ON THE MECHANICAL DESTABILIZATION OF A THREE-DIMENSIONAL DISPLACEMENT-SOFTENING PLANE OF WEAKNESS

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In previous studies, we investigated rupture instability on a two-dimensional plane of weakness that follows a nonlinear displacement-softening relation. We showed that if the strength weakens linearly with slip (displacement gap) on the plane, the critical length of the rupture region at instability is independent of any length scales entering into the description of the shape of the loading stress distribution. Here, By employing an energy approach, we study the stability of a three-dimensional plane following the linear displacement-softening constitutive law. The results, again independent of the shape of the loading stress distribution, are comparable with the two-dimensional ones, showing that the critical lengths are of the same order for both two- and three-dimensional cases.

Key Words : *rupture nucleation process, earthquake source mechanics, three-dimensional interface rupture, slip-weakening, rockburst*

1. Introduction

In order to explain the physical process of source nucleation, rupture propagation and wave radiation associated with earthquakes and rockbursts, a large number of laboratory experiments as well as theoretical and numerical models have been designed[1-17], and seismological observations utilizing the global seismic network and those performed in mines seem to support the idea that shallow earthquakes and rockbursts are caused by instabilities of planes of weakness (e.g., geological faults, joints): Understanding the transition from the quasi-static source nucleation to dynamic rupture propagation is believed to play a crucial role in understanding the mechanics of seismic events[1-5].

In earlier studies[6, 7], we investigated the behavior of a displacement-softening plane of weakness (interface) and evaluated the nucleation length that is relevant to interface instabilities and ensuing dynamic rupture. We considered two-dimensional interface rupture in an infinite, homogeneous elastic space subjected to a locally peaked loading stress (Fig.1),

$$\sigma_o = \sigma_p + Rt - q(x). \qquad (1)$$

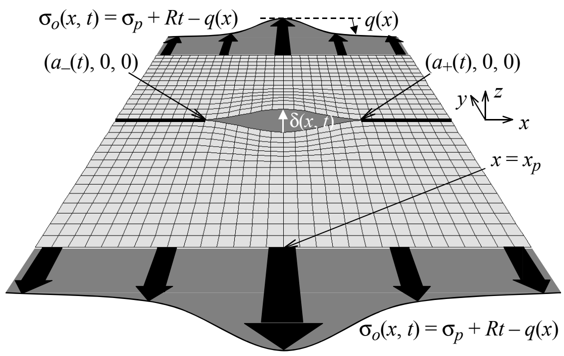

Fig.1 A displacement field associated with tensile rupture (mode I) rupture in an infinite, homogeneous, linear elastic space. The loading tensile stress $\sigma_o(x, t)$ is locally peaked in space and varies gradually with time, at rate R. Similarly, we can define the 2D problem for in- and anti-plane shear loading (modes II and III).

Here, the interface coincides with the x-z plane ($y = 0$) of a Cartesian coordinate system xyz, σ_p is the tensile (for mode I) or shear (for modes II and III) strength of the interface, and R (> 0 if the stress increases with time t) is the loading rate of the increasing stress (e.g., tectonic loading). The function $q(x)$ satisfies $q(x) > 0$ for $x \neq x_p$ and $q(x_p) = 0$. Thus $t = 0$ is the time when the peak value of loading stress, at x_p, first reaches σ_p so that rupture initiates at that point.

In the studies, as a constitutive law inside the rupture region, the displacement-softening law

$$\sigma = \sigma_p - W\delta^n, \quad (2)$$

is used where W and n are constants (W, $n > 0$). Figure 2 shows schematically this constitutive law (2). In the figure, the vertical axis denotes the strength σ while the horizontal one corresponds to the slip (displacement gap) δ. Slip, defined as $\delta(x, t) = u_y(x, 0^+, t) - u_y(x, 0^-, t)$ for mode I, $u_x(x, 0^+, t) - u_x(x, 0^-, t)$ for mode II, and $u_z(x, 0^+, t) - u_z(x, 0^-, t)$ for mode III, can occur if the local stress reaches the peak strength σ_p. The stress inside the rupture region of the interface (denoted by $\sigma(x, t)$ and coincides with $\sigma_y(x, 0, t)$ for mode I, $\sigma_{xy}(x, 0, t)$ for mode II, and $\sigma_{yz}(x, 0, t)$ for mode III) drops according to the relation (2). This nonlinear power-law relation has been suggested in the study of laboratory characterization of softening at (dynamic) slips from the mm to m range in dense sand[8] and quartz rocks[9] as well as in the investigation of seismological scaling observations for radiated energy and stress drop versus slip (slips 1-500 mm)[10].

