CHAPTER VII

SECONDARY STRESSES DUE TO THE RIGIDITY OF JOINTS

108. STRICTLY speaking, the stresses in all kinds of trusses are statically indeterminate; for it is only by assuming joints to be free of all constraints under all changes of form due to elastic deformation of members that we are generally enabled to calculate the stresses in trusses when external forces are known, while in reality all joints are more or less rigid and resistant to angular changes between truss-members. Even pin-connected trusses are to be considered to have rigid joints wherever the frictional resistance on the surface of the pin exceeds the torsional moment caused in the latter by the members which it connects.

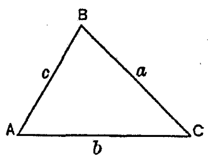

Fig. 78

In the following, we shall in the first place discuss the secondary stresses in the truss itself, and then those in the posts and lateral members due to rigid floor-beam and lateral connections.

SECONDARY STRESSES IN TRUSSES

109. Suppose a truss element ABC (Fig. 78) undergo deformations in its sides a, b and c due to stresses produced in them. If the joints do not offer any resistance to motion of members, then corresponding angular changes would take place forming a new triangle with

changed side-lengths. Representing by Δ the rate of linear and angular changes, + for extension or enlargement and − for the contrary, we have the following equations from the deformed triangle:

$$(b + \Delta b) \sin (A + \Delta A) = (a + \Delta a) \sin (B + \Delta B),$$

from which, by neglecting small quantities, we get

$$\Delta A = \left(\frac{\Delta a}{a} - \frac{\Delta b}{b}\right) \tan A + \Delta B \frac{\tan A}{\tan B}.$$

Similarly,

$$\Delta C = \left(\frac{\Delta c}{c} - \frac{\Delta b}{b}\right) \tan C + \Delta B \frac{\tan C}{\tan B};$$

and since

$$\Delta A + \Delta B + \Delta C = 0,$$

we get,

$$\Delta B = \frac{\Delta b}{b} (\cot A + \cot C) - \frac{\Delta a}{a} \cot C - \frac{\Delta c}{c} \cot A,$$

$$\Delta C = \frac{\Delta c}{c} (\cot A + \cot B) - \frac{\Delta a}{a} \cot B - \frac{\Delta b}{b} \cot A,$$

$$\Delta A = \frac{\Delta a}{a} (\cot B + \cot C) - \frac{\Delta b}{b} \cot C - \frac{\Delta c}{c} \cot B.$$

These equations would then give the amounts of angular changes due to changes in lengths of the members were the latter free to move at the joints. If, however, the joints were so rigid that angular changes could not take place, then each member would have to distort itself in such a way as to occupy its new position without producing angular changes at the joints. As a consequence, whatever positions the members may in this way assume, the tangents to their neutral axes at the joints should maintain the original angles between them. The constraint thus made active at each joint gives rise to bending in each member. Let a member $A\,C$ be cut off close to the joints (Fig. 79), and represent by M_a and M_c the moments acting at the ends, — positive for bending the piece anticlockwise and vice versâ. Further, let AC' represent the original position of AC, the former being tangent to the latter at A; and denote by α the displacement of C, — positive when it is clockwise and vice versâ.

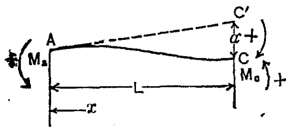

Fig. 79

Since the moment (positive when causing compression in upper fibres) at any point distant x from A is

$$- M_a + \frac{M_a + M_c}{L} x,$$

considering the moments only, we get for the internal work in the member,

$$\omega = \frac{1}{2\,EI}\int_0^L \left(- M_a + \frac{M_a + M_c}{L} x\right)^2 dx$$
$$= \frac{L}{6\,EI}(M_a^2 - M_a M_c + M_c^2),$$

in which I represents the moment of inertia of the section of the piece, and E the modulus of elasticity, both assumed to be uniform.

