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MATHEMATICAL MODELING OF MULTI-DIMENSIONAL OPEN CHANNEL FLOW

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INTRODUCTION

The flow in an open channel, natural or man-made, prismatic or non-prismatic, is three-dimensional. The flow component in the longitudinal direction is called "primary flow" while the other two components combine to form "secondary flow" in transverse cross sections of the channel. Cross sections of an alluvial channel are irregular and irregularly varying along the flow. Hydraulic variables, such as the slopes of channel, water surface and energy line, shear stress (including the boundary shear), velocities of primary and secondary flows, and sediment concentration, etc. all vary three-dimensionally. These hydraulic variables and the channel cross section interact each other and are, or in the process of, maintaining an equilibrium state. A change in one will, therefore, affects all the others.

Hydraulics of alluvial channel should be based on such concept and actually deal with the complex reality. The materials presented herein are based on results of a series of research conducted by the author and his students along this line.

Hydraulic analysis throughout this paper is presented in the framework of a curvilinear coordinate system consisting of isovels (curves along which the velocity is equal) of primary flow, marked as ξ-curves in Fig.1, and their orthogonal trajectories marked as η-curves.

DISTRIBUTION OF PRIMARY FLOW VELOCITY

In the $\xi-\eta$ coordinate system the distribution of primary flow velocity can be represented quite well by the following logarithmic equation(Ref.2,3,4):

$$u(x, y, z) = u(x, \xi) = \frac{\bar{u}_*}{\kappa} \ln \frac{\xi}{\xi_0} \quad \text{.................... (1)}$$

in which a suitable equation for the ξ-coordinate, representing a family of primary flow isovels, is

$$\xi = \left(\frac{y}{D}\right) \{(1 - \frac{|z|}{B_i}) \; e^{|z|/B_i}\}^{\beta_i} \quad \text{.................. (2)}$$

or

$$\xi = \left(\frac{y}{D}\right) [(1 - \frac{|z|}{B_i})]^{\beta_i} \quad \text{...................... (3)}$$