For the linear case (n = 1)[6], at $t > 0$, part of the interface opens (slips) and the stress in the rupture region decreases with the displacement-softening law (2), and at a later stage a critical nucleation length h_n is reached at which no further quasi-static solution exists for additional increase of the loading. That marks the onset of a dynamically controlled instability, and it has been shown, from the quasi-static mechanical equilibrium conditions, that the critical length h_n is independent of the shape of the loading stress distribution $q(x)$, and its universal value is given by the solution to an eigenvalue problem by

$$h_n \approx 1.158\, \mu^*/W, \quad (3)$$

if we consider sufficiently small slip δ so that the displacement-softening law (2) applies at least approximately for the range of slips which occur prior to instability. Here, $\mu^* = \mu$ (shear modulus) for mode III and $\mu/(1-\nu)$ for modes I and II, with ν being Poisson's ratio.

In the case of nonlinear power-law displacement-softening interface ($n \neq 1$)[7], there is no longer a universal critical length that is independent of the curvature of the loading stress distribution, and qualitative features of the slip development on the interface are considerably controlled by the power n: When n is larger than a threshold value n_{th} (= 2/3 for the quadratic remote loading $q(x) = \kappa x^2/2$), the behavior is qualitatively similar to that

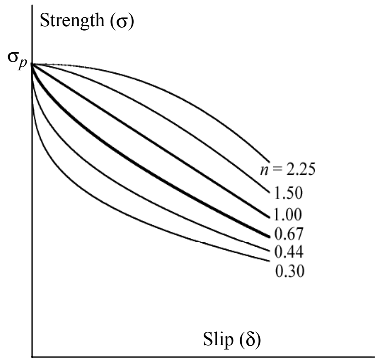

Fig.2 The nonlinear, power-law displacement-softening constitutive relation. The stress inside the rupture region of the interface obeys the relation given by $\sigma = \sigma_p - W\delta^n$, where W and n are positive constants.

for n = 1 (linear case). Slip develops gradually with increasing remote loading until it reaches the critical length above which no stable solution for increased loading exists; The case $n = n_{th}$ is transitional and the loading stress must in that case either always increase, or stay constant, or always decrease in order to expand the rupture region. Whichever occurs is controlled by another dimensionless parameter; When $0 < n < n_{th}$ (such as expected from dynamic frictional experiments and seismic inversions), the analysis indicates that upon slip initiation the loading must be decreased in order to quasi-statically expand the rupture region, suggesting instability at zero slip. When slip develops, the unstable equilibrium branch eventually stabilizes and the loading Rt must start to increase to expand the rupture region further. That may imply that as soon as the peak of loading reaches the strength σ_p of the interface, the rupture region nucleates (but actually, after a small slip, requiring a small further load increase, not represented in the power-law model) and then expands to a finite size at which arrest occurs.

In the following text, using a simplified energy approach[11-13], we shall further extend the analysis into a three-dimensional one. We consider the stability of an elliptical crack that follows the displacement-softening law (2) and is subjected to a locally peaked remote loading that increases with time. The discussion is for an open crack (opening mode) as well as for an elliptical crack under unidirectional shear loading (sliding mode).

2. Problem Statement

Here, in the three-dimensional context, we consider interface rupture in an infinite, homogeneous elastic space subjected to a locally peaked loading stress, $\sigma_o = \sigma_o(x, z, t)$. The planar interface coincides again with the x-z plane ($y = 0$). We define slip $\delta(x, z, t)$ on the interface as the displacement discontinuity $\delta(x, z, t) = u_y(x, 0^+, z, t) - u_y(x, 0^-, z, t)$ for tensile loading (opening mode), $u_x(x, 0^+, z, t) - u_x(x, 0^-, z, t)$ for unidirectional shear loading in the x-direction (sliding mode). The relevant (tensile or shear) stress on the interface is denoted by $\sigma(x, z, t)$ and coincides with $\sigma_y(x, 0, z, t)$ for the opening mode and $\sigma_{xy}(x, 0, z, t)$ for the sliding mode (Fig.3). It is assumed that the loading $\sigma_o(x, z, t)$, the stress that would act if the interface was constrained against any slip, increases with time quasi-statically, but retains its peaked character and takes the form

