CHAPTER V

ARCHES WITHOUT HINGES

73. IN this class of arches, since ends are fixed, there will be moments produced at these points whenever the resultant forces do not pass through them. Here we have, then, two more statically-indeterminate forces than in arches with two hinges.

Fig. 59 shows a symmetrical arch-rib loaded vertically with W.

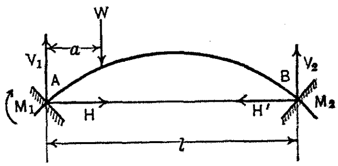

Fig. 59

Let M_1 and M_2 represent moments at A and B respectively. For all other designations, retaining those of the preceding chapter, we have, since the loading is vertical:

$$H - H' = 0,$$
$$V_1 + V_2 - W = 0.$$

For moment at any point distant x from A, we get,

$$m = M_1 + V_1 x - Hy, \text{ for } x < a,$$
$$m = M_1 + V_1 x - Hy - W(x - a), \text{ for } x > a;$$

for vertical shear,

$$V = V_1 \quad \text{for } x < a,$$
$$V = V_1 - W \text{ for } x > a;$$

and for the normal stress in the rib at x (Art. 43),

$$N = -(V \sin \phi + H \cos \phi).$$

Since the internal work in the arch-rib is generally (Art. 44)

$$\omega = \int_0^{l'} \frac{m^2 dc}{2\,EI} + \int_0^{l'} \frac{N^2 dc}{2\,AE},$$

substituting in this, the values of m and N, we get,

$$\omega = \int_0^{a'} \frac{(M_1 + V_1 x - Hy)^2\,dc}{2\,EI} + \int_{a'}^{l'} \frac{\{M_1 + V_1 x - Hy - W(x-a)\}^2\,dc}{2\,EI}$$

$$+ \int_0^{a'} \frac{(V_1 \sin\phi + H\cos\phi)^2 dc}{2\,EA} + \int_{a'}^{l'} \frac{\{(V_1 - W)\sin\phi + H\cos\phi\}^2\,dc}{2\,EA}.$$

Since H, M_1 and V_1 must successively make ω a minimum, we get for

$$\frac{d\omega}{dH} = 0, \qquad \frac{d\omega}{dM_1} = 0, \qquad \frac{d\omega}{dV_1} = 0,$$

the following equations:

$$M_1 \int_0^{l'} \frac{ydc}{I} + V_1\left(\int_0^{l'} \frac{xydc}{I} - \int_0^{l} \frac{\sin\phi dx}{A}\right) - H\left(\int_0^{l'} \frac{y^2dc}{I} + \int_0^{l} \frac{\cos\phi dx}{A}\right) - W\left(\int_{a'}^{l'} \frac{(x-a)\,ydc}{I} - \int_a^l \frac{\sin\phi dx}{A}\right) = 0. \quad (133)$$

$$M_1 \int_0 \frac{dc}{I} + V_1 \int_0^{l'} \frac{xdc}{I} - H \int_0^{l'} \frac{ydc}{I} - W \int_{a'}^{l'} \frac{(x-a)dc}{I} = 0. \quad (134)$$

$$M_1 \int_0^{l'} \frac{xdc}{I} + V_1\left(\int_0^{l'} \frac{x^2dc}{I} + \int_0^{l} \frac{\sin\phi dy}{A}\right) - H\left(\int_0^{l'} \frac{xydc}{I} - \int_0^{l} \frac{\cos\phi dy}{A}\right) - W\left(\int_{a'}^{l'} \frac{x(x-a)\,dc}{I} + \int_a^l \frac{\sin\phi dy}{A}\right) = 0. \quad (135)$$

These equations will give all the required values of M_1, H, and V_1, as soon as the form of the arch and mode of loading are known.

As to M_2 and V_2 we have,

$$M_2 = M_1 + V_1 l - W(l-a),$$
$$V_2 = W - V_1.$$

74. For obtaining expressions for H and M_1 only, it will be more convenient to assume two symmetrical loads (Fig. 60), as done in the case of two-hinged arches. Denoting the horizontal and vertical reactions and moments at A due to $2\,W$ by H', V', and M', we get,

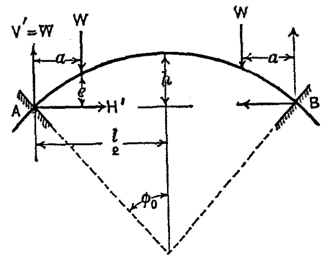

Fig. 60

$$H' = 2\,H,$$
$$M' = M_1 + M_2,$$
$$V' = V_1 + V_2 = W,$$

in which H, M_1, M_2, V_1, and V_2, denote the reactions and moments due to one W, as before.