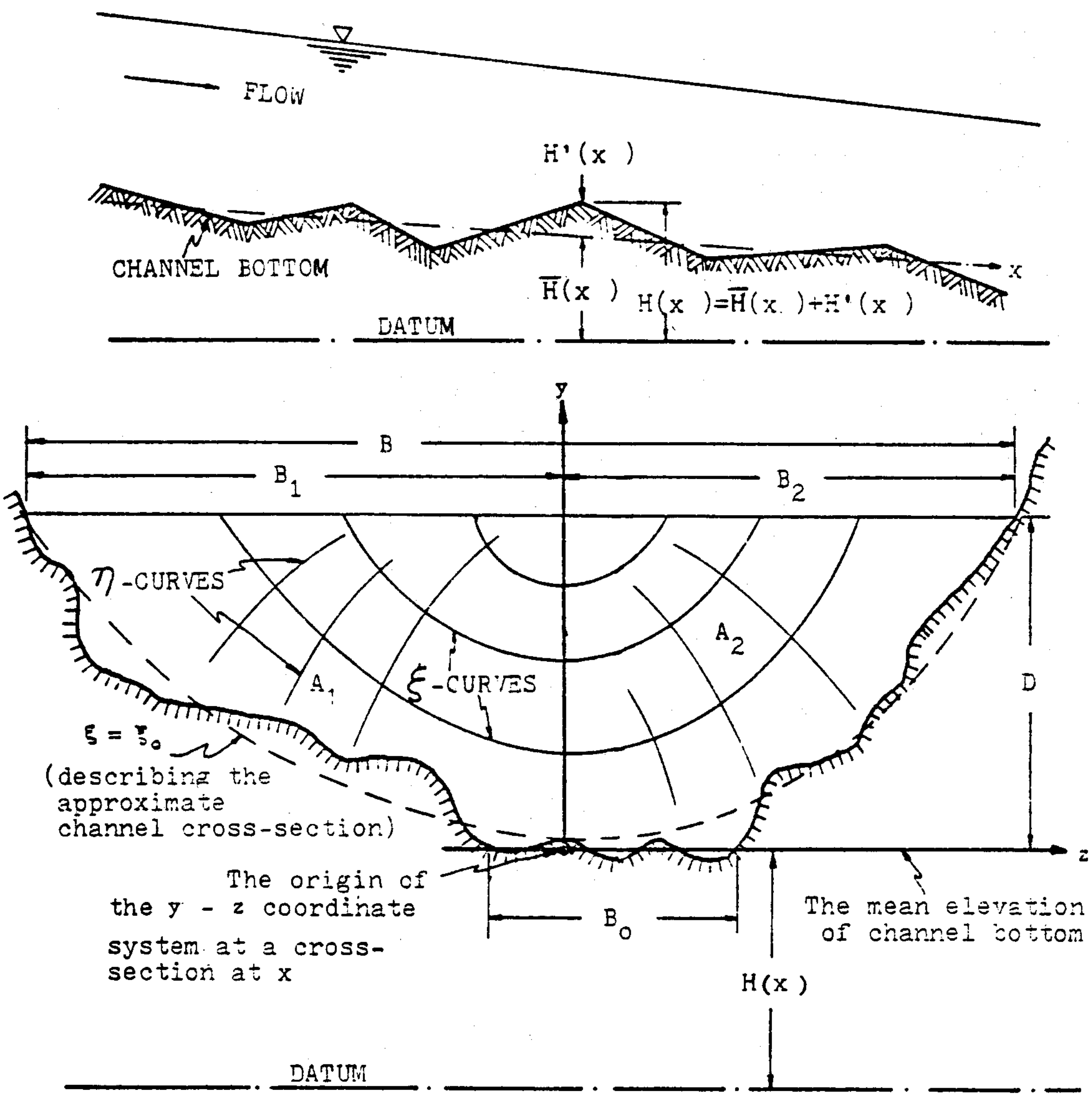

Figure 1. The $x - \xi - \eta$ Coordinate System

In Eqs.1 through 3, u = the point mean velocity of primary flow (in the x-direction); $\bar{u}_*$ = the mean shear velocity, evaluated as $\sqrt{gRS}$, at a cross section, in which g = the gravitational acceleration, R = the hydraulic radius, and S = the bed or channel slope; D = the water depth at the y-axis; B_i for i equal to either 1 or 2 = the transverse distance on the water surface between the y-axis and either the left or right bank of a channel cross section; z = the coordinate in the transverse direction; y = the coordinate in the vertical direction (The y-axis is selected such that it passes through the point of maximum primary flow velocity); ξ_0 and β_i = empirical coefficients which can be determined according to the variance reducing technique from the linear relation between u and ξ on a semi-log plot. Equation 1 has the same form as the Prandtl-von Karman universal velocity distribution equation. The value of κ, actually evaluated with observed data on u, also tends to be equal to about 0.4; thus, if no data are available, 0.4 can be used as a good approximation of κ.

The difference between Eqs. 2 and 3 is the exponential function, $e^{|z|/B_i}$, which preserves the continuity on the y-axis of ξ - curves on the two sides of y-axis, by making the slope of ξ - curves continuous on the y-axis. Figure 4(a) shows measured primary flow velocities along with isovels (ξ - curves) simulated by Eq.2. Without the exponential function (i.e.replacing it by one), Eq.3 still can give simulated isovels which are almost equally accurate (on the basis of error variance) as those given by Eq.2. Equation 3 has the advantage of being simple and, hence, convenient in deriving many useful equations: Using Eq.3, one can show that

$$r_i = \frac{A_i}{B_i} = [\, 1 - \frac{1}{1 - \beta_i} (\xi_0 - \beta_i\, \xi_0^{\,1/\beta_i}\,)] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

for i = 1 and 2; and

$$Q = BD\frac{\bar{u}_*}{\kappa} [\, - 1 - \ln\xi_0 + \xi_0 - \frac{1}{B}(B_1\, \beta_1\, r_1 + B_2\beta_2\, r_2)] \quad \ldots\ldots\ldots(5)$$