$$\sigma_o = \sigma_p + Rt - q(x, z). \qquad (4)$$

Here, σ_p is the tensile (for opening mode) or shear (for sliding mode) strength of the interface, and the loading function $q(x, z)$ satisfies $q(x, z) > 0$ for $(x, z) \neq (x_p, z_p)$ and $q(x_p, z_p) = 0$, and $t = 0$ corresponds to the time when the peak of loading stress first reaches σ_p and rupture starts at (x_p, z_p) (In this study, a loading function of the quadratic form $q(x, z) = \kappa\,(x^2 + m^2z^2)/2$, with κ [Pa/m^2] and m being positive constants, is assumed; in this case $x_p = z_p = 0$). At $t > 0$, some area of the interface opens (slips) and in the

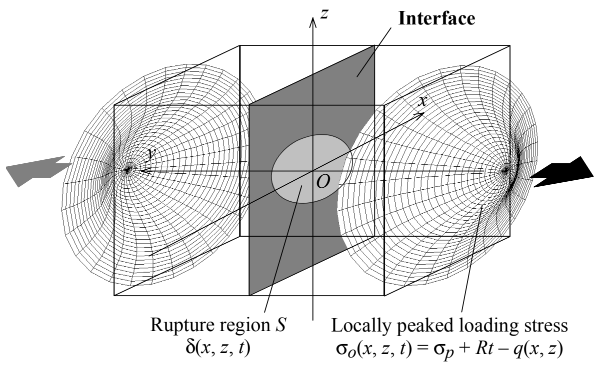

Fig.3 An infinite, homogeneous, linear elastic body under tensile loading. The loading stress $\sigma_o(x, z, t)$ is locally peaked in space [at $(x_p, z_p) = (0, 0)$ in this figure] and increases gradually with time, at rate R. Due to this increasing tensile loading, a rupture region S develops (opening mode). Here, we assume that $\sigma_o(x, z, t) = \sigma_p + Rt - q(x, z) = \sigma_p + Rt - \kappa\,(x^2 + m^2z^2)/2$, with m being a constant. Due to the increasing loading stress, an elliptical rupture region $x^2/a(t)^2+z^2/b(t)^2 < 1$ is assumed to expand with time and the stress inside this elliptical rupture region drops according to the displacement-softening law (2). Similarly, we can define the problem for unidirectional shear rupture (sliding mode).

rupture region the stress drops following the displacement-softening relation (2), with $n = 1$ (linear case). We consider sufficiently small slip δ so that a linear displacement-softening law with constant softening rate W applies at least until initiation of instability. The edge of the quasi-static rupture region where $\delta > 0$ are not specified *a priori* and will automatically be chosen so that the quasi-statically calculated $\sigma(x, z, t) = \sigma_p$ is satisfied at the edge, assumed later as $x^2/a(t)^2+z^2/b(t)^2 = 1$ (a, $b > 0$). Note that the aspect ratio of the ellipse $l' \equiv a(t)/b(t)$, determined by the quasi-static equilibrium condition at each time increase, is not equal to the aspect ratio of the loading function, m, and changes as the loading stress increases, except for special values of m [equation (7)]. At a late stage, critical nucleation lengths are reached where no more quasi-static solution exists for additional loading increase and dynamic instability starts. It can be shown that for a loading function of the quadratic form $q(x, z) = \kappa\,(x^2 + m^2z^2)/2$, the critical lengths for the three-dimensional, linear displacement-softening interface may be given again universally, depending only on the elastic parameters and the displacement-softening rate, regardless of the curvature of the loading κ. The analysis is performed based on a simplified energy approach[11-13], and summarized as follows:

Consider a one-degree-of-freedom slip distribution that induces no stress singularity at edge

$$\delta(x,z) = D\left(1-\frac{x^2}{a^2}-\frac{z^2}{b^2}\right)^{3/2}, \; (0 \le x^2/a(t)^2+z^2/b(t)^2 < 1) \qquad (5)$$

where a and b are the axes of the assumed elliptical rupture region, and D is a constant corresponding to the maximum slip. Then, By considering the quasi-static elastic equilibrium, we can express the stress on the interface $\sigma(x, z, t)$ as[14]

$$\sigma(x, z, t) = \sigma_o(x, z, t) - \frac{D}{a}\left(C - A\frac{x^2}{a^2} - B\frac{z^2}{b^2}\right), \qquad (6)$$

within $0 \le x^2/a(t)^2+z^2/b(t)^2 < 1$. Here, the constants A, B and C represent the effect of material properties (μ and ν) and the aspect ratio $k' \equiv b(t)/a(t)$ [or $l' \equiv a(t)/b(t)$].