Referring to the figure, we have for the total internal work in the arch,

$$\omega = 2\int_0^{a'} \frac{(M' + Wx - H'y)^2\,dc}{2\,EI} + 2\int_{a'}^{\frac{l'}{2}} \frac{(M' + Wa - H'y)^2\,dc}{2\,EI}$$

$$+ 2\int_0^{a'} \frac{(W\sin\phi + H'\cos\phi)^2\,dc}{2\,EA} + 2\int_{a'}^{\frac{l'}{2}} \frac{(H'\cos\phi)^2\,dc}{2\,EA},$$

whence for

$$\frac{d\omega}{dH'} = 0, \text{ and } \frac{d\omega}{dM'} = 0,$$

we get,

$$M' \int_0^{\frac{l'}{2}} \frac{ydc}{I} - H'\left(\int_0^{\frac{l'}{2}} \frac{y^2dc}{I} + \int_0^{\frac{l'}{2}} \frac{\cos^2\phi\,dc}{A}\right) + W\left(\int_0^{a'} \frac{xydc}{I} + \int_{a'}^{\frac{l'}{2}} \frac{aydc}{I} - \int_0^{a'} \frac{\sin\phi\cos\phi\,dc}{A}\right) = 0 \quad . \quad . \quad (136)$$

$$M' \int_0^{\frac{l'}{2}} \frac{dc}{I} - H' \int_0^{\frac{l'}{2}} \frac{ydc}{I} + W\left(\int_0^{a'} \frac{xdc}{I} + \int_{a'}^{\frac{l'}{2}} \frac{adc}{I}\right) = 0 \quad . \quad . \quad (137)$$

75. Temperature Stresses. — A uniform temperature change of t degrees would produce a change of $t\theta l$ in the span length of the arch — θ denoting the coefficient of expansion — were the end of the latter free to slide. Designating by

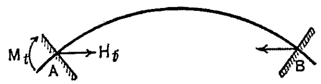

Fig. 61

H_t and M_t (Fig. 61)

the horizontal reaction and moment at A due to a temperature change — positive for rise — we get,

$$\omega = \int_0^{l'} \frac{(M_t - H_t y)^2\, dc}{2\, IE} + \int_0^{l'} \frac{(H_t \cos\phi)^2\, dc}{2\, AE}.$$

Since, according to the theorems of Castigliano (Art. 6),

$$\frac{d\omega}{dH_t} = t\theta l, \qquad \frac{d\omega}{dM_t} = 0,$$

we get

$$\int_0^{l'} \frac{-M_t y dc + H_t y^2 dc}{IE} + \int_0^{l'} \frac{H_t \cos^2\phi\, dc}{AE} = t\theta l,$$

$$\int_0^{l'} \frac{(M_t - H_t y)\, dc}{I} = 0,$$

from which

$$H_t = \frac{t\theta l E}{\int_0^{l'} \frac{y^2 dc}{I} + \int_0^{l} \frac{\cos\phi\, dx}{A} - \frac{\left(\int_0^{l'} \frac{ydc}{I}\right)^2}{\int_0^{l'} \frac{dc}{I}}}. \quad . \quad (138)$$

$$M_t = \frac{\int_0^{l'} \frac{ydc}{I}}{\int_0^{l'} \frac{dc}{I}} H_t . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

76. Stresses Due to Displacements of Supports. — The supports may sometimes yield to a certain extent, producing changes in their relative heights as well as the central angle and the span length of the arch.

