INTRODUCTION OF HYDRODYNAMIC BOUNDARY PERTURBATION METHOD - ITS APPLICATION ON SMALL AMPLITUDE WAVE THEORY

Hiroyoshi Shi-igai, AIT, TIT

## INTRODUCTION

There are many studies where many scholars tried to obtain the wave profile for a variable bottom profile. Some of them are listed on Bibliography.

The method presented here is rather general for many hydrodynamic problems; however, its application to wave problem seems to be very successful.

The situation which is generally considered is a governing equation such that

$$\nabla^2 \Phi(x,y) = 0 \tag{1}$$

or

$$\nabla^2 \psi(x,y) = 0 \tag{2}$$

where $\Phi$ is the velocity potential and $\psi$ is the stream function in two dimensional motion.

The Eqs. (1) and (2) mean that the fluid is ideal and incompressible and the motion is two dimensional and irrotational. There are neither real sinks nor sources in the flow field.

Suppose $\Phi$ or $\psi$ satisfies the following boundary condition

$$\Phi_y(x,b) = 0 \tag{3}$$

or

$$\psi(x,b) = 0 \tag{4}$$

where b must be constant. The problem here is to determine the solution if b is a weak function of x, say, $b_o + \varepsilon f(x)$, where $\varepsilon$ is a small parameter.

The standard procedure to obtain the solution is by assuming the solution in the power series form such that

$$\Phi = \Phi_o + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \varepsilon^3\Phi_3 + \ldots.. \tag{5}$$

and determine $\Phi_i$ step by step, where $\Phi_i$ must be exactly harmonic.

However, the general weak points of this method are:

1. Convergence may not be well verified.

2. The result is usually not easy to interpret, since it does appear in a series form.

For $\psi$, the similar method can be applied.

In order to avoid the above difficulty, the author proposes a new method which may be called a "Boundary Perturbation Solution", where the boundary condition is satisfied, while the governing equation is almost satisfied. The advantage of this method will be clear but, in short, the solution can easily be found if $\Phi$ or $\psi$ is given for the undisturbed boundary.

## BOUNDARY PERTURBATION SOLUTION

In order to avoid these difficulties, the boundary perturbation method is developed. We consider the function $F(x,y;\epsilon)$ such that

$$F(x,y) = \Phi(x,y-\epsilon f(x)) \tag{6}$$

Then $F_y = \Phi_y$ and

$$F_y|_{b_o+\epsilon f(x)} = \Phi_y(x,b_o) \tag{7}$$

We can choose a harmonic function $\Phi(x,y)$ which satisfies the following boundary condition.

$$\Phi_y(x,b_o) = 0 \tag{8}$$

F would then satisfy the boundary condition that $F_y|_{b_o+\epsilon f(x)} = 0$. The problem is whether or not F is harmonic.

Let

$$x = x_1, \qquad \xi = y - \epsilon f(x) \tag{9}$$

Then

$$F_y = F_\xi, \qquad F_x = \frac{\partial F}{\partial x_1}\frac{\partial x_1}{\partial x} + \frac{\partial F}{\partial \xi}\frac{\partial \xi}{\partial x}$$

$$= \frac{\partial F}{\partial x_1} - \epsilon f' \frac{\partial F}{\partial \xi} \tag{10}$$

and

$$\nabla^2 F = -\epsilon f'' \frac{\partial F}{\partial y} + \epsilon^2 f' \frac{\partial^2 F}{\partial y^2} - 2\epsilon f' \frac{\partial^2 F}{\partial x \partial y} \tag{11}$$

Eq. (11) implies that

$$\nabla^2 F = O(\epsilon,\epsilon^2) \tag{12}$$

Similarly,

$$\nabla^2 G = -\epsilon f'' G_y + \epsilon^2 f' G_{yy} - 2\epsilon f' \frac{\partial^2 G}{\partial x \partial y} \tag{13}$$

## FINITE ELEMENT CONDITION

It will be preferable if F or G satisfies the following finite element condition.

$$\int_s \nabla^2 F \, dS = 0 \tag{14}$$

$$\int_s \nabla^2 G \, dS = 0 \tag{15}$$

where s denotes a certain finite element.

The physical interpretation of Eq. (15) is that though there is a weak sink or source distribution in the flow field, the total discharge in a certain finite element is zero. Similarly, Eq. (16) implies that there is a weak vorticity distribution in the flow field, which may be cancelled in a finite element. In this case the motion is somewhat rotational. Note that these conditions are not essential, though preferable.

