# CHAPTER VI

## SUSPENSION BRIDGES

**99.** SUSPENSION bridges are lacking in rigidity besides being expensive of construction, and for that reason, those of moderate spans are but rarely constructed. It suffices to state that suspension bridges with trussed links are nothing more than inverted arches, and are to be treated in almost exactly the same manner as the latter of corresponding forms, with change in signs of stresses, and taking into consideration the motion of supports. This last condition brings the following term into the expression for work of resistance,

$$\tfrac{1}{2} H \Delta l,$$

in which $\Delta l$ represents the change in span length due to the motion of supports caused by the change in length of the anchor ties. In the suspension bridge of Fig. 73, we have,

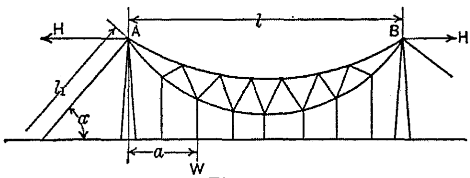

Fig. 73

$$\Delta l = 2 \Delta l_1 \sec \alpha.$$

Neglecting the frictional resistance to the motion of saddles on the towers or of the pin (in case the towers are hinged at the base), we get,

$$\Delta l_1 = \frac{H l_1 \sec \alpha}{E A_a},$$

in which $A_a$ represents the sectional area of the anchor tie. Whence

$$\tfrac{1}{2} H \Delta l = \frac{H^2 l_1 \sec^2 \alpha}{EA_a}.$$

Neglecting the compression of towers, we have, by referring to Art. 45, the following expression for $H$ due to one load $W$:

$$H = \frac{1}{2} \frac{\int_0^{a'} \frac{xydc}{I} + a \int_{a'}^{\frac{l'}{2}} \frac{ydc}{I} - \int_0^{a} \frac{\sin \phi \, dx}{A}}{\int_0^{\frac{l'}{2}} \frac{y^2dc}{I} + \int_0^{\frac{l}{2}} \frac{\cos \phi \, dx}{A} + \frac{l_1 \sec^2 \alpha}{A_a}} W. \quad . \quad . \quad . \quad (215)$$

For uniform temperature change $t$, since the actual horizontal displacements of $A$ and $B$ amount to

$$2\, t\theta l_1 \sec \alpha,$$

we have, by referring to Art. 46, the following equation:

$$H_t = \frac{t\theta E\,(l + 2\, l_1 \sec \alpha)}{\int_0^{l'} \frac{y^2dc}{I} + \int_0^{l} \frac{\cos \phi \, dx}{A}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (216)$$

**100.** For a long span, which is the proper field for a suspension bridge, the latter, as generally constructed, consists of cables and stiffening trusses, forming a composite system of construction.

**101. Cable Suspension Bridge with Continuous Stiffening Trusses.**—No mathematically correct method of calculating stresses in this kind of suspension bridge has, as yet, been made practicable,—most formulas used in practice being more or less rough approximations. The following method of calculation is another of the latter kind.

The cable—being given a uniform cross-section—forms, under its own weight, a catenary curve; since,



however, the weight of the floor system and stiffening trusses, which is generally uniformly distributed along the horizontal line, is far in excess of that of the cable itself, the curve actually assumed by the latter is closely

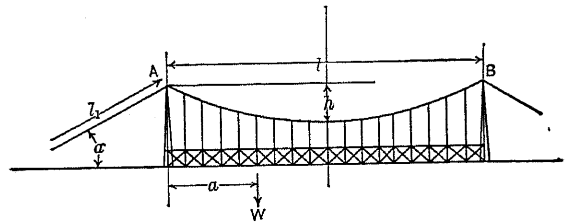


Fig. 74

allied to a parabola, especially as the suspender rods are so adjusted as to produce this form as nearly as possible. The entire weight of the stiffening girder in normal condition and at mean temperature, when not loaded, may thus be assumed to be borne by the cable, the ends of the girder touching the supports but producing no reaction. If now a load $W$ be placed on the girder, the elongation of the cable and suspenders will take place, and in consequence the stiffening girder will bear on the supports. A part of $W$ will then be borne by the stiffening girder and be transmitted directly to the supports, while the remainder goes through suspenders to the cable. For simplicity of calculation, it will be assumed in the following that the tension in suspenders due to any loading will be uniform, as in the case of dead loads. Such an assumption—although never strictly correct—is permissible considering the considerable rigidity generally given to the stiffening trusses.