In Eqs. 4 and 5 , A_i = the area of a channel cross section on each side of y-axis; Q = the volume rate of flow through an entire channel cross section; and $B = B_1+B_2$ or the total width of channel cross section on the water surface. With known (measured) values of D, Q, $\bar{u}_*$, κ(0.4), B_i and A_i for i = 1 and 2, Eqs.4 (for i = 1 and 2) and 5, containing three equations, can be solved simultaneously to determine the three unknowns, ξ_0, β_1, and β_2 for use in Eq.2 or 3 and, hence, Eq.1, if measured data of u are not available. There are other alternative methods available for evaluating the three velocity distribution parameters, without using actual data of u (2). It should be stated, however, that the best way is still to use the measured data of u to determine κ, ξ_0, β_1 and β_2 treating κ to be just another coefficient, without assuming it to be 0.4.

Orthogonal trajectories of ξ - curves (isovels of primary flow)can be obtained from Eq.2 as

$$\eta = y^2 + \frac{2B_i^2}{\beta_i} (\ln|z| - \frac{|z|}{B_i}) \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

or from Eq.3 as

$$\eta = y^2 - \frac{1}{\beta_i} (B_i - |z|)^2 \quad \ldots\ldots\ldots\ldots (7)$$

Pairing of Eqs.2 and 6, or Eqs.3 and 7, forms a curvilinear $\xi - \eta$ coordinate system.

CHANNEL CROSS SECTION

A cross section of an open channel can be approximated by the isovel, $\xi = \xi_0$, which is a curve described by

$$y = D\,\xi_0 [(1 - \frac{|z|}{B_i})\, e^{|z|/B_i}]^{-\beta_i} \quad \ldots\ldots\ldots\ldots (8)$$

or

$$y = D\,\xi_0 (1 - \frac{|z|}{B_i})^{-\beta_i} \quad \ldots\ldots\ldots\ldots (9)$$

along which u = 0. Equation 8 is based on Eq.2; and Eq.9 is from Eq.3.

The ratio, $r_i = A_i/(B_i D)$, given by Eq.4 can be used as a measure of geometrical shape of channel cross section on each side of y-axis. $r_i = 1$ means the rectangular shape; and $r_i = 1/2$ means the triangular shape. The cross sectional shape of a natural channel tends to be such that the value of r_i falls between 1/2 and 1. Use of a set of measured data on u in a rectangular channel can give ξ_0-curve which almost exactly coincides with the actual, rectangular cross section. Contrarily, if the rectangular cross section (with width, depth, discharge rate and slope given) are known, and the three coefficients, ξ_0, β_1 and β_2 are calculated without data on u, Eq.2 can give isovels of u quite accurately. Such a reversibility is also quite good if Eq. 8 or 9 is used for alluvial channels.

MEAN FLOW VELOCITY AND RESISTANCE COEFFICIENT

If Eq.5 is divided by the cross-sectional area, $A = A_1 + A_2 = rBD$, the mean flow velocity through the cross section can be expressed as

$$\bar{U} = \frac{\bar{u}_*}{\kappa r} [-1 - \ln \xi_0 + \xi_0 - \frac{1}{B} (B_1\beta_1 r_1 + B_2\beta_2 r_2)] \quad \ldots\ldots\ldots (10)$$

which can be used in lieu of such popular formulas as the Manning's. Some of the advantages of Eq.10 are that it relates the mean velocity to the distribution of point velocity, u, and the cross sectional geometry (as represented by r_1, r_2, B_1 and B_2).