Let $U[\delta(x, z)]$ be the energy functional of the displacement-weakened body with slip distribution $\delta(x, z)$, expressed as the sum of the energy when there is no slip, the energy change of the system outside the rupture region due to introduction of slip δ, and the energy of the slip plane due to δ. Then, the stationary condition $\partial U/\partial D = \partial U/\partial a = \partial U/\partial k' = 0$ indicates[13] that the critical lengths at instability are universal, i.e. independent of the curvature κ.

For the opening mode, Fig.4 shows the critical aspect ratio of the rupture ellipse at instability, k'_c (l'_c), for a given remote loading's aspect ratio m, and Fig.5 indicates the combination of critical lengths (a_c, b_c) for various values of the aspect ratio m of the remote loading. The critical length approaches the two-dimensional value (mode I) for extreme values of m. The results, shown in Figs 6 and 7 for the sliding mode, again indicate the universality of the critical lengths at instability, i.e., they do not depend on the curvature κ. For Poisson's ratio $\nu = 0.25$, Fig.6 indicates the relation between the remote loading's aspect ratio m and the critical aspect ratio of the rupture ellipse at instability, k'_c (l'_c), and Fig.7 shows the critical lengths a_c and b_c for varying aspect ratio m of the remote loading. We can see that critical lengths at instability approach the two-dimensional mode II or mode III value for extreme values of m.

As mentioned earlier, the aspect ratio of the rupture ellipse $l' \equiv 1/k' = a(t)/b(t)$ generally changes with the increase of loading stress, but it can be analytically shown that the aspect ratio l' is equal to that of the remote loading m, and remains the same regardless of time, t, when the condition

$$m = l' = 1, \quad \text{(for opening mode)}$$
$$m = l' = m_c \approx 1/(1-\nu), \quad \text{(for sliding mode)} \qquad (7)$$

is satisfied. This condition gives the minimum rupture area at instability ($\pi a_c b_c$) with

$$2a_c = 2b_c \approx 1.960 \frac{\mu}{W(1-\nu)}, \quad \text{(for opening mode)}$$

$$\begin{cases} 2a_c \approx 1.248 \dfrac{E(\sqrt{\nu(2-\nu)}) + (1-\nu)K(\sqrt{\nu(2-\nu)})}{(1-\nu)(2-\nu)} \dfrac{\mu}{W}, \\ 2b_c \approx 1.248 \dfrac{E(\sqrt{\nu(2-\nu)}) + (1-\nu)K(\sqrt{\nu(2-\nu)})}{2-\nu} \dfrac{\mu}{W}, \end{cases}$$
$$\text{(for sliding mode)} \quad (8)$$

where $K(k)$ [$E(k)$] is the complete elliptic integral of the first (second) kind, respectively. These critical conditions are indicated in Figs 5 and 7. For the opening mode, the condition $m = 1$ means axi-symmetric loading, and a circular crack develops on the interface. For the sliding mode, the condition $m = m_c \approx 1/(1-\nu)$ with $\nu = 0.25$ gives the critical lengths as $2a_c = 2.598\mu/W$ and $2b_c = 1.951\mu/W$.

3. Discussion

The minimum critical rupture lengths at instability obtained above for three-dimensional displacement-softening interfaces [equation (8)] compare with the two-dimensional one [equation (3): $1.158\mu/[W(1-\nu)]$ for modes I and II; $1.158\mu/W$ for mode III].