Let

$p$ = the pull in the suspenders per unit length of the span.
$H$ = the horizontal component of tension in the cable due to $p$.
$x, y$ = coördinates with origin at $A$.
$c$ = length measured along the cable.
$l$ = the distance between the towers.
$l'$ = the total length of the cable between the towers.
$l_1$ = the length of the anchor cable.
$A_c$ = the cross-section of the cables.
$I$ = the moment of inertia of the stiffening trusses.

Referring, then, to Fig. 74, we get the following works of resistance:

(1) Work in the cable. Passing a section through the centre of the span, and taking moment at $A$ or $B$, we have,

$$H = \frac{pl^2}{8h}.$$

Since $p$ is assumed to be acting vertically, $H$ will be constant throughout. The stress in the cable at any point $x$ will then be,

$$H\frac{dc}{dx},$$

and the work of resistance due to the same,

$$2\int_0^{\frac{l'}{2}} \frac{\left(H\frac{dc}{dx}\right)^2 dc}{2A_cE}.$$

Since

$$dc = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \frac{dy^2}{dx^2}},$$



introducing in this, the equation of parabola

$$y = \frac{4h}{l^2}x(l - x),$$

we get,

$$dc = \left\{1 + \frac{8h^2}{l^4}(l - 2x)^2\right\}dx, \text{ nearly.}$$

Substituting and integrating, we get for the total work of resistance in the cable the following approximate expression:

$$\frac{H^2(l^2 + 8h^2)}{2A_cEl},$$

or

$$\frac{\left(\frac{pl^2}{8h}\right)^2(l^2 + 8h^2)}{2A_cEl}.$$

(2) Work in anchor cables. If we neglect the resistance offered to the motion of the saddle, the tension in the anchor cable would be,

$$H \sec \alpha,$$

and the work of resistance in the same,

$$2\int_0^{l_1}\frac{(H \sec \alpha)^2 dc}{2EA_c} = \frac{p^2l^4l_1 \sec^2 \alpha}{64EA_ch^2}.$$

(3) Work in suspenders. Denoting with $L$ the mean length of all the suspenders, and with $A_s$ their total cross-sectional area, we at once get for the work in the suspenders the following expression:

$$\frac{(pl)^2L}{2A_sE}.$$

This term is generally so small compared with others, forming the total work of resistance, that it may be entirely neglected without appreciable error.

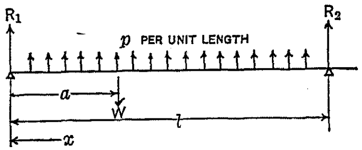


Fig. 75

(4) Work in stiffening trusses. Let $R_1$ and $R_2$ represent the end reactions (Fig. 75) of the stiffening truss. Taking moments at each support successively, we get,

$$R_1 = \frac{W(l-a)}{l} - \frac{pl}{2},$$

$$R_2 = \frac{Wa}{l} - \frac{pl}{2},$$

also

$$R_1 + R_2 + pl = W.$$

At any point distant $x$ from the left end, we have for moment,

$$m = R_1 x + \frac{px^2}{2} = \frac{W(l-a)}{l}x - \frac{px}{2}(l-x) \quad \text{for } x < a.$$

$$m = R_1 x + \frac{px^2}{2} - W(x-a) = \frac{Wa(l-x)}{l} - \frac{px(l-x)}{2} \quad \text{for } x > a.$$

Neglecting the deformation of web-members, we get for work of resistance, the following expression:

$$\int_0^l \frac{m^2 dx}{2IE} = \int_0^a \frac{\left\{\frac{W(l-a)}{l}x - \frac{px(l-x)}{2}\right\}^2 dx}{2IE} + \int_a^l \frac{\left\{\frac{Wa(l-x)}{l} - \frac{px(l-x)}{2}\right\}^2 dx}{2IE}.$$