A comparison of Eq.10 with the Manning's formula gives the Manning's resistance coefficient as

$$n = \frac{\kappa\, r\, R^{1/6}}{\sqrt{g}\,[\,-1 - \ln \xi_0 + \xi_0 - \frac{1}{B}(B_1\beta_1 r_1 + B_2\beta_2 r_2)\,]} \quad \ldots\ldots\ldots\ldots(11)$$

Equation 11 offers a rational way of evaluating the Manning's resistance coefficient, which is capable of including the effect of channel geometry (including the degree of asymmetry) and the interaction with the distribution of primary flow velocity(through ξ_0, $\beta_1$and β_2). Since the primary flow velocity distribution is related to such processes as secondary flow, dispersion and diffusion, and distribution of shear stress(including boundary shear), the Manning's "n" given by Eq.11 can be considered also related to or interacting with these processes.

SECONDARY FLOW

The momentum equation in the x-direction (longitudinal direction) in the x-ξ- η coordinate system directly gives the ξ-component of flow (i.e.the flow component perpendicular to the isovels of primary flow) as:

$$V_\xi = \left(\frac{\rho}{h_\xi}\frac{\partial u}{\partial \xi}\right)^{-1} \left\{ -\rho\frac{\partial u}{\partial t} - \rho u\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}(p + \rho g H) + \frac{\partial \sigma_x}{\partial x} + \frac{1}{h_\xi}\frac{\partial \tau_{\xi x}}{\partial \xi} + \frac{1}{h_\xi h_\eta}\frac{\partial h_\eta}{\partial \xi}\tau_{\xi x} \right\} \quad \ldots\ldots\ldots\ldots(12)$$

while the continuity equation gives the η-component (the flow component tangent to the isovels of primary flow):

$$V_\eta = V_{\eta *} - \frac{1}{h_\xi}\int_{\eta *}^{\eta} \left\{ h_\xi h_\eta \frac{\partial u}{\partial x} + \frac{\partial}{\partial \xi}(h_\eta V_\xi) \right\} d\eta \quad \ldots\ldots\ldots\ldots(13)$$

In Eqs.12 and 13, V_ξ and V_η = the ξ and η components of secondary flow velocity; h_ξ and h_η = the "scale factors" or the "metric coefficients" on the ξ and η curves; ρ = the fluid density; t = time; g = the gravitational acceleration; H = the mean elevation of the bottom of a transverse cross section of open channel; p = the hydrostatic pressure per unit area; $V_{\eta *}$ = the value of V_η at a boundary point where $\eta = \eta^*$; $\tau_{\xi x}$ is the shear stress in the x-direction in the plane perpendicular to the ξ -direction; and σ_x= the normal stress in the x-direction.

The ξ - and η -components of secondary flow velocity given by Eqs.12 and 13 are related to the y- and z- components in the (Cartesian) y-z system according to the following standard transformation rule:

$$V_\xi = h_\xi \frac{\partial \xi}{\partial y} v + h_\xi \frac{\partial \xi}{\partial z} w \quad \text{..................(14)}$$

$$V_\eta = h_\eta \frac{\partial \eta}{\partial y} v + h_\eta \frac{\partial \eta}{\partial z} w \quad \text{....................(15)}$$

To compute secondary currents, Eq.12 can be used first at every grid point of the $\xi - \eta$ coordinate network to obtain V_ξ. The other component, V_η, can then be obtained by the integral equation, Eq.13, along each η-curve starting from a boundary point. The boundary point can be selected on the water surface where the y-component of flow, v, is equal to zero. This boundary condition, v = 0, along with V_ξ from Eq.12 can be substituted into Eqs.14 and 15 to determine V_η and the z-component, w. With Eqs.12 and 13, along with these boundary conditions, the secondary flow velocities, V_ξ and V_η, can be computed at every grid point in the $\xi - \eta$ coordinate system, which can then be transformed into v and w in the Cartesian coordinate system By Eqs.14 and 15.