Since the force acting through α, reckoned in the same sense as the latter, is

$$\frac{M_a + M_c}{L},$$

it follows from the first theorem of Castigliano (Art. 6) that the first derivative of ω with respect to $\dfrac{M_a + M_c}{L}$ must be equal to α, so that with M_a as variable we have,

$$\frac{L}{6\,EI}(2\,M_a - M_c) = \frac{\alpha}{L} \quad . \quad . \quad . \quad . \quad . \quad (222)$$

Considering now the joint A (Fig. 80), and applying this equation to members AC and AB — denoting their lengths, end-displacements and moments by L, α, and M with corresponding suffixes as shown in the figure — we get,

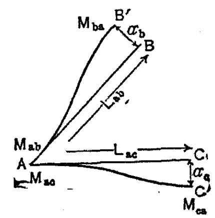

Fig. 80

$$\frac{L_{ac}}{6\,EI_{ac}}(2\,M_{ac} - M_{ca}) = \frac{\alpha_c}{L_{ac}} \quad . \quad . \quad . \quad . \quad (223)$$

$$\frac{L_{ab}}{6\,EI_{ab}}(2\,M_{ab} - M_{ba}) = \frac{\alpha_b}{L_{ac}} \quad . \quad . \quad . \quad . \quad (224)$$

Referring to Fig. 80, it will be seen that the angular change ΔA, which would have taken place had the angle A been free to change, to be equal to $BAC - B'AC'$, so that by paying attention to signs we may put

$$\Delta\,(BAC) = \frac{\alpha_c}{L_{ac}} - \frac{\alpha_b}{L_{ab}}.$$

Consequently we get,

$$6\,E\,\Delta\,BAC = \frac{L_{ac}}{I_{ac}}(2\,M_{ac} - M_{ca}) - \frac{L_{ab}}{I_{ab}}(2\,M_{ab} - M_{ba}) \quad . \quad . \quad . \quad (225)$$

Similarly for other angles, we obtain,

$$6\,E\,\Delta\,ABC = \frac{L_{ab}}{I_{ab}}(2\,M_{ab}-M_{ba}) - \frac{L_{bc}}{I_{bc}}(2\,M_{bc}-M_{cb}) \quad . \quad . \quad . \quad (226)$$

$$6\,E\,\Delta\,BCA = \frac{L_{cb}}{I_{cb}}(2\,M_{cb}-M_{bc}) - \frac{L_{ca}}{I_{ca}}(2\,M_{ca}-M_{ac}) \quad . \quad . \quad . \quad (227)$$

Since at each joint

$$\Sigma M = 0,$$

we have in the triangular frame ABC,

$$M_{ab} + M_{ac} = 0,$$
$$M_{ba} + M_{bc} = 0,$$
$$M_{cb} + M_{ca} = 0.$$

In this way, as many equations as there are unknown moments in a truss may be obtained by extending equations over all the triangles of the truss and setting the ΣM at each joint equal to zero. The following example will show the application of this method to the calculation of secondary stresses in a simple truss.

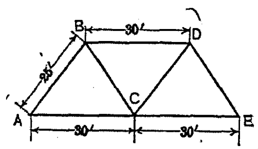

Fig. 81

EXAMPLE. — In the truss of Fig. 81 given:

$I_{ab} = 360$ in.⁴ $\quad L_{ab} = L_{bc} = 25$ ft.
$I_{bc} = 60$ " $\quad L_{bd} = L_{ac} = 30$ "
$I_{ac} = 40$ "
$I_{bd} = 360$ "

Supposing the members to sustain following intensities of direct stresses:

$\overline{AB} = \overline{DE} = -\ 7000$ lbs. per sq. in.
$\overline{AC} = \overline{CE} = +\ 7330$ lbs. per sq. in.
$\overline{BD} = -\ 7330$ lbs. per sq. in.
$\overline{BC} = \overline{CD} = +\ 8000$ lbs. per sq. in.

to calculate the secondary stresses induced in each member, assuming the joints to be perfectly rigid, and E taken at 30,000,000 lbs. per sq. in.

In consequence of the principal stresses the following deformations are produced:

In AB and DE . $\Delta L = -\frac{7000}{E} \times 25 \times 12 = -\ .070$ in.

AC and CE . $\Delta L = \frac{7330}{E} \times 30 \times 12 = +\ .088$ in.

BD . . . . $\Delta L = \frac{-\ 7330}{E} \times 30 \times 12 = -\ .088$ in.

BC and CD . $\Delta L = \frac{8000}{E} \times 25 \times 12 = +\ .080$ in.