From Fig.6, we find that, when $m \approx 0.832$ and $\nu = 0.25$, the aspect ratio of the shear rupture region at instability becomes 1, i.e., we have a circular critical rupture region in shear. In this case, the critical lengths are

$$2a_c = 2b_c \approx 2.286\ \mu/W. \quad (\text{for } m \approx 0.832, \nu = 0.25) \qquad (9)$$

Previous study on circular shear rupture instability[15] indicates

$$2a_c \approx (7\pi/12)\,(\sigma_o/\sigma_e)^2\,(\mu/W) \approx 1.833\,(\sigma_o/\sigma_e)^2\,(\mu/W), \qquad (10)$$

where the prestress is assumed to be uniform inside (σ_o) and outside (σ_e) the critical rupture region. Equation (10) has been derived by equating the available strain energy in the vicinity of the interface to the energy needed to propagate rupture at the edge[16], but there is an abrupt change of stresses at the edge of the rupture region. Similar expression has been obtained from a finite difference calculation of triggered rupture by suddenly loading a circular region[16]

$$2a_c \approx 1.6\,(\sigma_o/\sigma_e)^2\,(\mu/W). \qquad (11)$$

Equation (10) with $\sigma_o/\sigma_e \approx 1.247$ or equation (11) when $\sigma_o/\sigma_e \approx 1.429$ is roughly equivalent to equation (9), but the result obtained in (9) has stronger meaning in the sense that the size always coincides, no matter how peaked (or flat) is the loading stress, so long as we are in the linear range of the displacement-softening law.

Another numerical study on the dynamic instability of a crack of *a priori* fixed size under unidirectional uniform shear loading[17] implies the critical size is expressed as

$$(x^2+z^2)^{3/2} < \frac{(1.2 \pm 0.05)\pi}{2} \frac{\mu}{W} \left(\frac{x^2}{1-\nu} + z^2 \right), \qquad (12)$$

upon modifying the results found in the publication[17].

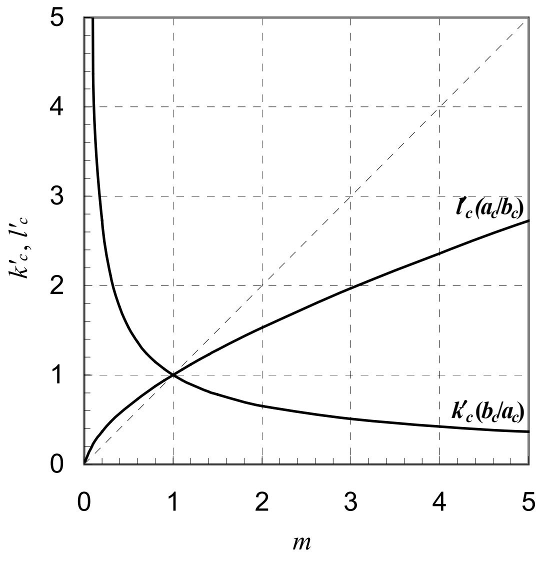

Fig.4 The relation between the remote loading's aspect ratio m and the critical aspect ratio of the rupture ellipse at instability, k'_c (l'_c), for the opening mode.

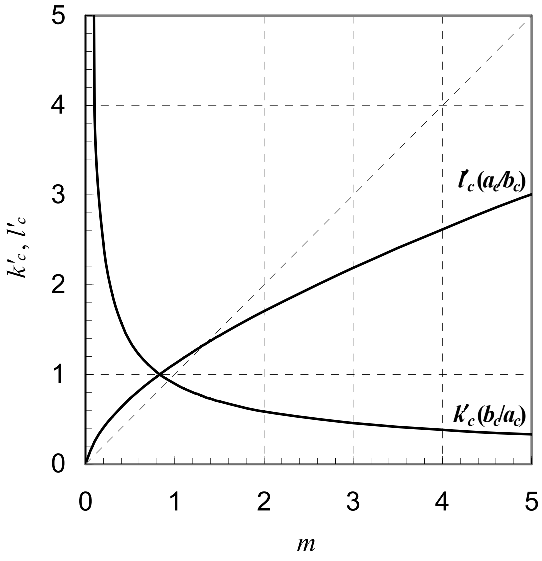

Fig.6 For a given remote loading's aspect ratio m, the critical aspect ratio of the rupture ellipse k'_c (l'_c) for the sliding mode can be obtained using this diagram (Poisson's ratio ν = 0.25).

