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Similarity Solution For Convection Near a Vertical Plate Without Using the Boussinesq Approximation

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It is well known that the Boussinesq approximation works well in the field of convection flow and heat transfer. However, the basis of this approximation is that temperature in the flow field varies little. Can we extend this assumption to the case in which large temperature difference exists? The engineering answer is definite so far, but from theoretical point of view, it is questionable. What can we do then? This is what we have attempted to do in this paper. In this paper, we propose a new approach to deal with convective heat transfer without using the Boussinesq assumption.

KEYWORDS: Convective flow, Boussinesq approximation, Similarity Solution

1 Introduction

Convection and convective heat transfer is a very widely investigated field. People already made so great progress that the number of published papers is very huge and covered almost all subject in this field. However, these analyses, without exception, are based on the Boussinesq assumption. It is doubtless that this approximation works well up to now. But we have a sense of crisis. Strictly speaking, the Boussinesq assumption is valid only for very small temperature difference in the flow field. Recently, we realized that under certain condition, we can drop the Boussinesq assumption and as an alternative, adopt another assumption which may be more suitable for the case of large temperature difference. Now, let us first take a look at the convective flow and heat transfer from a heated vertical plate.The convective flow along a vertical plate is of importance in a number of engineering applications. At the same time, it is one of the main model problems of convection. It helps us get better understanding of the laws governing convective heat transfer in other,more complex cases.

2 Analysis

2.1 Steady-state Convection near a Heated Vertical Wall

In this paper, we assume that boundary-layer approximation is applicable for two-dimensional flow along vertical plate, as found in the experimental observations of Schmidt and Beckman(1930) and as indicated by an order of magnitude analysis. Consequently, the basic equations for steady flow along a vertical flat surface with zero pressure gradient are:

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$$\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v = 0 \tag{1}$$

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = \rho g + \frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y}) \tag{2}$$

$$\rho u\frac{\partial T}{\partial x} + \rho v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}(\frac{K}{C_p}\frac{\partial T}{\partial y}) + \mu(\frac{\partial u}{\partial y})^2 \tag{3}$$

The second term in the right hand in eq(3) is the viscous dissipation which is often neglected in many papers. Now, we introduce transformation:

$$\xi = \frac{x}{L} \qquad \eta = \sqrt{\frac{Re}{c}}\frac{1}{L}\int_0^y \frac{\rho}{\rho_\infty}dy \qquad \psi = \sqrt{\frac{c}{Re}}\rho_\infty u_\infty LF(\xi,\eta)$$

$$T - T_\infty = (T_0 - T_\infty)G(\xi,\eta) \qquad Re = \frac{\rho_\infty u_\infty L}{\mu_\infty}$$

Where $L, u_\infty, \rho_\infty, \mu_\infty$, are characteristic length, velocity, density and viscosity. The meaning of the constant c will be explained later on.

By the definition of stream function, we have:

$$\rho u = \frac{\partial\psi}{\partial y} = \rho u_\infty \frac{\partial F}{\partial \eta}$$

$$\rho v = -\frac{\partial\psi}{\partial x} = -\sqrt{\frac{c}{Re}}\rho_\infty u_\infty \frac{\partial F}{\partial \xi}$$

Substituting these expressions into the governing equations, we got:

$$F_\eta F_{\xi\eta} - F_\xi F_{\eta\eta} = \frac{1}{c}\frac{\partial}{\partial\eta}(\frac{\rho\mu}{\rho_\infty\mu_\infty}\frac{\partial^2 F}{\partial\eta^2}) + \frac{Lg}{u_\infty{}^2}$$

$$F_\eta G_\xi d + F_\eta G d_\xi - F_\xi G_\eta d = \frac{d}{cP_r}\frac{\partial}{\partial\eta}(\frac{\rho\mu}{\rho_\infty\mu_\infty}\frac{\partial G}{\partial\eta}) + \frac{u_\infty{}^2}{c}\frac{\rho\mu}{\rho_\infty\mu_\infty}(\frac{\partial^2 F}{\partial\eta^2})^2$$

where, $d = d(\xi) = T_0 - T_\infty$ $\qquad$ T_0: temperature distribution along the plate.

We now assume that the product term

$$\rho\mu = const.$$

The basis of this assumption is that for gas, the dynamic viscosity increases when temperature increases, meanwhile, the density decreases when temperature increases, so that $\rho\mu$ can be regarded as constant. Let $\rho\mu/\rho_\infty\mu_\infty = c$; This c is just the one we used in the transformation.Following this assumption and assuming a similarity variable as well as a form of stream function.

$$Z_\eta = \eta b(\xi)$$

$$F(\xi,\eta) = C^*(\xi)f(Z_\eta)$$

If $G(\xi,\eta) = \phi(Z_\eta)$ holds; the equations become:

$$f'''(Z_\eta) + \frac{Lg}{C^* b^3 u_\infty{}^2} + \frac{C_\xi^*}{b}ff'' - (\frac{C_\xi^*}{b} + \frac{C^* b_\xi}{b^2})(f')^2 = 0$$

$$\frac{1}{P_r}\phi''(Z_\eta) + \frac{C_\xi{}^*}{b}f\phi' - \frac{d_\xi}{bd}f'\phi + \frac{(u_\infty C^* b)^2}{d}(f''')^2 = 0$$