Since

$$\cot BAC = .7500,$$
$$\cot ABC = .2915,$$
$$\cot BCA = .7500,$$

the angular changes, were they possible, would then be,

$$\Delta BAC = \frac{.080}{300}(.2915+.7500) - \frac{.088}{360}\times .75 + \frac{.07}{300}\times .2915 = .0001624,$$

$$\Delta ABC = \frac{.088}{360}(.7500+.7500) - \frac{.08}{300}\times .75 + \frac{.07}{300}\times .7500 = .0003416,$$

$$\Delta BCA = -\frac{.070}{300}(.7500+.2915) - \frac{.08}{300}\times .2915 - \frac{.088}{360}\times .7500$$
$$= -\ .0005040,$$

$$\Delta DBC = \frac{.080}{300}(.7500+.2915) - \frac{.08}{300}\times .2915 + \frac{.088}{360}\times .7500 = .0003835,$$

$$\Delta BCD = -\frac{.088}{360}(.7500+.7500) - \frac{.80}{300}\times .7500 - \frac{.08}{300}\times .7500$$
$$= -\ .0007670.$$

Applying Eqs. (225–227) to all the angles successively, we obtain,

$$1)\quad 6\,E\,.0001624 = \frac{360}{40}(2\,M_{ac} - M_{ca}) - \frac{300}{360}(2\,M_{ab} - M_{ba}).$$

$$2)\quad 6\,E\,.0003416 = \frac{300}{360}(2\,M_{ba} - M_{ab}) - \frac{300}{60}(2\,M_{bc} - M_{cb}).$$

$$3)\,-6\,E\,.0005040 = \frac{300}{60}(2\,M_{cb} - M_{bc}) - \frac{360}{40}(2\,M_{ca} - M_{ac}).$$

$$4)\quad 6\,E\,.0003835 = \frac{300}{60}(2\,M_{bc} - M_{cb}) - \frac{360}{360}(2\,M_{bd} - M_{db}).$$

$$-\,6\,E\,.0007670 = \frac{300}{60}(2\,M_{cd} - M_{dc}) - \frac{300}{60}(2\,M_{cb} - M_{bc}).$$

Since from symmetry,

$$M_{cd} = -M_{cb}, \qquad M_{dc} = -M_{bc}, \qquad M_{db} = -M_{bd},$$

the last equation becomes

$$5)\quad 6\,E\,.0007670 = \frac{600}{60}(2\,M_{cb} - M_{bc}).$$

Again, since, at each joint, $\Sigma M = 0$,

$$6)\quad M_{ab} + M_{ac} = 0.$$

$$7)\quad M_{ba} + M_{bc} + M_{bd} = 0.$$

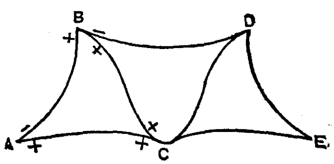

Fig. 82

From these equations we obtain the following values of Ms:

$M_{ac} = +\ \ 2{,}602$ in.-lbs.
$M_{ab} = -\ \ 2{,}602$ "
$M_{ba} = +\ 25{,}926$ "
$M_{bc} = +\ \ 2{,}454$ "
$M_{bd} = -\ 28{,}380$ "
$M_{cb} = +\ \ 8{,}130$ "
$M_{ca} = +\ 10{,}176$ "

Paying attention to the signs of the moments, it will be seen that the bending of the members takes place as shown exaggerated in Fig. 82.

The widths of members in the plane of the truss being as follows,

$AB = 12$ in.,
$BD = 12$ "
$BC = 4$ "
$AC = 4$ "

the secondary stress in each member may now be written,

In AB . . . . . $\frac{25926}{360} \times 6 = 432$ lbs. per sq. in.

BD . . . . . $\frac{28380}{360} \times 6 = 473$ " " " "

BC . . . . . $\frac{8130}{60} \times 2 = 271$ " " " "

AC . . . . . $\frac{10176}{40} \times 2 = 509$ " " " "

These stresses are to be added to the direct stresses already given, to obtain the maximum intensities of stresses in the members. Strictly speaking, the actual maximum in each member is less than the maximum thus found, by that portion of the direct stress taken up in producing the moments, but the difference is generally so slight as to be practically negligible.

SECONDARY STRESS IN THE POSTS DUE TO RIGID FLOOR-BEAM AND LATERAL CONNECTIONS

110. By Loading.—When the posts are connected rigidly at top by the lateral strut, and at bottom by the floor-beam,—as is most usually the case,—the bending of the floor-beam caused by loading on the same induces — by reason of constraints — the moments and thrust, producing deformations as shown exaggerated in Fig. 83.