Representing by

$$M_\Delta,\ V_\Delta,\ \text{and}\ H_\Delta,$$

the moment and the vertical and horizontal reactions at the left end of the arch, caused by such displacements, we get for the internal work in the arch,

$$\omega = \int_0^{l'} \frac{(M_\Delta + V_\Delta x - H_\Delta y)^2\, dc}{2\, EI} + \int_0^{l'} \frac{(H_\Delta \cos\phi + V_\Delta \sin\phi)^2 dc}{2\, EA}.$$

77. Let Δy = change in relative heights of supports — measured at the left support in the direction of the force, i.e., negative downward. Then, since the force acting through Δy is V_Δ only, according to the theorem of Castigliano, we have,

$$\frac{d\omega}{dV_\Delta} = \Delta y,$$

$$\frac{d\omega}{dH_\Delta} = 0,$$

$$\frac{d\omega}{dM_\Delta} = 0;$$

whence we get,

$$M_\Delta \int_0^{l'} \frac{xdc}{I} + V_\Delta \left(\int_0^{l'} \frac{x^2 dc}{I} + \int_0^{l'} \frac{\sin^2\phi dc}{A} \right) - H_\Delta \left(\int_0^{l'} \frac{xydc}{I} - \int_0^{l'} \frac{\cos\phi \sin\phi dc}{A} \right) = E\Delta y. \quad (140a)$$

$$- M_\Delta \int_0^{l'} \frac{ydc}{I} - V_\Delta \left(\int_0^{l'} \frac{xydc}{I} - \int_0^{l'} \frac{\sin\phi\ \cos\phi dc}{A} \right) + H_\Delta \left(\int_0^{l'} \frac{y^2 dc}{I} + \int_0^{l'} \frac{\cos^2\phi dc}{A} \right) = 0 \quad . \quad . \quad (140b)$$

$$M_\Delta \int_0^{l'} \frac{dc}{I} + V_\Delta \int_0^{l'} \frac{xdc}{I} - H_\Delta \int_0^{l'} \frac{ydc}{I} = 0 \quad . \quad . \quad . \quad . \quad (140c)$$

78. Next, let $\Delta\phi$ = total change of the central angle of the arch — measured at the left support in the sense of the moment, i.e., positive for the decrease of the central angle. Then, for similar reasons as before, we have,

$$\frac{d\omega}{dM_\Delta} = \Delta\phi,$$
$$\frac{d\omega}{dH_\Delta} = 0,$$
$$\frac{d\omega}{dV_\Delta} = 0;$$

whence,

$$M_\Delta\int_0^{l'}\frac{dc}{I} + V_\Delta\int_0^{l'}\frac{xdc}{I} - H_\Delta\int_0^{l'}\frac{ydc}{I} = E\Delta\phi \quad . \quad . \quad . \quad (141a)$$

$$-M_\Delta\int_0^{l'}\frac{ydc}{I} - V_\Delta\left(\int_0^{l'}\frac{xydc}{I} - \int_0^{l'}\frac{\sin\phi\cos\phi\, dc}{A}\right)$$
$$+H_\Delta\left(\int_0^{l'}\frac{y^2dc}{I} + \int_0^{l'}\frac{\cos^2\phi\, dc}{A}\right) = 0 \quad . \quad . \quad . \quad (141b)$$

$$M_\Delta\int_0^{l'}\frac{xdc}{I} + V_\Delta\left(\int_0^{l'}\frac{x^2dc}{I} + \int_0^{l'}\frac{\sin^2\phi\, dc}{A}\right)$$
$$-H_\Delta\left(\int_0^{l'}\frac{xydc}{I} - \int_0^{l'}\frac{\sin\phi\cos\phi\, dc}{A}\right) = 0 \quad . \quad (141c)$$

79. Finally, let Δl = total change in span length — measured at the left support in the direction of the force, i.e., positive for the decrease of the span length.

Here we have,

$$\frac{d\omega}{dH_\Delta} = \Delta l,$$
$$\frac{d\omega}{dV_\Delta} = 0,$$
$$\frac{d\omega}{dM_\Delta} = 0;$$

whence,

$$-M_\Delta\int_0^{l'}\frac{ydc}{I} - V_\Delta\left(\int_0^{l'}\frac{xydc}{I} - \int_0^{l'}\frac{\sin\phi\cos\phi\, dc}{A}\right)$$
$$+H_\Delta\left(\int_0^{l'}\frac{y^2dc}{I} + \int_0^{l'}\frac{\cos^2\phi\, dc}{A}\right) = E\Delta l. \quad . \quad (142a)$$

$$M_\Delta\int_0^{l'}\frac{xdc}{I} + V_\Delta\left(\int_0^{l'}\frac{x^2dc}{I} + \int_0^{l'}\frac{\sin^2\phi\, dc}{A}\right)$$
$$-H_\Delta\left(\int_0^{l'}\frac{xydc}{I} - \int_0^{l'}\frac{\sin\phi\cos\phi\, dc}{A}\right) = 0 \quad . \quad (142b)$$

$$M_\Delta\int_0^{l'}\frac{dc}{I} + V_\Delta\int_0^{l'}\frac{xdc}{I} - H_\Delta\int_0^{l'}\frac{ydc}{I} = 0 \quad . \quad . \quad . \quad . \quad . \quad (142c)$$