The scale factors on the ξ and η curves can be derived from Eqs.2 and 6 as

$$h_\xi = \frac{B_i^2 \; D}{\left(1 - \frac{|z|}{B_i}\right)^{\beta_i - 1} e^{\beta_i \frac{|z|}{B_i}} \sqrt{B_i^4 \left(1 - \frac{|z|}{B_i}\right)^2 + (\beta_i \, y \, |z|)^2}} \quad \text{...(16)}$$

$$h_\eta = \frac{\beta_i \; |z|}{2 \sqrt{B_i^4 \left(1 - \frac{|z|}{B_i}\right)^2 + (\beta_i \, y \, |z|)^2}} \quad \text{..............(17)}$$

If Eqs. 3 and 7 are used as $\xi - \eta$ coordinates,

$$h_\xi = \frac{B_i \; D}{\left(1 - \frac{|z|}{B_i}\right)^{\beta_i - 1} \sqrt{(B_i - |z|)^2 + \beta_i^2 y^2}} \quad \text{....................(18)}$$

$$h_\eta = \frac{\frac{1}{2} \; \beta_i}{\sqrt{\beta_i^2 y^2 + (B_i - |z|)^2}} \quad \text{..................................(19)}$$

SHEAR DISTRIBUTION

As indicated by Eq.12, the secondary flow interacts with the shear distribution. By definition of ξ -coordinate, $\tau_{x\eta}$ should be zero as the gradient of primary flow velocity in the η -direction (tangent to a ξ -curve) is zero; then, $\tau_{\xi x}$ should represent the total shear at a point in a ξ -η plane(y-z plane, or a transverse cross section of a channel). If the wind effect is neglected, the shear stress, $\tau_{\xi x}$ or $\tau_{x\xi}$ on the water surface is zero. On the channel bed (along which $\xi = \xi_0$) the secondary flow velocity, V_ξ , which is perpendicular to the channel bed, is zero; then, Eq.12 gives another boundary condition,

$$\frac{\partial \tau_{\xi x}}{\partial \xi} + \frac{1}{h_\eta} \frac{\partial h_\eta}{\partial \xi} \tau_{\xi x} = -\rho g S h_\xi \quad \text{..............(20)}$$

An equation for $\tau_{\xi x}$ that satisfies these boundary conditions on the water surface and the channel bed, under a steady, uniform flow, is

$$\frac{\tau_{\xi x}(\xi, \eta)}{\bar{\tau}_0} = \alpha(\eta) \left[\xi_D(\eta) - \xi \right]^2 + \frac{D}{R} \left[\xi_D(\eta) - \xi \right] \quad \text{.......(21)}$$

in which

$$\alpha(\eta) = \frac{h_\xi}{R\,[\xi_D(\eta) - \xi_0]} \; \frac{1 - \dfrac{D}{h_\xi}\left\{1 - \dfrac{1}{h_\eta}\dfrac{\partial h_\eta}{\partial \xi}\left[\xi_D(\eta) - \xi_0\right]\right\}}{2 - \dfrac{1}{h_\eta}\dfrac{\partial h_\eta}{\partial \xi}\left[\xi_D(\eta) - \xi_0\right]} \quad \text{...(22)}$$

In Eqs.21 and 22, $\xi_D(\eta)$= the value of ξ on the water surface along an η -curve (i.e. on each η -curve the value of ξ varies from ξ_0 to $\xi_D(\eta)$), which can be calculated once the primary flow isovels and, hence, the ξ-η coordinates are determined. $\bar{\tau}_0$ = the mean boundary shear at a section, which can be evaluated as ρgRS . In Eq.22 all quantities, except $\xi_D(\eta)$ are evaluated at (ξ_0, η) on the channel bed.

For a "wide channel", without effects of the sides, $h_\xi = D$, $\partial h_\eta/\partial\xi = 0$, $R \to D$, $\xi_D(\eta) = 1$, $\xi = y/D$, and $\alpha(\eta) = 0$, so that

$$\frac{\tau_{\xi x}(\xi,\eta)}{\bar{\tau}_0} = 1 - \frac{y}{D} \quad \text{............................(23)}$$

which is widely used as a shear distribution equation in open channel hydraulics where the effect of two sides and secondary currents are neglected. Equation 21 includes the interaction of shear stress distribution with secondary flow. With $\xi = \xi_0$, Eq.21 gives the boundary shear (the shear stress on the channel bed).