Representing by M_0, M_1, M_2, and M_3 (Fig. 84) the moments produced at the four corners of the frame, — positive when producing compression on the outer fibre,

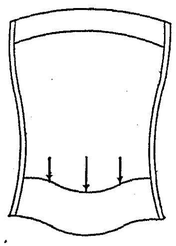

Fig. 83

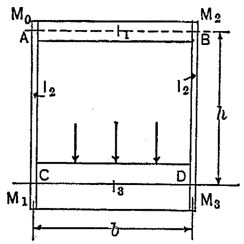

Fig. 84

and vice versâ, — it is evident that, for all loads symmetrically disposed about the centre of the floor-beam,

$$M_0 = M_2,$$
$$M_1 = M_3.$$

Let m_1, m_2, and m_3 represent moments at any point in the upper strut AB, post AC, and floor-beam CD respectively. Then, neglecting the influence of direct stresses and considering moments alone, we get for the work of resistance in the frame,

$$\omega = \int_0^b \frac{m_1^2 dx}{2 EI_1} + 2 \int_0^h \frac{m_2^2 dx}{2 EI_2} + \int_0^b \frac{m_3^2 dx}{2 EI_3},$$

I_1, I_2, and I_3 representing the moments of inertia of the sections of strut, post, and floor-beam respectively. Since

$$m_1 = M_0,$$
$$m_2 = M_0 + \frac{M_1 - M_0}{h} x, \text{ origin of } x \text{ taken at } A,$$
$$m_3 = M_1 + M,$$

in which M represents the moment due to loading, assuming the floor-beam to be simply supported at both ends.

Substituting these values of m in the above expression for work, and integrating by assuming the cross-sections of members to be so many constants, we get,

$$\omega = \frac{M_0^2 b}{2 EI_1} + \frac{3 M_0 M_1 h + (M_1 - M_0)^2 h}{3 EI_2} + \frac{1}{2 EI_3}\left(M_1^2 b + 2 M_1 \int_0^b M dx + \int_0^b M^2 dx\right).$$

Since the values of M_0 and M_1 must be such as to make the total work of resistance in the frame a minimum, setting the first derivatives of ω taken successively with respect to M_0 and M_1 equal to zero, we get,

$$\frac{M_0 b}{I_1} + \frac{M_1 h + 2 M_0 h}{3 I_2} = 0,$$
$$\frac{2 M_1 h + M_0 h}{3 I_2} + \frac{M_1 b}{I_3} + \int_0^b \frac{M dx}{I_3} = 0;$$

from which

$$M_0 = -\frac{\frac{h}{3 I_2}}{\frac{b}{I_1} + \frac{2 h}{3 I_2}} M_1 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (228)$$

Eliminating M_1,

$$M_0 = \frac{\frac{h}{3 I_2} \int_0^b \frac{M dx}{I_3}}{\left(\frac{b}{I_1} + \frac{2 h}{3 I_2}\right)\left(\frac{2 h}{3 I_2} + \frac{b}{I_3}\right) - \left(\frac{h}{3 I_2}\right)^2} \quad . \; . \; . \; . \quad (229)$$

111. The integral $\int_0^b Mdx$ depends on the mode of symmetrical loading. For a *single-track railway bridge* loaded as shown in Fig. 85, we have,

$$\int_0^b Mdx = -2\int_0^{\frac{b-a}{2}} Wxdx - \int_{\frac{b-a}{2}}^{\frac{b+a}{2}} W\left(\frac{b-a}{2}\right)dx = -\frac{W}{4}(b^2-a^2),$$

so that we get,

$$M_0 = -\frac{\frac{h}{12 I_2 I_3}(b^2-a^2)}{\left(\frac{b}{I_1}+\frac{2h}{3I_2}\right)\left(\frac{2h}{3I_2}+\frac{b}{I_3}\right)-\left(\frac{h}{3I_2}\right)^2}W. \quad . \quad (230)$$

$$M_1 = \frac{\frac{b^2-a^2}{4I_3}\left(\frac{b}{I_1}+\frac{2h}{3I_2}\right)}{\left(\frac{b}{I_1}+\frac{2h}{3I_2}\right)\left(\frac{2h}{3I_2}+\frac{b}{I_3}\right)-\left(\frac{h}{3I_2}\right)^2}W \quad . \quad . \quad (231)$$

112. In a *highway bridge*, since the live and dead loads may be assumed to be uniformly distributed along the floor-beam length, designating by w the load per unit length, we get,