Assuming $I$ and $E$ to be uniform throughout, we get for the integrals,

$$\frac{40W^2a^2(l-a)^2 - 10Wpla(l^3 - 2a^2l + a^3) + p^2l^6}{240IEl}.$$

If we neglect the deformations produced in the towers and the anchorages, we have for the total work of resistance in the structure,

$$\omega = \frac{p^2l^3(l^2 + 8h^2)}{128h^2A_cE} + \frac{p^2l^4l_1\sec^2\alpha}{64A_cEh^2} + \frac{40W^2a^2(l-a)^2 - 10Wpla(l^3 - 2a^2l + a^3) + p^2l^6}{240IEl}.$$

Setting the first derivative of $\omega$ with respect to $p$ equal to zero, we get,

$$p = \frac{a(l^3 - 2a^2l + a^3)}{\frac{3l^3I}{8h^2A_c}(l^2 + 8h^2 + 2ll_1\sec^2\alpha) + \frac{l^5}{5}}W \quad . \quad . \quad (217)$$

For a uniform load $w$ per unit length, we have but to put $wda$ instead of $W$, and integrate between given limits of loading. Thus, for the load extending for $a_1$ from the left support, we have,

$$p = \frac{a_1^2(5l^3 - 5a_1^2l + 2a_1^3)w}{\frac{15l^3I}{4h^2A_c}(l^2 + 8h^2 + 2ll_1\sec^2\alpha) + 2l^5} \quad . \quad . \quad (218)$$

**102.** The following stresses may now be written: The *stress in the cable* — maximum at the towers,

$$H\sec\phi_0 = \frac{1}{8}\frac{pl^2}{h}\sec\phi_0.$$

$\phi_0$ being the inclination of the tangent at towers to the horizontal.

The *stress in the anchor cable,*

$$\frac{1}{8}\frac{pl^2}{h}\sec\alpha, \text{ approximately.}$$

The *stress in the suspender,*

$$\frac{pl}{n},$$

$n$ denoting the number of panels into which equi-distant suspenders divide the span.

In the *stiffening truss* at any point distant $x$ from the left end,

$$\text{Shear} \begin{array}{ll} = R_1 + px & \text{for } x < a \\ = R_1 + px - W & \text{for } x > a \end{array} \Bigg\} \text{ due to } W.$$

$$\begin{array}{ll} = R_1 + (p - w)\,x & \text{for } x < a_1 \\ = R_1 + px - wa_1 & \text{for } x > a_1 \end{array} \Bigg\} \text{ due to } w.$$

$$\text{Moment} \begin{array}{ll} = R_1x + \frac{px^2}{2} & \text{for } x < a \\ = R_1x + \frac{px^2}{2} - W(x - a) & \text{for } x > a \end{array} \Bigg\} \text{ due to } W.$$

$$\begin{array}{ll} = R_1x + \frac{(p - w)}{2}x^2 & \text{for } x < a_1 \\ = R_1x + \frac{px^2}{2} - a_1w\left(x - \frac{a_1}{2}\right) & \text{for } x > a_1 \end{array} \Bigg\} \text{ due to } w.$$

**103. Temperature Stresses.** — It hardly requires explanation that a rising temperature tends to strain the stiffening trusses and at the same time relieve the cables, while falling temperature strains both cables and trusses.