INTERACTIONS

In the preceding mathematical models of primary flow velocity distribution, channel cross section, flow resistance, secondary flow, and shear distribution, common building blocks have been used that facilitate investigating and gaining insights into interactions among them. It is not difficult to see from these models that a change in one of them will affect the others.

Figures 2 and 3 show:(a)the distribution of primary flow velocity, (b)the distribution of boundary shear on the channel bottom, computed by Eq.21 with $\xi = \xi_0$, (c)directions of secondary flow, and (d)the magnitude of secondary flow velocity (the vector sum of v and w), computed for a section of Rio Grande Conveyance Channel, near Bernardo, New Mexico (5) with Q =1280 cfs, and for a section of a straight, rectangular flume (1) with Q = 16.88 cfs. These Figs. indicate that the maximum boundary shear is from about 2.5 to 3.0 times $\bar{\tau}_0$ (the mean boundary shear), and tends to occur near the edges of channel bottom, near the corners of channel cross section above which relatively strong secondary currents prevail.

Figure 2 further indicates a similarity between the boundary shear distribution and the shape of channel cross section (See the measured channel bed indicated by the dotted line in Fig.2(a). The similarity explains the tendency of scour and erosion to occur near the edges of channel bottom and an eventual fall or erosion of channel banks. The interaction of secondary currents, boundary shear and channel cross section is unmistakable. If the rectangular channel shown in Fig. had loose boundary, the channel would be eroded and the cross section would become similar to the measured channel cross section of alluvial channel shown in Fig. . Simulation of such an interaction, using the mathematical models presented herein, enables gaining insights into unobservable outcomes under a range of alternative designs and controls for channel stability. It should serve as a decision-making tool in selection of design and control for management of streams and rivers.

SUMMARY AND CONCLUSION

A unified approach to mathematical modeling of open channel flow has been presented which provides explicit links among the interacting variables such as the distribution of primary flow velocity, channel cross section, discharge rate, water depth, mean flow velocity, resistance coefficient, secondary flow, and shear stress distribution. The approach should be especially useful in study of flow and other transport processes in alluvial channels.

ACKNOWLEDGMENTS

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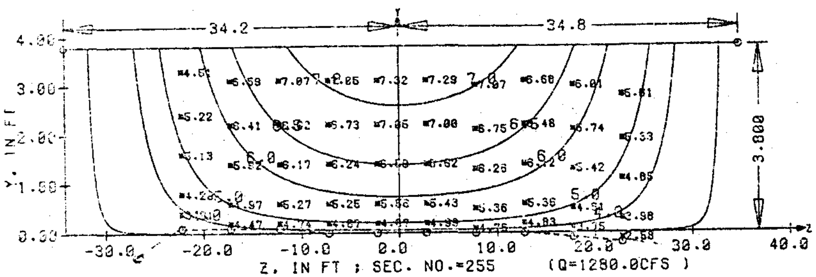
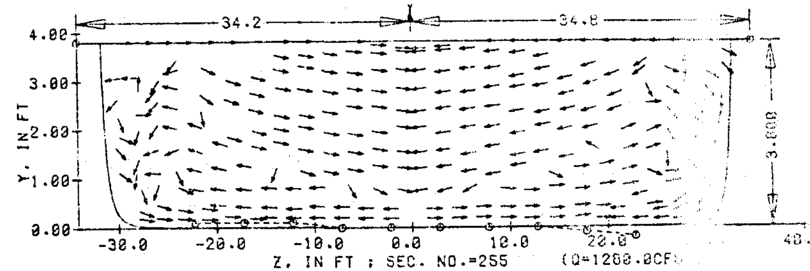
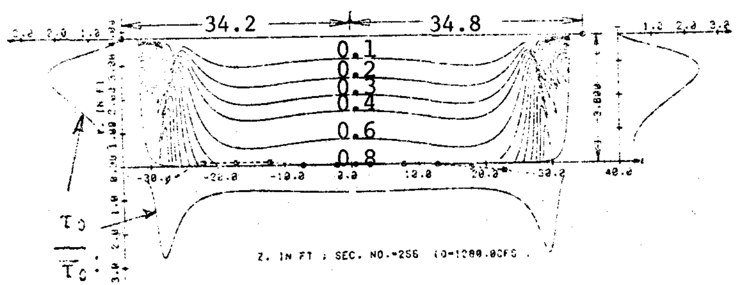