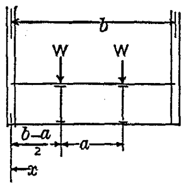

Fig. 85

$$\int_0^b Mdx = -\frac{wb^3}{12};$$

so that from Eqs. (228) and (229) we obtain for this case,

$$M_0 = -\frac{\frac{hb^3}{36 I_2 I_3}w}{\left(\frac{b}{I_1}+\frac{2h}{3I_2}\right)\left(\frac{2h}{3I_2}+\frac{b}{I_3}\right)-\left(\frac{h}{3I_2}\right)^2} \quad . \quad . \quad (232)$$

$$M_1 = \frac{\frac{b^3}{12 I_3}\left(\frac{b}{I_1}+\frac{2h}{3I_2}\right)w}{\left(\frac{b}{I_1}+\frac{2h}{3I_2}\right)\left(\frac{2h}{3I_2}+\frac{b}{I_3}\right)-\left(\frac{h}{3I_2}\right)^2} \quad . \quad . \quad . \quad (233)$$

113. Knowing M_0 and M_1, the moment at any point in each member can at once be found. In the post the maximum fibre stress calculated from such moment is to be combined with the intensity of direct stress which the post receives as the member of the truss, to obtain the greatest intensity of compression in the same.

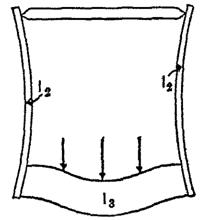

Fig. 86

The *upper strut* is subjected to a compression of

$$\frac{M_1 - M_0}{h},$$

and a bending of M_0 throughout its length. The *floor-beam* sustains, beside a bending of

$$M_1 + M,$$

a direct stress of the same amount but of opposite kind as that in the upper strut. The cross-section of the floor-beam is, however, generally so great that the direct stress produced in the same need hardly ever to be taken into account.

If the upper strut were either hinged at both ends (Fig. 86) or its section so small that I_1 may be entirely neglected, then M_0 would disappear and

$$M_1 = -\frac{\int_0^b \frac{Mdx}{I_3}}{\left(\frac{2h}{3I_2}+\frac{b}{I_3}\right)} \quad . \quad . \quad . \quad . \quad . \quad (234)$$

The upper strut would be free of moment, and the direct stress simply equals

$$\frac{M_1}{h}.$$

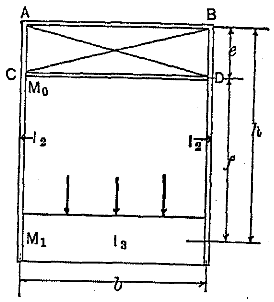

Fig. 87

114. In the bracing of Fig. 87 with designations as given, by neglecting the effect of all direct stresses, we get for the work of resistance,

$$\omega = \frac{1}{EI_2}\int_0^f\left(M_1 + \frac{M_0 - M_1}{f}x\right)^2 dx + \frac{1}{EI_2}\int_0^e\left(\frac{M_0}{e}x\right)^2 dx$$
$$+ \frac{1}{2\,EI_3}\int_0^b (M_1 + M)^2\,dx.$$

Differentiating this with respect to M_0 and M_1 successively, and setting the differential coefficients equal to zero, we get,

$$fM_1 + 2\,(f + e)\,M_0 = 0,$$

or

$$M_0 = -\frac{f}{2\,h}M_1,$$

and

$$\frac{f}{I_2}\left(\frac{M_0}{3} + \frac{2\,M_1}{3}\right) + \frac{1}{I_3}\left(bM_1 + \int_0^b Mdx\right) = 0.$$

Combining the two equations, we obtain,

$$M_1 = -\frac{\frac{1}{I_3}\int_0 Mdx}{\frac{f(3\,h + e)}{6\,hI_2} + \frac{b}{I_3}} \quad . \quad . \quad . \quad . \quad (235)$$

$$M_0 = \frac{\frac{f}{2\,hI_3}\int_0^b Mdx}{\frac{f(3\,h + e)}{6\,hI_2} + \frac{b}{I_3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (236)$$

The integral $\int_0^b Mdx$ is the same as in the preceding case.

The stresses in the struts are,

$$\overline{AB} = -\frac{M_0}{e},$$

$$\overline{CD} = \frac{M_0}{e} + \frac{M_0 - M_1}{f}.$$

EXAMPLE. — (1) In a single-track railway bridge with the cross-section of Fig. 88, given:

$$I_1 = 1600 \text{ in.}^4$$
$$I_2 = 800 \text{ in.}^4$$
$$I_3 = 4800 \text{ in.}^4$$
$$W = 60{,}000 \text{ lbs.}$$
Ext. width of post = 15 in.

required to find the maximum fibre stress caused in the posts.