Since,

$$l' = \int_0^{l'} dc = l\left(1 + \frac{8h^2}{3l^2}\right), \text{ nearly (see page 143),}$$



a uniform change of $t$ degrees — positive for rise — changes the length of the main-span cable by

$$t\theta l' = t\theta l\left(1 + \frac{8h^2}{3l^2}\right), \text{ nearly.}$$

Representing by $h_1$ the sag of the deformed cable, we have,

$$l(1 + t\theta)\left(1 + \frac{8h^2}{3l^2}\right) = l\left(1 + \frac{8h_1^2}{3l^2}\right),$$

from which

$$h_1 = h\left\{1 + \frac{t\theta}{2}\left(1 + \frac{3}{8}\frac{l^2}{h^2}\right)\right\}, \text{ nearly.}$$

The deformed cable would then deflect upward or downward by about

$$\frac{ht\theta}{2}\left(1 + \frac{3}{8}\frac{l^2}{h^2}\right) = \frac{3}{16}\frac{t\theta l^2}{h}, \text{ approximately.}$$

The change in the length of anchor cables due to $t$ being $2\,t\theta l_1$, will change the main-span length by

$$-\,2\,t\theta l_1 \sec\alpha.$$

Representing by $h_2$, the sag of the cable with changed span length, we have,

$$(l - 2\,\theta t l_1 \sec\alpha)\left\{1 + \frac{8h_2^2}{3(l - 2\,\theta t l_1 \sec\alpha)^2}\right\} = l\left(1 + \frac{8h^2}{3l^2}\right),$$

so that the deflection of the cable due to this cause alone would be

$$\frac{3}{8}\frac{l}{h}\theta t l_1 \sec\alpha, \text{ approximately.}$$

The total deflection of the cable, due to a temperature change of $t$, were the latter free to deflect, would then be,

$$\delta = \frac{3}{16}\frac{l}{h}t\theta(l + 2\,l_1\sec\alpha) \quad \ldots \ldots \quad (219)$$

This, however, could not take place owing to the rigidity of the truss; but as the difference is generally slight, we shall assume $\delta$ to be the total deflection.

Neglecting the changes in lengths of suspenders, and further assuming the stress produced in the latter by the temperature change to be uniform, and calling it $p_t$ per unit length of the span, we get,

$$p_t = \frac{384\,EI}{5\,l^4}\,\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (220)$$

This uniform pull on the cable produces tension in the latter, whose horizontal component is

$$H_t = \frac{48\,EI}{5\,hl^2}\,\delta,$$

and in the stiffening truss a moment of

$$\frac{48\,EI}{5\,l^2}\,\delta$$

at the centre of the span.

Each suspender sustains a pull due to this cause of

$$\frac{p_t l}{n}.$$

**104.** There is still another factor, not considered in the preceding discussion, viz.: the change in the amount of $H$ due to variation of the deflection of the cable from whatever cause. Thus, if at the normal temperature

$$H = \frac{pl^2}{8\,h},$$

a change of $t$ degrees would give

$$H = \frac{pl^2}{8\,h\left\{1 + \frac{t\theta}{2}\left(1 + \frac{3}{8}\frac{l^2}{h^2}\right)\right\}} \quad . \quad . \quad . \quad . \qquad (221)$$



TRUSSES WITH REDUNDANT MEMBERS

**105.** The different forms of trusses with redundant members may be indefinitely multiplied. They are generally decomposable into as many simple trusses as there are systems of such members to form them. The following few simple cases will sufficiently explain a mode of procedure for calculating the stresses in this kind of trusses.

**106.** Fig. 76 shows an ordinary lattice truss. The

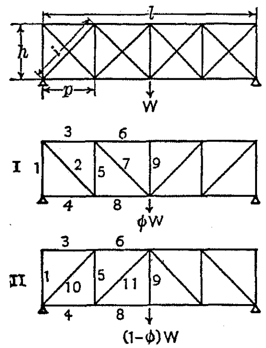


Fig. 76

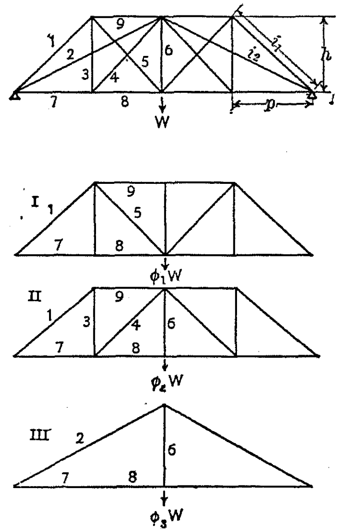


Fig. 77

truss is evidently decomposable into two simple trusses $I$ and $II$, — all the members excepting diagonals being common to both trusses.