(a)Distribution of primary flow velocity(fps)

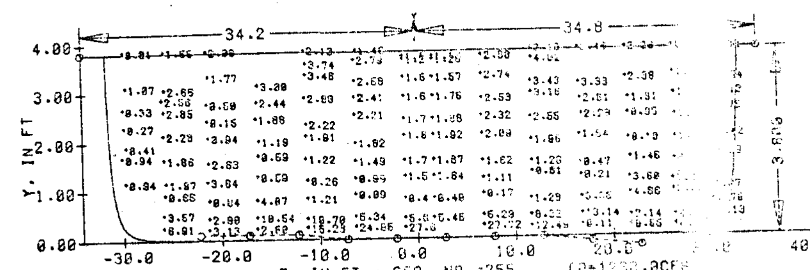


(c) Direction of secondary currents

Note:In (a),(c) and (d):---O---=Measured channel bed(Ref.7)
———— =Approximated channel bed ($\xi = \xi_0$)

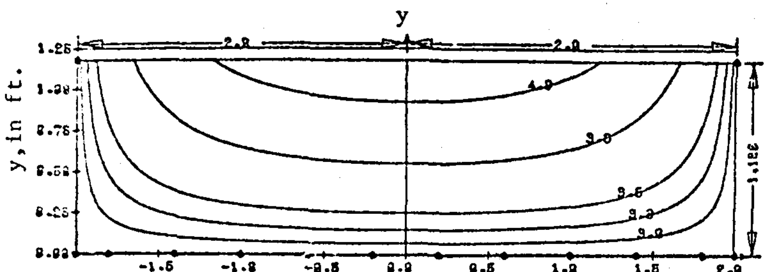


(b) Distribution of shear in the flow and along the channed boundary,calculated by Eq.21.

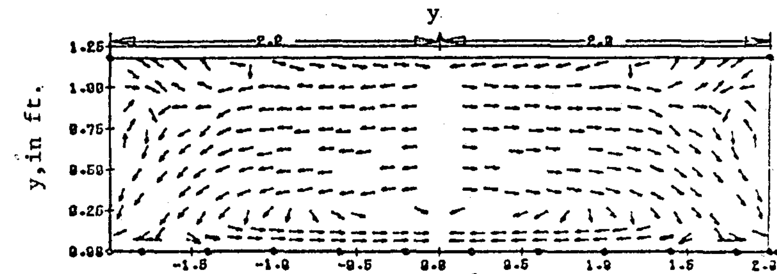


(d)Magnitude of secondary flow velocity, in percentage of mean primary flow velocity(5.2 fps)

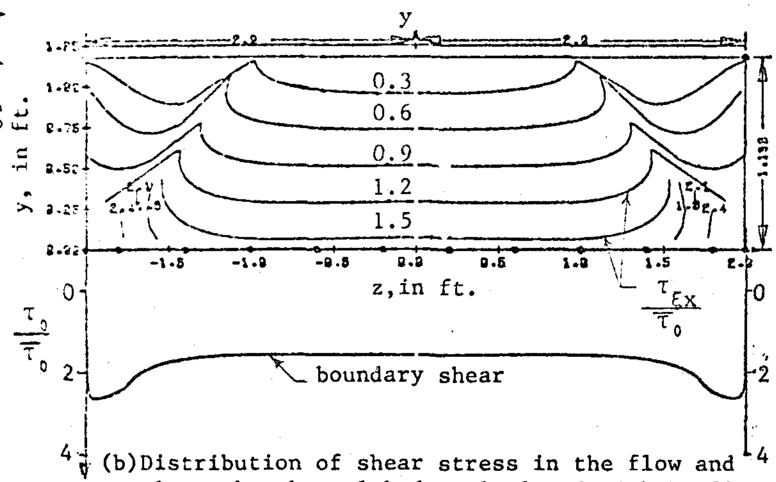
Figure 2. Simulation of Flow and Boundary shear in Rio Grande Conveyance Channel (Sec.#255 ,Ref.7) near Bernardo, N.M., Q = 1280 cfs