From Eqs. (230) and (231),

$$M_0 = -\frac{\frac{20\,(16^2 - 6^2)}{12 \times 800 \times 4800}}{\left(\frac{16}{1600} + \frac{40}{2400}\right)\left(\frac{40}{2400} + \frac{16}{4800}\right) - \left(\frac{20}{2400}\right)^2}60{,}000$$
$$= -12{,}350 \text{ ft.-lbs.}$$

$$M_1 = \frac{\frac{256 - 36}{4 \times 4800}\left(\frac{16}{1600} + \frac{40}{2400}\right)}{\frac{668}{1{,}440{,}000}}60{,}000 = 39{,}520 \text{ ft.-lbs.}$$

Consequently the greatest fibre stress in the post is in this case found near to its lower end, and is in amount,

$$\frac{39{,}520 \times 12 \times 7.5}{800} = 3700 \text{ lbs. per sq. in.}$$

(2) In a highway bridge with cross-section as shown in Fig. 89, to calculate the stresses in the post and struts due to

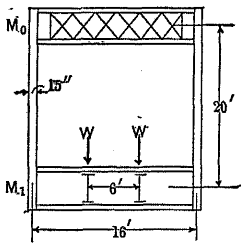

Fig. 88

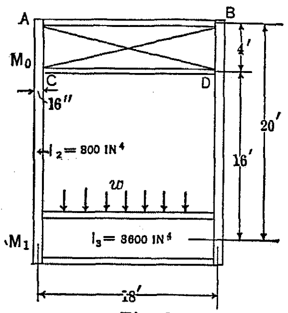

Fig. 89

constraint caused by a uniform load of $w = 3000$ lbs. per ft. run, the dimensions being as given in the figure.

From Eqs. (235) and (236),

$$M_1 = \frac{\dfrac{3000 \times 18^3}{12 \times 3600}}{\dfrac{16\,(60+4)}{6 \times 20 \times 800} + \dfrac{18}{3600}} = +\,25{,}850 \text{ ft.-lbs.}$$

$$M_0 = -\frac{\dfrac{16 \times 3000 \times 18^3}{2 \times 12 \times 20 \times 3600}}{\dfrac{16\,(60+4)}{6 \times 20 \times 800} + \dfrac{18}{3600}} = -\,10{,}340 \text{ ft.-lbs.}$$

The greatest fibre stress in the post will then be,

$$\frac{25{,}850 \times 12 \times 8}{800} = 3102 \text{ lbs. per sq. in.}$$

The stresses in the upper struts are,

$$\overline{AB} = \frac{10{,}340}{4} = +\,2585 \text{ lbs.}$$

$$\overline{CD} = -\frac{10{,}340}{4} - \frac{10{,}340 + 25{,}850}{16} = -\,4847 \text{ lbs.}$$

(3) Suppose, in the case of Fig. 88 and Example (1), the upper strut to be hinged at both ends; then from Eq. (234) we get,

$$M_1 = \frac{\dfrac{W}{4\,I_3}(b^2 - a^2)}{\dfrac{2\,h}{3\,I_2} + \dfrac{b}{I_3}} = \frac{\dfrac{60{,}000\,(16^2 - 6^2)}{4 \times 4800}}{\dfrac{2 \times 20}{3 \times 800} + \dfrac{16}{4800}} = 34{,}375 \text{ ft.-lbs.},$$

showing that the posts receive somewhat greater stresses from constraint, when the upper strut is firmly fixed to them.

115. By Wind Pressure. — Wind pressure also produces moment and direct stresses in posts forming a rigid frame, whenever the panel wind pressure is to be transferred from one chord to another, — the distortion of the frame being as shown in Fig. 90, in which the wind pressures, acting at the upper panel points are being brought down to the lower. This is simply a somewhat more general case of wind pressure discussed in Art. 20. From what has been there said, it will be easy to see that here again the points of contraflexure in the posts may be assumed to be at the same height.

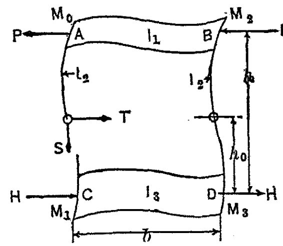

Fig. 90

Assuming the wind pressures $2P$ to be resisted equally at C and D, we have

$$H = P.$$

Passing now a section through the point of contraflexure in the left-side post, and representing with T and S the tangential and direct stresses acting at the section, we have as before,

$$T = H.$$

Taking moments successively at A, C, O, B, and D, we get,

$$\begin{aligned} M_0 &= - T(h - h_0) = - H(h - h_0), \\ M_1 &= Hh_0, \\ - Sb - 2P(h - h_0) &= 0, \\ M_2 &= - T(h - h_0) - Sb = H(h - h_0) = - M_0, \\ M_3 &= Th_0 - Sb - 2Ph = - Hh_0 = - M_1. \end{aligned}$$

Represent by

m_1 the moment at any point in the upper strut.
m_2 the moment at any point in the post.
m_3 the moment at any point in the floor-beam.