From the physical point of view, the use of $\psi$ looks more reasonable, since it is rather difficult to imagine the distribution of sinks and sources in the flow field, while a weak vorticity distribution is quite acceptable.

In case of small amplitude wave motion, we can obtain the surface profile as well as wave velocity from F or G, where the boundary condition means the bottom condition in this paper. The surface is represented as y=0 while $y = -h + \epsilon f(x)$ describes the bottom boundary.

As $v = F_y = \Phi_y$

$$\eta = \int F_y|_{y=0} dt = \int \Phi_y(x,-\epsilon f(x))dt \qquad (16)$$

The integration can be carried out for many cases without much difficulty. Similarly from G, we can obtain $\eta$,

$$\eta = \int \psi_x(x,-\epsilon f(x))dt \qquad (17)$$

The two results are slightly different from each other, since the assumptions are different as was mentioned before.

For very small $\epsilon$, $\Phi_y(x,-\epsilon f(x)) = \Phi_y(x,o) - \epsilon f(x)\Phi_{yy}(x,o)$

$$\eta = \int\Phi_y(x,o)dt - \epsilon f(x) \int\Phi_{yy}(x,o)dt$$

or

$$\eta = \eta_o - \epsilon f(x) \int\Phi_{yy}(x,o)dt \qquad (18)$$

where $\eta_o$ is the wave profile for the undisturbed boundary.

Similarly, for $\psi$ we obtain

$$\eta = \eta_o - \epsilon f(x) \int\psi_{xy}(x,o)dt \qquad (19)$$

Therefore, the results are exactly the same for the first order of $\epsilon$, since $\Phi_{yy} = \psi_{xy}$.

For wave velocity we have to use the dispersion relation as follows.

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \quad \text{on the surface, or at } y = 0. \qquad (20)$$

On the surface, therefore,

$$F_{xt} = -g\frac{\partial}{\partial x}\int\Phi_y(x,-\epsilon f)dt \qquad (21)$$

while

$$\frac{\partial^2\Phi(x,-\epsilon f)}{\partial x\partial y} = \frac{\partial}{\partial t}[\Phi_x - \epsilon f'\Phi_y] = \Phi_{xt} - \epsilon f'\Phi_{yt} = -g\int[\Phi_{yx} - \epsilon f'\Phi_{yy}]dt \qquad (22)$$

Again, the difference in the result is order of $\epsilon$. If we use $\psi$, the result is, in a similar manner, as follows:

$$\frac{\partial^2 G}{\partial y\partial t} = -g\frac{\partial}{\partial x}\int\psi_x(x,-\epsilon f)dt \quad \text{on the surface,} \qquad (23)$$

or

$$\psi_{yt} + g \int \psi_{xx} dt - g\varepsilon f' \int \psi_{xy} dt = 0 \quad \text{on the surface.} \tag{24}$$

Note that in the above expression $\Phi$ and $\psi$, also their derivatives, are $\Phi(x,\eta-\varepsilon f)$ and $\psi(x,\eta-\varepsilon f)$; however, for small amplitude waves and small changes in bottom configuration, we can always use $\Phi(x,o)$ and $\psi(x,o)$. If we apply this method, for instance, for the case where the small amplitude waves are passing over a wavy bottom expressed by $y = -h(1-\varepsilon \sin mx)$, the wave profile and velocity are given by

$$\eta = a \cos\chi - \frac{a\varepsilon mh}{2} \coth kh[\sin\{(k+m)x-wt\} + \sin\{(k-m)x-wt\}] \tag{25}$$

$$C^2 = \frac{g}{k} \tanh kh - \frac{ghm^2}{k^2} \sin mx \qquad (m \gg k) \tag{26}$$

## CONCLUSION

The boundary perturbation method is proposed and the results are very interesting. The several applications one of which is presented here show that the method is useful for small amplitude wave theory. The author cannot see a serious difficulty in the future application of the method for the higher order wave theory, because of the generality of this method.

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## LIST OF SYMBOLS

| | | | |
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| $\Phi$ | velocity potential | $\psi$ | stream function |
| F | modified velocity potential | $\chi$ | Kx-wt |
| K | wave number | w | frequency (angular) |
| h | water depth (constant) | m | wave number of wavy bottom |

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