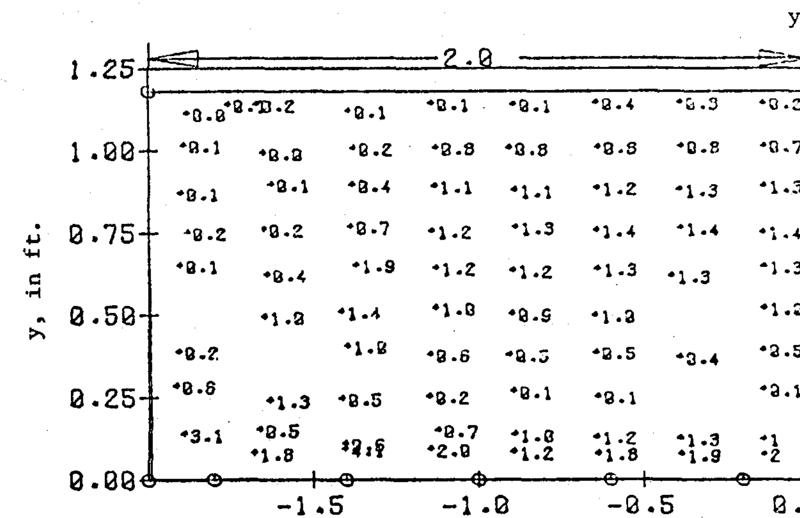
(a)Distribution of primary flow velocity(fps)



(c)Direction of secondary currents



(b)Distribution of shear stress in the flow and along the channel bed, calculated with Eq.21



(d)Magnitude of secondary flow velocity, in percentage of mean primary flow velocity(3.58fps)

Figure 3.Simulation of Flow and Shear Stress in a Rectangular Flume (Ref.2), Q = 16.88cfs.

APPENDIX I.- REFERENCES

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APPENDIX II.- NOTATION

The following symbols are used in this paper:

A = cross-sectional area;
A_i = cross-sectional area on one side of y-axis;
B = top width of cross section;
B_i = top width of cross section on one side of y-axis;
D = water depth along y-axis at cross section;
g = gravitational acceleration;
H = mean elevation of channel bottom at x;
h_ξ, h_η = scale factors in coordinate transformation;
i = index, denoting either left or right side of y-axis;
n = Manning's resistance coefficient;
p = pressure;
Q = volume rate of flow
r = a geometrical factor, A/BD;
S = bed or channel slope;
S_e = energy slope;
S_w = water surface slope;
t = time;
u = x-component of mean flow velocity at a point in cross section;
$\bar{U}$ = mean velocity of flow at a cross section;
u_* = mean shear velocity at a cross section;
v, w = y and z components of secondary flow velocity, respectively;
V_ξ, V_η = ξ and η components of secondary flow velocity, respectively;
x,y,z = Cartesian coordinates along longitudinal, vertical, and transverse directions of channel flow, respectively;

α = a coefficient in shear distribution formula;
β_i = constant in velocity distribution equation;
κ = constant in logarithmic velocity distribution equation;
ξ, η = curvilinear coordinates based on isovels of primary flow;
ξ_0 = value of ξ for which u = 0;
ξ_D = value of ξ on along an η -curve on the water surface;
ρ = density of water;
$\tau_{\xi x}$ = shear stress in the x-direction in the plane perpendicular to ξ - direction;
$\bar{\tau}_0$ = the mean boundary shear at a cross section; and
τ_0 = boundary shear at a point on the channel bed at a cross section.