Neglecting the influence of all direct stresses as being inconsiderable when compared with moments, the total work of resistance due to the latter will be,

$$\omega = \int_0^b \frac{m_1^2 dx}{2EI_1} + 2\int_0^h \frac{m_2^2 dx}{2EI_2} + \int_0^b \frac{m_3^2 dx}{2EI_3}.$$

Referring to Fig. 90 and to the preceding equations, we have,

$$m_1 = M_0 + \frac{M_2 - M_0}{b} x = H(h - h_0)\left(\frac{2x}{b} - 1\right), \text{ origin of } x \text{ taken at } A.$$

$$m_2 = M_0 + \frac{M_1 - M_0}{h} x = H(x + h_0 - h), \text{ origin of } x \text{ taken at } A.$$

$$m_3 = M_1 + \frac{M_3 - M_1}{b} x = Hh_0\left(1 - \frac{2x}{b}\right), \text{ origin of } x \text{ taken at } C.$$

Substituting these values of m in the expression for work, and integrating, we get,

$$\omega = \frac{H^2(h - h_0)^2 b}{6EI_1} + \frac{H^2 h(h^2 - 3hh_0 + 3h_0^2)}{3EI_2} + \frac{H^2 h_0^2 b}{6EI_3}.$$

Since the value of h_0 must be such as to make the internal work a minimum, the first derivative of ω with respect to h_0 set equal to zero will at once give,

$$h_0 = \frac{\dfrac{b}{I_1} + \dfrac{3h}{I_2}}{\dfrac{b}{I_1} + \dfrac{6h}{I_2} + \dfrac{b}{I_3}} h \quad . \quad . \quad . \quad . \quad . \quad (237)$$

This value of h_0 substituted in the foregoing equations of moments, will give moments at all points of each member. The direct stress in the post is nothing else than

$$S = \frac{2P(h - h_0)}{b}.$$

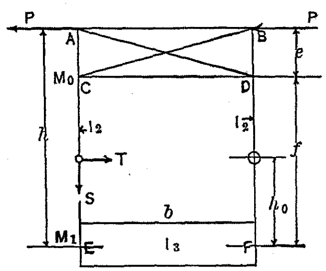

Fig. 91

116. With the bracing of Fig. 91, by referring back to Art. 22, it will at once be seen that here again

$$T = H = \frac{P + P}{2},$$

and taking moments at C, E, and O,

$$\begin{aligned} M_0 &= - T(f - h_0) = - H(f - h_0), \\ M_1 &= Th_0 = Hh_0, \\ - Sb - 2P(h - h_0) &= 0. \end{aligned}$$

Then at any point distant x from E we have the following moments in the post and floor-beam:

$$E \text{ to } C \quad . \quad . \quad M_1 + \frac{M_0 - M_1}{f} x = H\,(h_0 - x),$$

$$C \text{ to } A \quad . \quad . \quad M_0 - \frac{M_0}{e}(x - f) = H\,(f - h_0)\left(\frac{x-f}{e} - 1\right),$$

$$E \text{ to } F \quad . \quad . \quad M_1 - \frac{2\,M_1}{b} x = Hh_0\left(1 - \frac{2\,x}{b}\right).$$

Neglecting the influence of all direct stresses, the total internal work may now be written,

$$\omega = \frac{H^2}{EI_2}\left\{\int_0^f (h_0 - x)^2\,dx + \int_f^h (f - h_0)^2\left(\frac{x - h^2}{e}\right)dx\right\}$$
$$+ \frac{H^2 h_0^{\,2}}{2\,EI_3}\int_0^b \left(1 - \frac{2\,x}{b}\right)^2 dx.$$

Integrating, we get,

$$\omega = \frac{H^2}{EI_2}\left\{f\left(h_0^{\,2} - h_0 f + \frac{f^2}{3}\right) + \frac{e}{3}(f - h_0)^2\right\} + \frac{H^2 h_0^{\,2} b}{6EI_3}\,.$$

Setting the first derivative of ω with respect to h_0 equal to zero, we get,

$$h_0 = \frac{\dfrac{f}{I_2}(2\,h + f)}{\dfrac{2}{I_2}(h + 2f) + \dfrac{b}{I_3}} \quad . \quad . \quad . \quad . \quad . \quad (238)$$

If we make $I_3 = \infty$ in this, the equation would revert to that of the case of posts with the lower ends fixed — discussed in Art. 22.