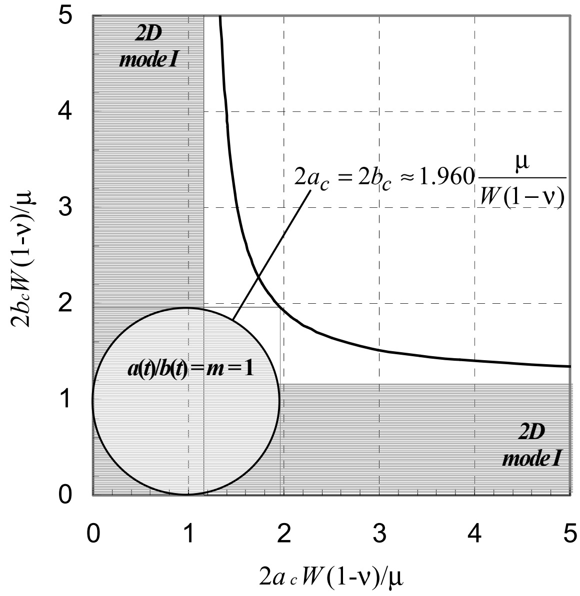

Fig.5 The critical lengths (a_c, b_c) at instability associated with an elliptical crack that follows the linear displacement-softening relation (opening mode). The lengths are obtained for varying aspect ratio m of the remote loading, and they approach the two-dimensional mode I value for extreme values of m. When $m = 1$ (axi-symmetric loading), the aspect ratio of the rupture region is always 1 (circular crack) and the minimum rupture area at instability is given.

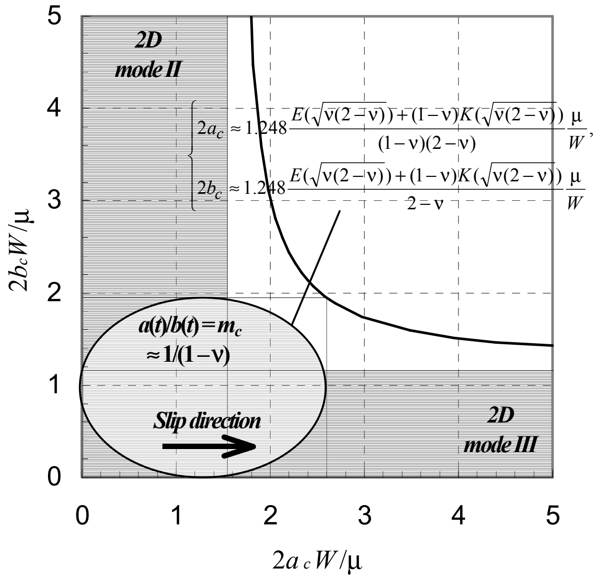

Fig.7 The combination of critical lengths (a_c, b_c) at instability related to a linear displacement-softening elliptical rupture (sliding mode; ν = 0.25). The lengths are shown for various values of the aspect ratio m of the remote loading, and for extreme values of m, they approach the two-dimensional mode II or mode III value. When $m \approx 1/(1-\nu)$, the aspect ratio of the rupture region is always constant and the rupture area at instability ($\pi a_c b_c$) becomes minimum.

The sign ± appears because of the accuracy of the finite difference method with orthogonal grid used in the analysis. The edge of the rupture region (12) is pseudoelliptical and approximately corresponds to the boundary of the critical ellipse (8) in shear, and for instance, for ν = 0.25, we can confirm that the elliptical contour associated with the second equation of (8) is located inside the "band" of the contours corresponding to "±0.05"in (12).

4. Conclusions

The purpose of this study was to show the critical size at instability that is relevant to a linear displacement-softening interface under a locally peaked, increasing stress field in the three-dimensional context. By employing a simplified energy approach and assuming a loading stress of a quadratic form and an elliptical rupture region, we showed the following results: (1) Like the two-dimensional case, the critical size is independent of the curvature of the quadratic loading stress distribution; (2) For the opening (tensile) mode, if the remote loading is axi-symmetric, the aspect ratio of the induced rupture region does not change with time and is equal to 1 (circular crack). The corresponding critical diameter is 1.960 $\mu/[W(1-\nu)]$, with μ being shear modulus, W linear displacement-softening slope, and ν Poisson's ratio; (3) For the sliding mode (unidirectional shear), the aspect ratios of the remote loading m and the rupture region remain constant if $m \approx 1/(1-\nu)$. The corresponding critical rupture size is written in terms of μ, ν and W. For example, when $\nu = 0.25$, the critical lengths are $2.598\mu/W$ and $1.951\mu/W$; and (4) Comparison of these results with the two-dimensional one ($1.158\mu/[W(1-\nu)]$ for modes I and II; $1.158\mu/W$ for mode III) shows that the critical lengths are of the same order for all two- and three-dimensional cases. The problem considered here is quite simplified, but it still retains the basic features that may play a crucial role in understanding the rupture nucleation process related to earthquakes and rockbursts.