PARABOLIC ARCH WITHOUT HINGES

80. Assuming, as in the case of two-hinged arches (Art. 48), the cross-section of the rib to so vary from the crown toward each end that at any section

$$I = I_0 \sec\phi,$$
$$A = A_0 \sec\phi,$$

(I_0 and A_0 denoting the moment of inertia and cross-section of the rib at the crown), and introducing these together with the equation of parabola

$$y = \frac{4h}{l^2}x(l - x)$$

in Eqs. (136) and (137), and integrating

$$\int_0^{\frac{l'}{2}}\frac{ydc}{I} = \frac{hl}{3I_0},$$
$$\int_0^{\frac{l'}{2}}\frac{y^2dc}{I} = \frac{4h^2l}{15I_0},$$
$$\int_0^{\frac{l'}{2}}\frac{\cos^2\phi\, dc}{A} = \frac{l^2\phi_0}{8hA_0},$$
$$\int_0^{a'}\frac{xydc}{I} = \frac{ha^3(4l - 3a)}{3l^2I_0},$$
$$\int_{a'}^{\frac{l'}{2}}\frac{aydc}{I} = \frac{ah(l^3 - 6a^2l + 4a^3)}{3l^2I_0},$$
$$\int_0^{a'}\frac{\sin\phi\cos\phi\, dc}{A} = \frac{l^2e}{(l^2 + 16h^2)A_0}, \text{ nearly,*}$$

* Howe, Treatise on Arches.

$$\int_0^{\frac{l'}{2}} \frac{dc}{I} = \frac{l}{2\,I_0},$$

$$\int_0^{a'} \frac{x\,dc}{I} = \frac{a^2}{2\,I_0},$$

$$\int_{a'}^{\frac{l'}{2}} \frac{a\,dc}{I} = \frac{a\,(l-2\,a)}{2\,I_0},$$

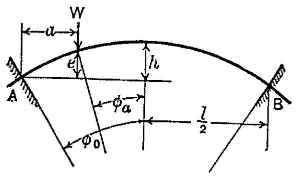

Fig. 62

and putting, as before,

$$\frac{I_0}{A_0} = i^2,$$

we get,

$$M'\frac{hl}{3} - H'\left(\frac{4\,h^2 l}{15} + \frac{l^2 i^2 \phi_0}{8\,h}\right) + W\left\{\frac{ah(l^3 - 2a^2 l + a^3)}{3\,l^2} - \frac{l^2 i^2 e}{l^2 + 16\,h^2}\right\} = 0.$$

$$M'\frac{l}{2} - H'\frac{hl}{3} + W\frac{a\,(l-a)}{2} = 0.$$

Eliminating M', and remembering that

$$e = \frac{4\,h}{l^2}\,a\,(l-a),$$

we obtain,

$$H'\left(\frac{h^2 l}{45} + \frac{l^2 i^2 \phi_0}{16\,h}\right) = W\left\{\frac{a^2 h\,(l-a)^2}{6\,l^2} - \frac{2\,ah\,(l-a)\,i^2}{l^2 + 16\,h^2}\right\}.$$

Consequently, for *one load* W (Fig. 62), we get,

$$H = \tfrac{1}{2}H' = \frac{60\,h^2}{16\,h^3 l + 45\,l^2 i^2 \phi_0}\left\{\frac{a^2\,(l-a)^2}{l^2} - \frac{12\,a\,(l-a)\,i^2}{l^2 + 16\,h^2}\right\}W. \quad (143)$$