A load $W$ hung at the middle panel point must — according to the principle of least work — be divided between trusses $I$ and $II$ in such a way that the internal work performed in the truss will be a minimum.

Representing by $\phi$ the ratio of the load borne by truss $I$ to the whole, we have the following stresses:

| Members. | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Trusses. I | $-\frac{1}{2}\phi W$ | $+\frac{i}{2h}\phi W$ | $-\frac{p}{2h}\phi W$ | 0 | $-\frac{1}{2}\phi W$ | $-\frac{p}{h}\phi W$ |
| II | 0 | . . . | 0 | $+\frac{p}{2h}(1-\phi) W$ | $-\frac{1}{2}(1-\phi) W$ | $-\frac{p}{2h}(1-\phi) W$ |

| Members. | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|
| Trusses. I | $+\frac{i}{2h}\phi W$ | $+\frac{p}{2h}\phi W$ | 0 | . . . . . | . . . . |
| II | . . . | $+\frac{p}{h}(1-\phi) W$ | $+(1-\phi) W$ | $-\frac{i}{2h}(1-\phi) W$ | $-\frac{i}{2h}(1-\phi) W$ |

Denoting by $A$ with corresponding suffixes the cross-sectional areas of members, we get for the internal work in the trusses the following:

$$\frac{\Sigma S^2 L}{2AE} = \frac{W^2}{4Eh^2}\left\{\phi^2\left(\frac{h^3}{A_1}+\frac{i^3}{A_2}+\frac{i^3}{A_7}+\frac{p^3}{A_3}\right)+(1-\phi)^2\left(\frac{2h^3}{A_9}+\frac{i^3}{A_{10}}+\frac{i^3}{A_{11}}+\frac{p^3}{A_4}\right)\right.$$
$$\left.+\frac{h^3}{A_5}+(1+\phi)^2\frac{p^3}{A_6}+(2-\phi)^2\frac{p^3}{A_8}\right\} = \omega.$$

Since the value of $\phi$ must be such as to make the internal work a minimum, differentiating $\omega$ with respect to $\phi$, and setting the differential coefficient equal to zero, we at once obtain,

$$\phi = \frac{h^3\left(\frac{2}{A_9}\right)+i^3\left(\frac{1}{A_{10}}+\frac{1}{A_{11}}\right)+p^3\left(\frac{1}{A_4}-\frac{1}{A_6}+\frac{2}{A_8}\right)}{h^3\left(\frac{1}{A_1}+\frac{2}{A_9}\right)+i^3\left(\frac{1}{A_2}+\frac{1}{A_7}+\frac{1}{A_{10}}+\frac{1}{A_{11}}\right)+p^3\left(\frac{1}{A_3}+\frac{1}{A_4}+\frac{1}{A_6}+\frac{1}{A_8}\right)}.$$



**107.** Fig. 77 shows another case of a truss with redundant members, formed evidently by a combination of Pratt, Howe and King-post trusses. In order to find the stresses in different members of the truss, decompose the latter, as before, into elementary trusses $I$, $II$ and $III$, and denote by $\phi_1$, $\phi_2$ and $\phi_3$ the ratios of distribution of $W$ in $I$, $II$ and $III$ respectively, so that

$$\phi_1 + \phi_2 + \phi_3 = 1.$$

The following stresses may now be written:

| Members. | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Trusses. I | $-\frac{i_1}{2h}\phi_1 W$ | . . . | . . . | . . . . | $+\frac{i_1}{2h}\phi_1 W$ |
| II | $-\frac{i_1}{2h}\phi_2 W$ | . . . | $+\frac{1}{2}\phi_2 W$ | $-\frac{i_1}{2h}\phi_2 W$ | . . . . |
| III | . . . . | $-\frac{i_2}{2h}\phi_3 W$ | . . . . | . . . | . . . . |

| Members. | 6 | 7 | 8 | 9 | |
|---|---|---|---|---|---|
| Trusses. I | . . . . | $+\frac{p}{2h}\phi_1 W$ | $+\frac{p}{2h}\phi_1 W$ | $-\frac{p}{h}\phi_1 W$ | . . . . |
| II | $+\phi_2 W$ | $+\frac{p}{2h}\phi_2 W$ | $+\frac{p}{h}\phi_2 W$ | $-\frac{p}{2h}\phi_2 W$ | . . . . |
| III | $+\phi_3 W$ | $+\frac{p}{h}\phi_3 W$ | $+\frac{p}{h}\phi_3 W$ | . . . . | . . . . |