References

1) Campillo, M. and Ionescu, I.R.: Initiation of antiplane shear instability under slip dependent friction. *J. Geophys. Res.*, 102, pp.20,363–20,371, 1997.
2) Ionescu, I.R. and Campillo, M.: Influence of the shape of friction law and fault finiteness on the duration of initiation. *J. Geophys. Res.*, 104, pp.3013–3024, 1999.
3) Dascalu, C., Ionescu, I.R. and Campillo, M.: Fault finiteness and initiation of dynamic shear instability. *Earth Planet. Sci. Lett.*, 177, pp.163–176, 2000.
4) Lapusta, N., Rice, J.R., Ben-Zion, Y. and Zheng, G.: Elastodynamic analysis for slow tectonic loading with spontaneous rupture episodes on faults with rate- and state-dependent friction. *J. Geophys. Res.*, 105, pp.23,765–23,789, 2000.
5) Lapusta, N. and Rice, J.R.: Nucleation and early seismic propagation of small and large events in a crustal earthquake model. *J. Geophys. Res.*, 108(B4), cn:2205, doi:10.1029/2001JB 000793, 2003.
6) Uenishi, K.: On the mechanical instability of a displacement-softening plane of weakness. *Proceedings of the 37th Symposium on Rock Mechanics*, pp.371–374, Committee on Rock Mechanics, The Japan Society of Civil Engineers, Tokyo, 2008.
7) Uenishi, K.: Rupture instability on a nonlinear slip-weakening plane of weakness. *Proceedings of the 12th Japan Symposium on Rock Mechanics and the 29th Western Japan Symposium on Rock Engineering*, pp.379–384, Japanese Committee for Rock Mechanics of ISRM (International Society for Rock Mechanics), Tokyo, 2008.
8) Chambon, G., Schmittbuhl, J. and Corfdir, A.: Laboratory gouge friction: seismic-like slip weakening and secondary rate- and state-effects. *Geophys. Res. Lett.*, 29(10), cn:1366, doi:10.1029/2001GL014467, 2002.
9) Goldsby, D.L. and Tullis, T.E.: Low frictional strength of quartz rocks at subseismic slip rates. *Geophys. Res. Lett.*, 29(17), cn:1844, doi:10.1029/2002GL015240, 2002.
10) Abercrombie, R.E. and Rice, J.R.: Can observations of earthquake scaling constrain slip weakening? *Geophys. J. Int.*, 162(2), pp.406–424, doi:10.1111/j.1365-246X.2005.02579.x, 2005.
11) Rice, J.R.: Dislocation nucleation from a crack tip: an analysis based on the Peierls concept. *J. Mech. Phys. Solids*, 40(2), pp.239–271, 1992.
12) Rice, J.R. and Beltz, G.E.: The activation energy for dislocation nucleation at a crack. *J. Mech. Phys. Solids*, 42(2), pp.333–360, 1994.
13) Uenishi, K. and Rice, J.R.: Three-dimensional rupture instability of a slip-weakening fault under heterogeneous loading. *Eos Trans. AGU*, 85(46), Fall Meet. Suppl., Abstract S13E-04, 2004.
14) Bui, H.D.: An integral equations method for solving the problem of a plane crack of arbitrary shape. *J. Mech. Phys. Solids*, 25(1), pp.29–39, 1977.
15) Day, S.M.: Three-dimensional simulation of spontaneous rupture: The effect of nonuniform prestress. *Bull. Seismol. Soc. Am.*, 72(6), pp.1881–1902, 1982.
16) Madariaga, R. and Olsen, K.B.: Criticality of rupture dynamics in 3-D. *Pure Appl. Geophys.*, 157(11-12), pp.1981–2001, 2000.
17) Favreau, P., Campillo, M. and Ionescu, I.R.: Initiation of shear instability in three-dimensional elastodynamics. *J. Geophys. Res.*, 107(B7), cn:2147, doi:10.1029/2001JB 000448, 2002.