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ON THE MECHANICAL INSTABILITY OF A DISPLACEMENT-SOFTENING PLANE OF WEAKNESS

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We investigate the nucleation of instability on a displacement-softening plane of weakness (interface) subjected to a locally peaked, gradually increasing loading stress. Rupture initiates when the peak of the loading stress first reaches the strength level of the interface to start displacement softening. Then the size of the rupture region grows under increased loading stress until finally a critical nucleation length is reached, which marks the onset of a dynamically controlled instability (e.g., earthquakes, rockbursts). We prove that the nucleation length is independent of the shape of the loading stress distribution. Its universal value is proportional to an elastic modulus and inversely proportional to the linear displacement-softening rate, and is given by the solution to an eigenvalue problem.

Key Words : *rupture nucleation process, earthquake source mechanics, interface rupture, slip-weakening, rockburst*

1. Introduction

In order to understand the physical process of loss of stability of cracked or partially contacting solid structures, many mechanical models have been proposed and studied. For example, analytically, using the potential energy principle in the framework of quasi-static elasticity and a linear displacement-softening law, a boundary eigenvalue problem is derived for a cohesive crack model[1)]. It was shown that under a critical condition related to the smallest eigenvalue, the corresponding eigenfunction represents the non-unique part of the displacement solution and the critical load can be determined via that eigenfunction. Also, by variational analysis, the condition of stability loss of an elastic structure with a growing cohesive crack is obtained where the stress is specified as a decreasing function of the crack opening displacement (displacement-softening)[2, 3)]. That condition was transformed into an eigenvalue problem for a homogeneous Fredholm integral equation, with the structure size as the eigenvalue, and solved for the maximum load as well as the maximum deflection that is carried by the structure, explicitly in terms of the eigenfunction associated with the integral equation.

Other examples are related to the study of earthquake source process: A spectral method was used to investigate the initiation of dynamic anti-plane slip instabilities of a slip-weakening (analogous to displacement-softening for tensile rupture) geological plane of weakness (e.g., fault, joint) in a homogeneous linear elastic medium that is pre-stressed uniformly up to the frictional threshold[4-6)]. Using an eigenvalue analysis, an analytical expression of the slip was given and that slip was divided into two parts: the solution associated with positive eigenvalues ("dominant part") and negative eigenvalues ("wave part"). It was shown that the dominant part, characterized by an exponential growth with time, controls the development of the instability and the wave part becomes rapidly negligible when the instability develops. The effect of displacement-softening rate on the duration of the quasi-static phase and the critical crack length was evaluated[4, 5)]. The analysis was further extended to instabilities of a finite crack of (*a priori*) fixed length[6)]. However, no explanation regarding the physical meaning of the (*a priori*) fixed crack length and uniform loading has been given in these analyses. Since the rigorous numerical treatments of the complete earthquake cycle[7, 8)] show clearly that a region of initially quasi-static crack grows in size in a quasi-static manner before dynamic breakout of the rupture, models that have quasi-statically extending cracks are needed.

In the following, based on the quasi-static elastic equilibrium condition and the linear displacement-softening law, we show that the nucleation length relevant to the instability of a plane of weakness (interface) depends only on elastic modulus of the medium and the displacement-softening rate[9)].

2. Problem Statement

We investigate the behavior of a linear displacement-softening interface and evaluate the nucleation length that is relevant to interface instabilities and ensuing dynamic rupture. We consider two-dimensional interface rupture in an infinite, homogeneous elastic space subjected to a locally peaked loading stress (Fig.1)

$$\sigma_o(x,t)=\sigma_p+Rt-q(x). \tag{1}$$

Here, the interface coincides with the x-z plane ($y=0$) of a Cartesian coordinate system xyz, σ_p is the tensile (for mode I) or shear (for modes II and III) strength of the interface, and R (> 0 if the stress increases with time t) is the loading rate of the increasing stress (e.g., tectonic loading). The function $q(x)$ satisfies $q(x)>0$ for $x\neq x_p$ and $q(x_p)=0$. Thus $t=0$ is the time when the peak value of loading stress, at position x_p, first reaches σ_p so that slip initiates at that point.