Similarly, by carrying out the integrations in Eqs. (134) and (135), we get,

$$M_1 l + V_1\frac{l^2}{2} - H\frac{2}{3}hl - W\frac{(l-a)^2}{2} = 0,$$

$$M_1\frac{l^2}{2} + V_1\left\{\frac{l^3}{3} + \frac{i^2\,(4\,hl - l^2\,\phi_0)}{4\,h}\right\} - H\frac{hl^2}{3}$$
$$- W\left[\frac{(l-a)^2\,(2\,l+a)}{6} + \frac{i^2\{8\,h\,(l-a) - l^2\,(\phi_a + \phi_0)\}}{8\,h}\right] = 0.$$

Eliminating V_1 and M_1 successively, and putting

$$\frac{4\,hl - l^2\,\phi_0}{4\,h} = n,$$

$$\frac{8\,h\,(l-a) - l^2\,(\phi_a + \phi_0)}{8\,h} = m,$$

$$M_1 = \frac{1}{l^4 + 12\,lni^2}\left[H\left(\frac{2\,hl^4}{3} + 8\,hlni^2\right)\right.$$
$$\left. - W\,\{(al^2 - 6\,ni^2)\,(l-a)^2 + 6\,l^2 mi^2\}\right] \quad \ldots\ldots \quad (144)$$

$$V_1 = \frac{1}{l^3 + 12\,ni^2}\{(l-a)^2\,(l+2\,a) + 12\,mi^2\}W \quad \ldots\ldots \quad (145)$$

ϕ_a and ϕ_0 denoting the inclination of tangents at a and A respectively.

Neglecting the effect of axial stress, — since the terms containing i^2 ought then to disappear, — we get,

$$H = \frac{15\,a^2\,(l-a)^2}{4\,hl^3}W \quad \ldots\ldots \quad (146)$$

$$M_1 = \frac{(l-a)^2\,(5\,a^2 - 2\,al)}{2\,l^3}W \quad \ldots\ldots \quad (147)$$

$$V_1 = \frac{(l-a)^2\,(l+2\,a)}{l^3}W \quad \ldots\ldots \quad (148)$$

81. Temperature Stress. — For a uniform temperature change of t, introducing in Eqs. (138) and (139) the equation of parabola and the expressions for I and A already given, and integrating the terms severally, we obtain,

$$H_t = \frac{t\theta EI_0}{\dfrac{4\,h^2}{45} + \dfrac{i^2 l\phi_0}{4\,h}} \quad \ldots\ldots \quad (149)$$

$$M_t = \tfrac{2}{3}hH_t \quad \ldots\ldots \quad (150)$$

Neglecting the axial stress, these equations become,

$$H_t = \frac{45\, t\theta E I_0}{4\, h^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

$$M_t = \frac{15\, t\theta E I_0}{2\, h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

82. Displacement Stresses. — For a change of Δy in relative heights of supports (Art. 77), we get, by carrying out the integrations in Eqs. (140) and combining them,

$$V_\Delta = \frac{E I_0 \Delta y}{l\left\{\frac{l^2}{12} + \frac{(4\, h - l\, \phi_0)\, i^2}{4\, h}\right\}} \quad . \quad . \quad (153)$$

$$M_\Delta = -\, V_\Delta \frac{l}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (154)$$

$$H_\Delta = 0.$$

For a change of $\Delta\phi$ in the central angle (Art. 78), similarly we get from Eq. (141) the following equations:

$$H_\Delta = \frac{2\, h E I_0 \Delta\, \phi}{l\left(\frac{4\, h^2}{15} + \frac{3\, l\, \phi_0 i^2}{4\, h}\right)} \quad . \quad . \quad . \quad . \quad (155)$$

$$V_\Delta = -\, \frac{2\, E I_0\, \Delta\phi}{\frac{l^2}{3} + i^2\left(\frac{4\, h - l\, \phi_0}{h}\right)} \quad . \quad . \quad . \quad (156)$$

$$M_\Delta = -\, V_\Delta \frac{l}{2} + H_\Delta \frac{2\, h}{3} + \frac{E I_0\, \Delta\phi}{l} \quad . \quad (157)$$

For a change of Δl in span length (Art. 79), we get from Eq. (142),

$$H_\Delta = \frac{E I_0\, \Delta\, l}{l\left(\frac{4\, h^2}{45} + \frac{i^2\, l\, \phi_0}{4\, h}\right)} \quad . \quad . \quad . \quad . \quad (158)$$

$$M_\Delta = H_\Delta \frac{2}{3} h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (159)$$

$$V_\Delta = 0.$$

CIRCULAR ARCH WITHOUT HINGES

(Uniform Cross-Section)

