if β is set to one.

If the coefficient β is set to be one, the problem then is to find the momentum coefficient. While no doubt that the momentum coefficient itself varies in space, and it is desirable to find the appropriate function, we assume here a constant momentum coefficient for the present study.

3.4 Sediment wave celerity

It is known that during flood seasons in natural rivers, there is the so-called kinematic wave propagating to the downstream (see for example, Henderson 1963). The suspended-load concentration measured at a certain station during a flood event, as well as the flow discharge, can be expressed in the form of a histogram (Fig.1). From the histograms shown in Fig.1, it might be reasoned that the suspended load also participates (as does the flow discharge), with a certain speed - here designated as the *sediment wave celerity* - in the kinematic wave motion.

The thus defined sediment wave celerity in general should be different from the kinematic-wave celerity, C. However, if the concentration is low and the sediment particles are small, as is assumed in the present analysis, then this difference is negligible. So we have (Tu and Graf 1992): $\frac{\partial c}{\partial x} = -\frac{1}{C}\frac{\partial c}{\partial t}$. Further, from the assumption made above that u is equal to V, we may write: $-\frac{\partial(uc)}{\partial x} = \frac{V}{C}\frac{\partial c}{\partial t}$.

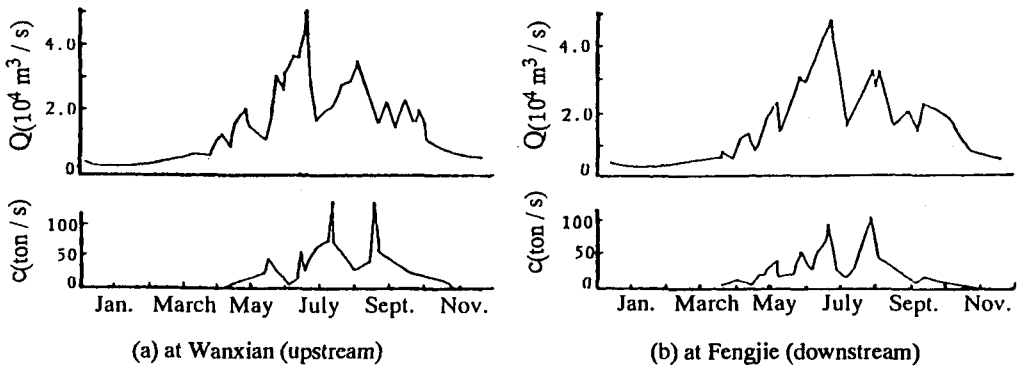(a) at Wanxian (upstream)   (b) at Fengjie (downstream)

Fig.1 Hydrograph and Suspended-load Curves from the Yangtze River
(Ning Chien et al.1987, p.75)

3.5 Settling velocity of sediment particles

Since it is assumed in our analysis that the density of the water-sediment mixture is about equal to the clear-water density, the turbulent flow fluctuations and sediment concentration should bear little effect on the settling velocity of the sediment particles. In other words, the settling velocity can be assumed as constant.

4 Initial and Boundary Conditions

4.1 Initial condition

We assume that at a certain station under investigation, the initial vertical profile of the suspended load is known beforehand as:

at $t = 0$, $c = f(y)$ for $0 < y < D$ (3)

The given function $f(y)$, of course, may represent an equilibrium or non-equilibrium distribution profile, depending on the actual situation. Generally speaking, before the flood arrives, flow is mostly steady, and the diffusion of suspended load might have well reached an equilibrium condition in the reach or the station under study. When the flood comes, this equilibrium state will be destroyed. The evolution of the suspended-load profile in unsteady flow is the principal aim of our analysis.

4.2 Bottom condition

In almost all of the past investigations on suspended-load diffusion, the final results invariably involve a certain reference concentration, which is the sediment concentration at a certain height near the bed. The selection of the reference height, and particularly, the measurement or determination of the concentration at that height, are very demanding tasks. There is no satisfying solution at the present state of knowledge. We assume that close to (not at) the bottom the pick-up rate equals the rate of deposit under equilibrium condition, c_a. That is:

$$\varepsilon \left. \frac{\partial c}{\partial y} \right|_{y \to 0} = - \omega \, c_a \qquad (4)$$
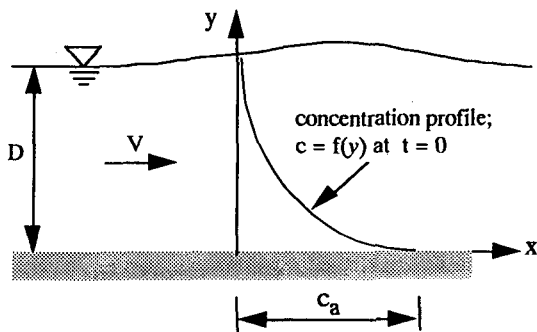

Fig.2 Definition Sketch

4.3   Surface condition

It is evident that there should be no movement of the suspended load across the water surface. Expressed mathematically, this boundary condition is given as:

$$\varepsilon \left. \frac{\partial c}{\partial y} \right|_{y = D} = - \omega \, c \Big|_{y = D} \qquad (5)$$

A sketch showing the boundary conditions is shown in Fig.2.