In the study, as a constitutive law inside the rupture region, the displacement-softening law

$$\sigma(x,t)=\sigma_p-W\delta(x,t), \tag{2}$$

is used where W ($W>0$) is a constant. Figure 2 shows schematically this constitutive law (2). In the figure, the ordinate denotes the strength σ and the abscissa corresponds to the slip (displacement gap) δ. Slip, defined as $\delta(x,t)=u_y(x,0^+,t)-u_y(x,0^-,t)$ for mode I, $u_x(x,0^+,t)-u_x(x,0^-,t)$ for mode II, and $u_z(x,0^+,t)-u_z(x,0^-,t)$ for mode III, can occur if the local stress reaches the peak strength σ_p. The stress inside the rupture region of the interface (denoted by $\sigma(x,t)$ and coincides with $\sigma_y(x,0,t)$ for mode I, $\sigma_{xy}(x,0,t)$ for mode II, and $\sigma_{yz}(x,0,t)$ for mode III) drops according to the relation (2).

Figure 3 shows schematically the development of the rupture region by the increasing loading stress. Figure 3(a) pertains to the situation at time $t=0$, where the loading stress $\sigma_o(x,0)$ reaches the strength of the interface, σ_p, to start displacement softening. Prior to this stage, no slip has occurred. The function $q(x)$ can be identified as the difference between the straight horizontal line $\sigma=\sigma_p$ and the curve $\sigma=\sigma_o(x,0)$. At $t>0$ [Fig.3(b)], part of the interface slips and the stress inside the rupture region drops according to the displacement-softening law (2). Note that the extremities of the quasi-static rupture region where $\delta>0$ (i.e., the support of the slip distribution) are not specified *a priori* and will automatically be chosen so that the quasi-statically calculated $\sigma(x,t)=\sigma_p$ is satisfied at those extremities, $x=a_\pm(t)$. At least that will hold so long as a quasi-static solution actually exists. We would like to know when it just fails to exist; this situation gives the nucleation length of earthquake rupture. Figure 3(c) shows such a situation. At a late stage, a critical nucleation length h_n is reached at which no further quasi-static solution exists for additional increase of the loading. That marks the onset of a dynamically controlled instability. In the following, we will prove that for the linear displacement-softening law, the nucleation length is independent of the shape of the loading stress distribution, that is, it is independent of the mathematical form of $q(x)$. Its universal value is proportional to an elastic modulus and inversely proportional to the displacement-softening rate, and is given by the solution to an eigenvalue problem.

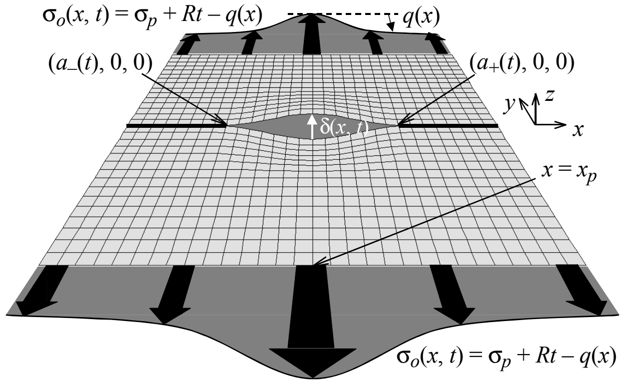

Fig.1 A displacement field associated with tensile (mode I) rupture in an infinite, homogeneous, linear elastic space. The loading shear stress $\sigma_o(x,t)$ is locally peaked in space and increases gradually with time, at rate R. Similarly, we can define the problem for in-plane (mode II), or for anti-plane shear (mode III) rupture.

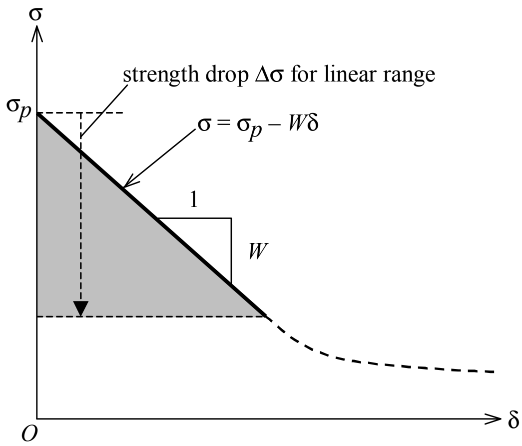

Fig.2 The (initially) linear displacement-softening constitutive law. The stress inside the rupture region of the interface obeys the linear relation $\sigma=\sigma_p-W\delta$, at least when the strength drop is less than $\Delta\sigma$. The displacement-softening rate W is a constant ($W>0$).