Examples. — In a through bridge 18 feet in width, an upper panel wind pressure amounting to 18,000 lbs. is to be brought down to the lower panel point at an intermediate

point in the truss. To calculate the stresses produced in the posts and bracing.

In the case of Fig. 92 from Eq. (237),

$$h_0 = \frac{\dfrac{18}{3600} + \dfrac{3 \times 18.5}{2400}}{\dfrac{18}{3600} + \dfrac{6 \times 18.5}{2400} + \dfrac{18}{12{,}000}} \times 18.5 = 9.9 \text{ ft.},$$

so that

$$M_0 = -\,9000\,(18.5 - 9.9) = -\,77{,}400 \text{ ft.-lbs.},$$
$$M_1 = 9000 \times 9.9 = 89{,}100 \text{ ft.-lbs};$$

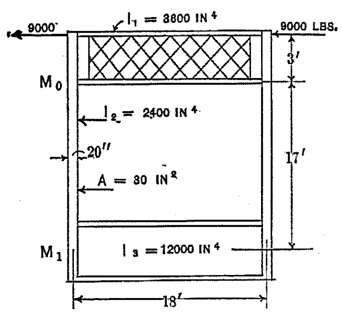

Fig. 92

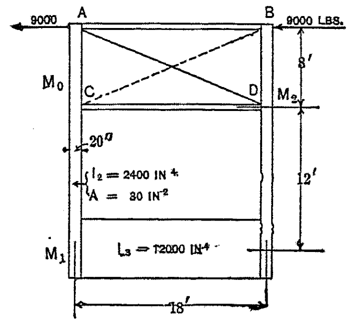

Fig. 93

and since, by using the designation in Fig. 90,

$$-\,Sb - 2\,P\,(20 - h_0) = 0,$$

$$S = -\,\frac{18{,}000\,(20 - 9.9)}{18} = 10{,}100 \text{ lbs.}$$

The maximum stress in the post will therefore be

$$\frac{M_1}{I_2} \times 10 \pm \frac{S}{A} = \frac{89{,}100 \times 12}{2400} \times 10 + \frac{10{,}100}{30} = 4792 \text{ lbs. per sq. in.}$$

The stress in the upper strut due to bending is

$$\frac{77{,}400 \times 12}{3600} \times 18 = 4644 \text{ lbs. per sq. in.}$$

In the case of Fig. 93, we have, from Eq. (238),

$$h_0 = \frac{\frac{12}{2400}(40 + 12)}{\frac{2}{2400}(20 + 24) + \frac{18}{12,000}} = 6.8 \text{ ft.},$$

so that

$$M_0 = -9000(12 - 6.8) = -46,800 \text{ ft.-lbs.},$$
$$M_1 = 9000 \times 6.8 = 61,200 \text{ ft.-lbs.},$$
$$S = -\frac{18,000(20 - 6.8)}{18} = 13,200 \text{ lbs.}$$

The maximum stress in the post will therefore be

$$\frac{61,200 \times 12}{2400} \times 10 + \frac{13,200}{30} = 3500 \text{ lbs. per sq. in.}$$

The stresses in the bracing are obtained in the following manner:

Passing a section through bracings and one of the points of contraflexure, as shown in Fig. 94, and considering the left portion, we have, by taking moment at A,

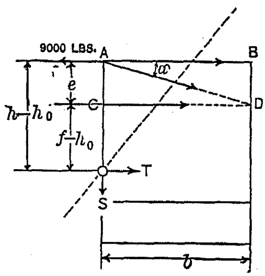

Fig. 94

$$-T(h - h_0) - \overline{CD}e = 0,$$
$$\overline{CD} = -\frac{9000(20 - 6.8)}{8} = -14,850 \text{ lbs.}$$

Taking moment at D,

$$-T(f - h_0) - Sb - 9000e + \overline{AB}e = 0,$$
$$-H(f - h_0) + 2H(h - h_0) - 9000e + \overline{AB}e = 0,$$
$$\overline{AB} = -\frac{9000(20 - 6.8)}{8} = -14,850 \text{ lbs.}$$

Since at the section Σ vert. forces $= 0$,

$$-S - \overline{AD}\sin\alpha = 0,$$
$$\overline{AD} = \frac{2H(h - h_0)}{b\sin\alpha} = \frac{18,000(20 - 6.8)}{18 \times .4062} = 32,500 \text{ lbs.}$$