Denoting by $A$ with suffixes the cross-sections of the members, we get for $\omega$,

$$\frac{\Sigma S^2L}{2AE} = \frac{W^2}{2\,EH^2}\left\{\frac{i_1^{\,3}(\phi_1+\phi_2)^2}{A_1} + \frac{i_2^{\,3}(1-\phi_1-\phi_2)^2}{A_2} + \frac{h^3\phi_2^{\,2}}{A_3} + \frac{i_1^{\,3}\phi_2^{\,2}}{A_4} + \frac{i_1^{\,3}\phi_1^{\,2}}{A_5}\right.$$
$$\left. + \frac{h^3(1-\phi_1)^2}{A_6} + \frac{p^3(2-\phi_1-\phi_2)^2}{A_7} + \frac{p^3(2-\phi_1)^2}{A_8} + \frac{p^3(2\,\phi_1+\phi_2)^2}{A_9}\right\} = \omega.$$

Making

$$\frac{d\omega}{d\phi_1} = 0,$$
$$\frac{d\omega}{d\phi_2} = 0,$$

we obtain at once the following equations:

$$\phi_1\left\{i_1^{\,3}\left(\frac{1}{A_1}+\frac{1}{A_5}\right)+\frac{i_2^{\,3}}{A_2}+\frac{2\,h^3}{A_6}+p^3\left(\frac{1}{A_7}+\frac{1}{A_8}+\frac{4}{A_9}\right)\right\}+\phi_2\left\{\frac{i_1^{\,3}}{A_1}+\frac{i_2^{\,3}}{A_2}\right.$$
$$\left.+p^3\left(\frac{1}{A_7}+\frac{2}{A_9}\right)\right\}-\left\{\frac{i_2^{\,3}}{A_2}+\frac{2\,h^3}{A_6}+2\,p^3\left(\frac{1}{A_7}+\frac{1}{A_8}\right)\right\}=0,$$

$$\phi_2\left\{\frac{i_1^{\,3}}{A_1}+\frac{i_2^{\,3}}{A_2}+p^3\left(\frac{1}{A_7}+\frac{2}{A_9}\right)\right\}+\phi_2\left\{i_1^{\,3}\left(\frac{1}{A_1}+\frac{1}{A_4}\right)+\frac{i_2^{\,3}}{A_2}+p^3\left(\frac{1}{A_7}+\frac{2}{A_9}\right)\right\}$$
$$-\left(\frac{i_2^{\,3}}{A_2}+\frac{2\,p^3}{A_7}\right)=0.$$

Representing the terms within the brackets by $\alpha$, $\beta$, $\gamma$, $\delta$ and $\epsilon$ in order, we get

$$\phi_1\alpha + \phi_2\beta = \gamma,$$
$$\phi_1\beta + \phi_2\delta = \epsilon,$$

from which

$$\phi_1 = \frac{\beta\epsilon - \gamma\delta}{\beta^2 - \alpha\delta},$$

$$\phi_2 = \frac{\beta\gamma - \alpha\epsilon}{\beta^2 - \alpha\delta},$$

$$\phi_3 = \frac{\beta(\beta-\gamma) + \epsilon(\alpha-\beta) + \delta(\gamma-\alpha)}{\beta^2 - \alpha\delta}.$$



As this chapter is the least important of all, considering the comparative rarity of the kinds of structures treated, it is deemed sufficient merely to indicate a mode of procedure in the calculation of stresses by means of the method of work.