83. Referring to Fig. 63, we have for circular arc with origin of coördinates at A the following relations:

$$x = r\,(\sin\phi_0 - \sin\phi),$$
$$dx = -\, r\cos\phi\, d\,\phi,$$
$$y = r\,(\cos\phi - \cos\phi_0),$$
$$dy = -\, r\sin\phi\, d\,\phi,$$
$$dc = -\, r d\,\phi,$$
$$a = r\,(\sin\phi_0 - \sin\phi_a),$$

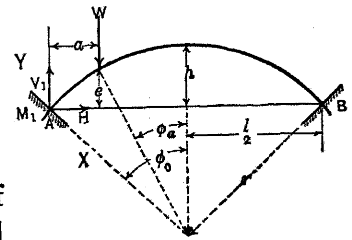

Fig. 63

ϕ denoting the inclination of the tangent to the horizontal at any point xy.

Introducing these in Eqs. (133), (134), and (135), and integrating the terms severally,

(*a*)

$$\int_0^{l'} y dc = 2\, r^2\,(\sin\phi_0 - \phi_0\cos\phi_0).$$

$$\int_0^{l'} xy dc = 2\, r^3 \sin\phi_0\,(\sin\phi_0 - \phi_0\cos\phi_0).$$

$$\int_0^{l} \sin\phi\, dx = 0.$$

$$\int_0^{l'} y^2 dc = r^3\,\{\phi_0\,(1 + 2\cos^2\phi_0) - 3\sin\phi_0\cos\phi_0\}.$$

$$\int_0^{l} \cos\phi\, dx = r\,(\phi_0 + \sin\phi_0\cos\phi_0).$$

$$\int_{a'}^{l'} (x - a)\, y dc = r^3\,\{\tfrac{1}{2}(\sin\phi_0 + \sin\phi_a)^2 - (\phi_0 + \phi_a)\sin\phi_a\cos\phi_0$$
$$+ \cos\phi_0\,(\cos\phi_0 - \cos\phi_a)\}.$$

$$\int_a^{l} \sin\phi\, dx = \frac{r}{2}(\sin^2\phi_a - \sin^2\phi_0).$$

(*b*)

$$\int_0^{l'} dc = 2\,r\,\phi_0.$$

$$\int_0^{l'} x dc = 2\,r^2\,\phi_0 \sin\phi_0.$$

$$\int_0^{l'} y dc = 2\,r^2\,(\sin\phi_0 - \phi_0\cos\phi_0).$$

$$\int_{a'}^{l'} (x-a)\,dc = r^2\,\{(\phi_0+\phi_a)\sin\phi_a + \cos\phi_a - \cos\phi_0\}.$$

(*c*)

$$\int_0^{l'} x dc = 2\,r^2\,\phi_0 \sin\phi_0.$$

$$\int_0^{l'} x^2 dc = r^3\,\{\phi_0\,(1 + 2\sin^2\phi_0) - \cos\phi_0\sin\phi_0\}.$$

$$\int_0^{l} \sin\phi\,dy = r\,(\phi_0 - \cos\phi_0\sin\phi_0).$$

$$\int_0^{l'} xy dc = 2\,r^3\sin\phi_0\,(\sin\phi_0 - \phi_0\cos\phi_0).$$

$$\int_0^{l} \cos\phi\,dy = 0.$$

$$\int_{a'}^{l'} (x-a)\,x dc = r^3\,\{(\phi_0+\phi_a)\,(\tfrac{1}{2} + \sin\phi_0\sin\phi_a)$$
$$+ (\sin\phi_0 + \sin\phi_a)(\cos\phi_a - \cos\phi_0) - \tfrac{1}{2}(\cos\phi_0\sin\phi_0 + \sin\phi_a\cos\phi_a)\}.$$

$$\int_a^{l} \sin\phi\,dy = \frac{r}{2}\,\{(\phi_0+\phi_a) - \cos\phi_0\sin\phi_0 - \cos\phi_a\sin\phi_a\}.$$