3. Displacement-Softening Nucleation Length

By considering the quasi-static elastic equilibrium, we can express the stress on the interface $\sigma(x,t)$ only in terms of the slip $\delta(x,t)$ as[10]

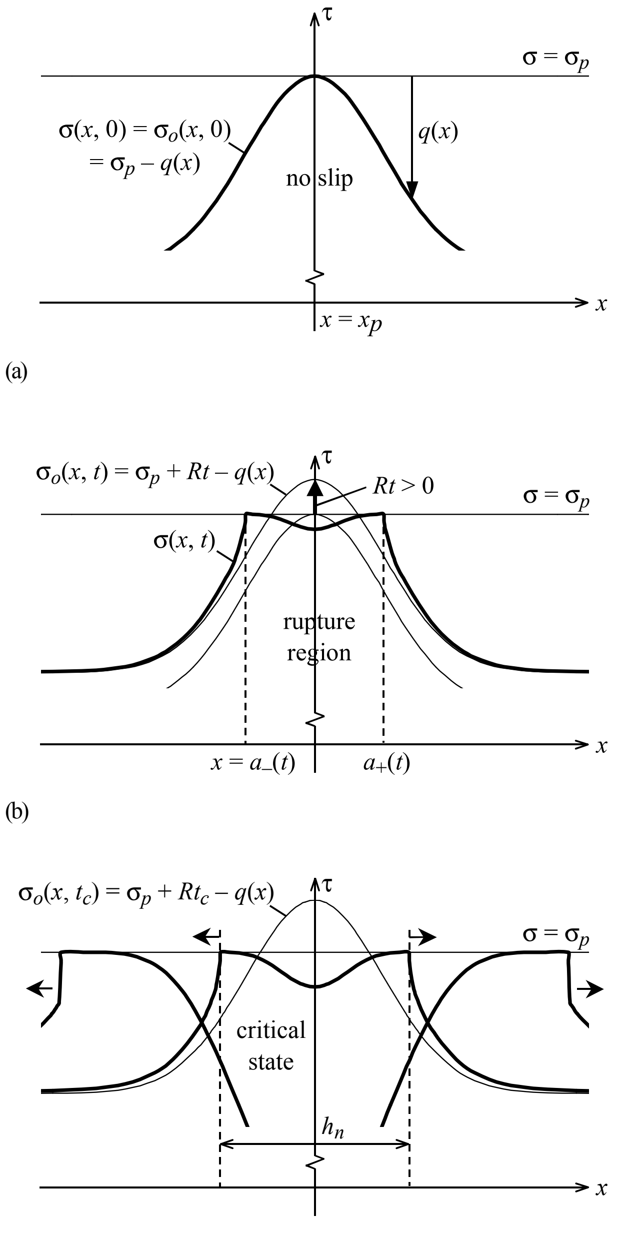

Fig.3 Development of the rupture region induced by the increasing loading stress. (a) At time $t = 0$, the peak of the stress distribution reaches the peak strength of the interface, σ_p. Prior to this stage, no slip has occurred; (b) At $t > 0$, part of the interface slips and the stress inside the rupture region drops according to the displacement-softening law; and (c) At a later stage $t = t_c$, when the length of the rupture region reaches a critical value, h_n, the interface system becomes unstable and the rupture region will expand even without any increase of the loading stress. We will show that h_n is independent of R and $q(x)$.

$$\sigma(x,t) = \sigma_o(x,t) - \frac{\mu^*}{2\pi}\int_{a_-(t)}^{a_+(t)} \frac{\partial\delta(\xi,t)/\partial\xi}{x-\xi}\,d\xi. \tag{3}$$

Here, $\mu^* = \mu$ (shear modulus) for mode III and $\mu/(1-\nu)$ for modes I and II, with ν being Poisson's ratio. Using this elastic equilibrium condition (3) and the displacement-softening law (2), together with equation (1) and differentiation with respect to time, we obtain

$$-WV(x,t) = R - \frac{\mu^*}{2\pi}\int_{a_-(t)}^{a_+(t)} \frac{\partial V(\xi,t)/\partial\xi}{x-\xi}\,d\xi$$

$$\text{for } a_-(t) < x < a_+(t). \tag{4}$$

Here, $V(x, t)$ is slip rate $V(x, t) \equiv \partial\delta(x, t)/\partial t$. By introducing $a(t) \equiv [a_+(t) - a_-(t)]/2$, $b(t) \equiv [a_+(t) + a_-(t)]/2$, $X \equiv [x - b(t)]/a(t)$ and $v(X, t) \equiv V(x, t)/[\sqrt{2}V_{rms}(t)]$, where $V_{rms}(t)$ is the root-mean-square slip rate