5   Solution

The mathematical deduction procedures here are in principle similar to those adopted by other investigators in the past, the major difference being that the flow considered is unsteady and the concept of sediment-wave celerity is introduced.

Considering what have been discussed in the preceding section, the governing equation for the diffusion of suspended load in unsteady open-channel flows can now be rewritten as:

$$\frac{\partial c}{\partial t} = W \frac{\partial c}{\partial y} + E \frac{\partial^2 c}{\partial y^2} \qquad \text{where:} \qquad W = \omega/(1 - \frac{V}{C}); \qquad E = \varepsilon/(1 - \frac{V}{C}) \qquad (6)$$

with the boundary conditions given by Eqs.5, 6 and 7.

From kinematic-wave theory (Henderson 1963), the wave celerity (C) is proportional to the depth-averaged velocity (V). The ratio of C/V, being different for different channel shapes, can be assumed as constant in the present case. Consequently, the coefficients, W and E in Eq.6, are constants.

The final solution of Eq.6 is composed of a particular solution and a general solution. The particular solution is obtained by assuming $\partial c/\partial t = 0$, i.e., when the diffusion process is in equilibrium and the suspended-load concentration does not vary with time. In this case, one has from Eq.6:

$$W \frac{\partial c}{\partial y} + E \frac{\partial^2 c}{\partial y^2} = 0 \qquad (7)$$

for which the solution is:

$$c = c_a \, e^{- (W/E) y} \qquad (8)$$

Next we search for a general solution of Eq.6, which is done in a classic way. Assuming that the solution for Eq.6 is a product of two functions - one a function of y alone and the other a function of t alone:

$$c = Y(y) \, T(t) \qquad (9)$$

and replacing Eq.9 into Eq.6, one has:

$$\frac{T'}{T} = \frac{E\,Y'' + W\,Y'}{Y} = -k^2 \tag{10}$$

Solving Eq.10 for T and Y, separately, we have:

$$T = a_0\, e^{-k^2 t}$$

$$Y = e^{-0.5(W/E)y}\ [a_{11} \cos \alpha y + a_{22} \sin \alpha y\ ]$$

from which we have:

$$c = e^{-k^2 t}\ e^{-0.5(W/E)y}\ [a_1 \cos \alpha y + a_2 \sin \alpha y\ ] \tag{11}$$

which satisfies Eq.6. In Eq.11, a_1 and a_2 are coefficients, and the relationship between k and α is given in:

$$\alpha^2 = k^2/E - W^2/(4E^2) \tag{12}$$

Combining Eqs.10 and 13, we have a general solution for Eq.6;

$$c = c_a\, e^{-(W/E)y} + e^{-k^2 t}\ e^{-0.5(W/E)y}\ [a_1 \cos \alpha y + a_2 \sin \alpha y\ ] \tag{13}$$

From Eq.13 and the boundary condition given in Eq.4, for $y \to 0$, one has:

$$a_2 = \frac{W}{2E\alpha}\, a_1 \tag{14}$$

And with the boundary condition given in Eq.5, for $y = D$, one obtains:

$$\mathrm{ctg}\alpha D = \frac{a_1\,\alpha - \dfrac{W}{2E}\,a_2}{a_2\,\alpha + \dfrac{W}{2E}\,a_1}$$

Replacing Eq.14 into the above expression, we derive the following equation from which α can be calculated:

$$2\mathrm{ctg}\alpha D = (\alpha D) / \left(\frac{WD}{2E}\right) - \left(\frac{WD}{2E}\right) / (\alpha D) \tag{15}$$

Using Eq.14, a solution for Eq.6 that satisfies the boundary conditions (but not the initial condition) is deduced from Eq.13:

$$c = c_a\, e^{-(W/E)y} + e^{-k^2 t}\ e^{-0.5(W/E)y}\ a_1 \left(\cos \alpha y + \frac{W}{2E\alpha} \sin \alpha y \right) \tag{16}$$

where α is given in Eq.15. With α, in turn, one can calculate k readily from Eq.12.