Therefore,the necessary existence conditions for the similarity solution are given as follows:

$$C^* b^3 = A_1 \qquad \frac{C_\xi}{b} = A_2 \qquad \frac{C_\xi{}^*}{b} + \frac{C^* b_\xi}{b^2} = A_3$$

$$\frac{d_\xi C^*}{bd} = B_1 \qquad \frac{(u_\infty C^* b)^2}{b} = B_2$$

From these conditions, we can obtain:

$$C^*(\xi) = D_1 \xi^{3/4} \qquad d(\xi) = D_3 \xi \qquad b^*(\xi) = D_2 \xi^{-1/4} \qquad D_i : constant$$

If we neglect the effect of viscous dissipation as did in many previous papers, the similarity solution, of course, exists, too. So, we can conclude that inclusion of viscous dissipation term doesn't spoil the similarity in the case considered here. Moreover, the present study shows that the similarity solutionsexist only for linear surface (at y=0) temperature distributions. This conclusion is different from previous studies which have shown that similarity solutionsexist both for a power law and exponetial surface temperature distributions. Now, let us calculate the heat flux from the surface.The local flux is given by

$$q''(x) = -K(\frac{\partial T}{\partial y})_{y=0} = \frac{C_p}{p_r}\sqrt{\frac{Re}{c}}\frac{1}{L}\frac{\rho\mu}{\rho_\infty}d(\frac{\partial \phi}{\partial Z_\eta})_{Z_\eta=0} = K_1 x \qquad K_1 : constant$$

The total heat transfer from zero to x per unit width is given by

$$q(x) = \int_0^x q''(x)dx = \frac{K_1}{2}x^2$$

This result implies that there is very strong heat exchange on the solid-fluid interface.Our method can be extended to cases of time-dependent convective flow and convective flow near a horizontal plate.

2.2 Transient Free Convection near a Heated Vertical surface

The problem of unsteady free convection in the vicinity of a doubly infinite vertical flat plate (i.e,extending to both plus and minus infinity) and the semi-infinite vertical flat plate (i.e,extending to plus infinity above a leading edge) has been studied extensively. As first pointed out by Siegel, the initial behavior of the temperature and velocity fields for the semi-infinite vertical flat plate are the same as for the doubly infinite vertical flat plate. Upon commencement of the transient, one may consider the fluid moving up from the leading edge as a wave, in front of which the velocity and temperature are only functions of the time and the distance y, from the plate. Behind the wave there must be a dependence on the vertical coordinate,x. The governing equations are following for the doubly infinite vertical plate whose surface temperature or heat flux is any function of time. They also apply to the semi-infinite vertical plate, with the same boundary conditions, at all positions which lie above the point of maximum penetration of the leading-edge effect.

$$\rho\frac{\partial u}{\partial t} = \rho g + \frac{\partial}{\partial y}(\mu\frac{\partial u}{\partial y}) \qquad \rho\frac{\partial T}{\partial t} = \frac{\partial}{\partial y}(\frac{k}{C_p}\frac{\partial T}{\partial y})$$

In this case, we can use a simpler transformation, as follows:

$$\eta = \int_0^y \frac{\rho}{\rho_\infty}dy \qquad t = t$$

by using this transformation and assuming $\rho\mu/\rho_\infty{}^2 = a^2$;equations become

$$\frac{\partial u}{\partial t} = g + a^2\frac{\partial^2 u}{\partial \eta^2} \qquad \frac{\partial T}{\partial t} = \frac{1}{P_r\rho_\infty{}^2}\frac{\partial^2 T}{\partial \eta^2}$$

Let $u^* = u - gt$, then we obtain;

$$\frac{\partial u^*}{\partial t} = a^2\frac{\partial^2 u^*}{\partial \eta^2} \qquad \frac{\partial T}{\partial t} = \frac{1}{P_r\rho_\infty{}^2}\frac{\partial^2 T}{\partial \eta^2}$$

Both equations are standard heat conduction equation. They are easy to solve. Then, for transient convective flow along a vertical plate, we can get analytic solution for various boundary condition without using the Boussinesq approximation. With regard to the case of horizontal plate, when Pr number is of order of unity, the fluid dynamic boundary layer is approximately equal to thermodynamic boundary layer, as indicated by pioneering researcher. Therefore, the Boundary-Layer approximation is valid for such situation, then, our method can be applied to this case, too.

3 Conclusion

The foregoing results are obtained by using a new assumption instead of the Boussinesq approximation. The method is meaningful when we want to deal with the case in which large temperature difference exists in the flow field. Meanwhile, we retained the viscous dissipation in the the energy equation, so that our results will permit its calculation even for process in which dissipation is dominant. One thing should be mentioned is that for quiescent surrounding, our method is also applicable, however, it needs some special treatment.

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