And eliminating M_1, V_1, and H successively from the three equations, we obtain,

$$H = \frac{\left\{\begin{array}{l}\sin\phi_0(\cos\phi_a - \cos\phi_0) + \sin\phi_a(\phi_a\sin\phi_0 - \phi_0\sin\phi_a) \\ \quad + \dfrac{\phi_0}{2}\left(1 + \dfrac{i^2}{r^2}\right)(\sin^2\phi_a - \sin^2\phi_0)\end{array}\right\}}{\phi_0\left(1 + \dfrac{i^2}{r^2}\right)(\phi_0 + \cos\phi_0\sin\phi_0) - 2\sin^2\phi_0}\,W. \quad (160a)$$

or, since

$$\sin\phi_0 = \frac{l}{2\,r}, \qquad \cos\phi_0 = \frac{r-h}{r},$$

$$\sin\phi_a = \frac{l-2\,a}{2\,r}, \qquad \cos\phi_a = \frac{r-h+e}{r},$$

$$H = \frac{le + \left(\dfrac{l}{2} - a\right)\{\phi_a l - \phi_0\,(l - 2\,a)\} - \phi_0\left(1 + \dfrac{i^2}{r^2}\right)(l-a)\,a}{\phi_0\left(1 + \dfrac{i^2}{r^2}\right)\{2\,r^2\,\phi_0 + l\,(r-h)\} - l^2}\,W. \quad (160b)$$

$$M_1 = \frac{r}{\phi_0}(\sin\phi_0 - \phi_0\cos\phi_0)H + \frac{r}{2\left(1 + \dfrac{i^2}{r^2}\right)(\phi_0 - \cos\phi_0\sin\phi_0)} \times$$
$$\Big[\sin\phi_a(\sin\phi_0\cos\phi_0 - \sin\phi_0\cos\phi_a + \phi_0) - \phi_a\sin\phi_0$$
$$- \frac{1}{\phi_0}\left(1 + \frac{i^2}{r^2}\right)(\cos\phi_0\sin\phi_0 - \phi_0)\,(\phi_a\sin\phi_a - \phi_0\sin\phi_0$$
$$+ \cos\phi_a - \cos\phi_0) + \frac{i^2}{r^2}\,\{\sin\phi_a\,(\cos\phi_a\sin\phi_0$$
$$- \cos\phi_0\sin\phi_0 + \phi_0) - \phi_a\sin\phi_0\}\Big]\,W \quad . \quad . \quad . \quad . \quad . \quad . \quad (161a)$$

or

$$M_1 = \left(\frac{l}{2\,\phi_0} - r + h\right)H + \frac{1}{4\,\phi_0\left(1 + \dfrac{i^2}{r^2}\right)\{2\,r^2\,\phi_0 - l\,(r-h)\}} \times$$
$$\Big[(l - 2\,a)\,(2\,r^2\,\phi_0 - le)\,\phi_0 - 2\,\phi_0\phi_a\,r^2 l$$
$$- \left(1 + \frac{i^2}{r^2}\right)\{2\,r^2\,\phi_0 - l\,(r-h)\}\{2\,\phi_a\,a + l\,(\phi_0 - \phi_a) - 2\,e\}$$
$$+ \frac{i^2}{r^2}\phi_0\,\{(l - 2\,a)\,(2\,r^2\,\phi_0 + le) - 2\,r^2\,\phi_a\,l\}\Big]\,W \quad . \quad . \quad . \quad (161b)$$

$$V_1 = \left\{\frac{1}{2} + \frac{\left(1 + \dfrac{i^2}{r^2}\right)(\phi_a - \sin\phi_a\cos\phi_a) + 2\sin\phi_a(\cos\phi_a - \cos\phi_0)}{2\left(1 + \dfrac{i^2}{r^2}\right)(\phi_0 - \cos\phi_0\sin\phi_0)}\right\}W. \quad (162a)$$

or

$$V_1=\left[\frac{1}{2}+\frac{\left(1+\frac{i^2}{r^2}\right)\{2\phi_a r^2-(l-2a)(r-h+e)\}+2e(l-2a)}{2\left(1+\frac{i^2}{r^2}\right)\{2r^2\phi_0-l(r-h)\}}\right]W. \quad (162b)$$