$$V_{rms}(t) \equiv \sqrt{\frac{1}{a_+(t)-a_-(t)}\int_{a_-(t)}^{a_+(t)} V^2(x,t)\,dx}\ , \tag{5}$$

we can normalize equation (4). Thus, suppressing explicit reference to the time-dependence of $a(t)$, $V_{rms}(t)$ and $v(X, t)$,

$$-\frac{aW}{\mu^*}v(X) = \frac{a}{\sqrt{2}\mu^*}\frac{R}{V_{rms}} - \frac{1}{2\pi}\int_{-1}^{+1}\frac{v'(s)}{X-s}\,ds$$

$$\text{for } -1 < X < +1, \tag{6}$$

where $'$ denotes the first derivative of a function. The rupture region keeps growing in time but since $V_{rms}/(aR/\mu^*)$ diverges as the nucleation condition is approached, at the critical length, $aR/(\mu^* V_{rms})$ becomes zero. That length is the nucleation length that we seek. At that length the above integral equation for $v(X)$ becomes

$$\frac{aW}{\mu^*}v(X) = \frac{1}{2\pi}\int_{-1}^{+1}\frac{v'(s)}{X-s}\,ds \quad \text{for } -1 < X < +1. \tag{7}$$

The critical length is thus given as the length such that the eigen equation (7) has a nontrivial solution for $v(X)$. Equation (7) implies that a solution is given when $a(t)W/\mu^*$ reaches the smallest eigenvalue $a_c W/\mu^* = \lambda_0 \approx 0.579$ and $v(X, t_c)$ is equal to the associated eigenfunction $v_0(X)$. Thus, the critical length h_n is given by

$$h_n = 2a_c \approx 1.158\,\mu^*/W. \tag{8}$$

Note that, the critical length depends only on the (generalized) shear modulus μ^* and the displacement-softening rate W and is independent of the rate and the shape of the loading, that is, of R and $q(x)$.

In our formula for the critical nucleation length (8), the displacement-softening rate W plays a crucial part. This parameter in the linear softening law (2) is given by $W = \Delta\sigma/D_c$ if one assumes linear softening for slips $\delta < D_c$, and no further softening for $\delta > D_c$.

Examples: Consider the displacement-softening process in the post-failure stage of laboratory tests of initially intact samples. Typical laboratory testing of shear fracture of initially intact Fichtelbirge granite specimens at different confining pressures (7.5 to 300MPa) in a stiff, servo-controlled triaxial apparatus[11, 12] suggests that $\mu = 30$GPa and $\nu = 0.25$ (i.e., $\mu^* = 40$GPa), and the displacement-softening process is approximately linear: In the relatively low range of the interface-normal compressive stress, 60MPa < σ_n < 120MPa, the displacement-softening distance is approximately D_c = 440μm and the strength drop $\Delta\sigma$ = 20MPa. In this case, the displacement-softening rate $W = \Delta\sigma/D_c \approx$ 50GPa/m and the nucleation length is $h_n \approx$ 0.9m. For σ_n = 140MPa, the displacement-softening distance is about D_c = 460μm and the strength drop is found to be $\Delta\sigma$ = 30MPa. We have $W \approx$ 70GPa/m and $h_n \approx$ 0.7m in this instance. Under relatively high interface-normal compression 250MPa < σ_n < 600MPa, D_c is some 800μm and the strength drop scales with σ_n, $\Delta\sigma = 0.04\sigma_n$ + 50MPa[12]. Therefore, in this range of σ_n, $\Delta\sigma \approx$ 60 to 75MPa and the displacement-softening rate is $W \approx$ 75 to 90GPa/m. Hence we have $h_n \approx$ 0.5 to 0.6m.

4. Conclusions

The purpose of this contribution was to show the critical length that is relevant to (linear) displacement-softening interface instabilities. By considering the quasi-static elastic equilibrium condition, we have indicated that the critical length can be expressed in terms of the smallest eigenvalue (of the reduced problem), the elastic modulus and the displacement-softening rate only. It should be noted that the fundamental nature of rupture instabilities remains the same even when the type (mode I, II, or III) and the shape of the loading change. Although the problem investigated in this study is quite simplified, it still retains the essential characteristics that are believed to play an important part during the rockburst / earthquake nucleation process.

Acknowledgments

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