Further, we see that Eq.15, with any given values of W, D and E, renders an infinite number of real positive roots for α, all of which can be used in Eq.16. So the most general solution for Eq.6 is:

$$c = c_a\, e^{-(W/E)y} + e^{-0.5(W/E)y} \sum_{n=1}^{\infty} e^{-(\alpha_n^2 E + 0.25 W^2/E)t}\ a_n \left(\cos \alpha_n y + \frac{W}{2E\alpha_n} \sin \alpha_n y \right) \tag{17}$$

The remaining unknown coefficient, a_n, is determined using the initial condition (Eq.3), such that:

$$f(y) = c_a \, e^{-(W/E)y} + e^{-0.5(W/E)y} \sum_{n=1}^{\infty} a_n \left(\cos \alpha_n y + \frac{W}{2E\alpha_n} \sin \alpha_n y \right)$$

from which the expression for a_n can be derived (see Dobbins 1943):

$$a_n = \frac{2\alpha_n}{(\alpha_n^2 + \frac{W^2}{4E^2})D + \frac{W}{E}} \int_0^D \left\{ [\, f(y) - c_a \, e^{-(W/E)y}\,] \, e^{0.5(W/E)y} \left(\cos \alpha_n y + \frac{W}{2E\alpha_n} \sin \alpha_n y \right) \right\} dy$$

(18)

Equation 19 is the solution for suspended-load diffusion in unsteady, two-dimensional open-channel flows. Once the water depth, the settling velocity, the diffusion coefficient, the initial and boundary conditions are known beforehand, Eq.17, together with Eqs.15 and 18, gives the vertical profile of suspended-load concentration at any time instant in unsteady flow. For example, if $f(y) = c_0$, then Eq.17 would predict the diffusion processes after a sediment-water mixture with a known concentration is released into the flow. Further, if $c_0 = 0$, that would be a case of scouring, when in the beginning only water (clear of sediment) is supplied at the station under study. Another example would be for the rate of deposit under equilibrium condition to be zero ($c_a = 0$), simulating thus the case where appears settling out from equilibrium concentration to zero concentration at all water depths. For all these three particular initial or boundary conditions, the solution can be readily obtained (and simplified) from Eqs.17, 19 and 20.

6   Conclusion

In this analysis, first the similarities between the histogram of suspended-load concentration and those of other hydraulic parameters are examined, then the concept of sediment-wave celerity is introduced. Subsequently, the diffusion of suspended load in unsteady open-channel flows is investigated analytically. For a given station in a channel flow, Eq.17, together with Eqs.15 and 18, may be used to predict the suspended-load concentration at water height, y, and time instant, t. However, since several assumptions, particularly that of a constant velocity and a constant diffusion coefficient, were adopted in the analysis, further improvements are necessary before the results obtained could be used in practice. For this very reason, we envisage using a variable diffusion coefficient in our future studies.

7   References

Apmann, R. P. and R. R. Rumer (1970), Diffusion of Sediment in Developing Flow, Proc. Am. Soc. Civil Engrs., Vol.96, No. HY1.

Camp, T. R. (1945), Sedimentation and the Design of Settling Tanks, Trans. Am. Soc. Civil Engrs.,Vol.111.

Dobbins, W. E. (1943), Effect of Turbulence on Sedimentation, Trans. Am. Soc. Civil Engrs., Vol.109.

Goda, T. (1953), A Study on the Mechanism of Transportation of Suspended Sediment, Mem. Faculty Eng., Kyoto Univ., Vol.15, No.14.

Graf, W.H. (1984), Hydraulics of Sediment Transport, Water Resources Publications, Colorado, p.513.

Henderson, F.M. (1963), Flood Waves in Prismatic Channels, Proc., Am. Soc. Civ. Engrs, J. Hyd. Div., Vol.89, No.HY4, pp.39-67.

Hjelmfelt, A. T. and C. W. Lenau (1970), Non-Equilibrium Transport of Suspended Sediment, Proc. Am. Soc. Civil Engrs., Vol.96, No. HY7.

Kalinske A. A. (1940), Suspended Material Transportation under Non-Equilibrium Conditions, Trans. Am. Geophys. Union, Vol.21.

Mei, C. C. (1969), Non-uniform Diffusion of Suspended Sediment, Proc. Am. Soc. Civil Engrs., Vol.65, No. HY1.

Ning Chien, R. Zhang and C. Chow (1987), Fluvial Processes, Science Press, Beijing, p.584 (in Chinese).

Rouse, H. (1937), Modern Conceptions of the Mechanics of Turbulence, Trans. Am. Soc. Civil Engrs., Vol.102.

Tu Haizhou and W. H. Graf (1992),Vertical distribution of shear stress in unsteady open-channel flow, Proc. Instn Civ. Engrs Wat., Mart. & Energy, London, Vol.96, June, pp.63-69.