Neglecting the effect of axial stress, we get,

$$H=\frac{\left\{\begin{array}{l}\sin\phi_0(\cos\phi_a-\cos\phi_0)+\sin\phi_a(\phi_a\sin\phi_0-\phi_0\sin\phi_a)\\ \quad+\frac{\phi_0}{2}(\sin^2\phi_a-\sin^2\phi_0)\end{array}\right\}}{\phi_0(\phi_0+\cos\phi_0\sin\phi_0)-2\sin^2\phi_0}W. \quad (163a)$$

or

$$H=\frac{le+\left(\frac{l}{2}-a\right)\{\phi_a l-\phi_0(l-2a)\}-\phi_0 a(l-a)}{\phi_0\{2r^2\phi_0+l(r-h)\}-l^2}W. \quad (163b)$$

$$M_1=\frac{r}{\phi_0}(\sin\phi_0-\phi_0\cos\phi_0)H+\frac{r}{2(\phi_0-\cos\phi_0\sin\phi_0)}\times$$
$$\Big[\sin\phi_a(\sin\phi_0\cos\phi_0-\sin\phi_0\cos\phi_a+\phi_0)-\phi_a\sin\phi_0$$
$$-\frac{1}{\phi_0}(\cos\phi_0\sin\phi_0-\phi_0)(\phi_a\sin\phi_a-\phi_0\sin\phi_0+\cos\phi_a-\cos\phi_0)\Big]W. \quad (164a)$$

or

$$M_1=\left(\frac{l}{2\phi_0}-r+h\right)H+\frac{1}{4\phi_0\{2r^2\phi_0-l(r-h)\}}\times$$
$$\Big[(l-2a)(2r^2\phi_0-le)\phi_0-2\phi_0\phi_a r^2 l$$
$$-\{2r^2\phi_0-l(r-h)\}\ \{2\phi_a a+l(\phi_0-\phi_a)-2e\}\Big]W. \quad (164b)$$

$$V_1=\left\{\frac{1}{2}+\frac{(\phi_a-\sin\phi_a\cos\phi_a)+2\sin\phi_a(\cos\phi_a-\cos\phi_0)}{2(\phi_0-\cos\phi_0\sin\phi_0)}\right\}W. \quad (165a)$$

or

$$V_1=\left[\frac{1}{2}+\frac{\{2\phi_a r^2-(l-2a)(r-h+e)\}+2e(l-2a)}{2\{2r^2\phi_0-l(r-h)\}}\right]W. \quad (165b)$$

84. Temperature Stresses. — For a uniform temperature change of t — positive for rise — we get, by carrying out the integrations as before in Eqs. (138) and (139),

$$H_t=\frac{t\theta lEI\phi_0}{r^3\left[\phi_0(\phi_0+\sin\phi_0\cos\phi_0)\left(1+\frac{i^2}{r^2}\right)-2\sin^2\phi_0\right]} \quad (166a)$$

$$=\frac{2t\theta lEI\phi_0}{r\left[\phi_0\{2\phi_0 r^2+l(r-h)\}\left(1+\frac{i^2}{r^2}\right)-l^2\right]} \quad (166b)$$

$$M_t=H\frac{(\sin\phi_0-\phi_0\cos\phi_0)r}{\phi_0} \quad (167a)$$

$$=H\frac{l-2\phi_0(r-h)}{2\phi_0} \quad (167b)$$

Neglecting axial stress, we get,

$$H_t=\frac{t\theta lEI\phi_0}{r^3[\phi_0(\phi_0+\sin\phi_0\cos\phi_0)-2\sin^2\phi_0]} \quad (168a)$$

$$=\frac{2t\theta lEI\phi_0}{r[\phi_0\{2\phi_0r^2+l(r-h)\}-l^2]} \quad (168b)$$

85. Displacement Stresses. — For a change of Δy (Art. 77), we get, by carrying out the integrations in Eq. (140),

$$V_\Delta=\frac{EI\,\Delta y}{r^3\left(1+\frac{i^2}{r^2}\right)(\phi_0-\cos\phi_0\sin\phi_0)} \quad (169a)$$

$$=\frac{2EI\,\Delta y}{r\left(1+\frac{i^2}{r^2}\right)\{2r^2\phi_0-l(r-h)\}} \quad (169b)$$

$$M_\Delta=-V_\Delta r\sin\phi_0=-V_\Delta\frac{l}{2} \quad (170)$$

$$H_\Delta=0.$$

For a change of $\Delta\phi$ in the central angle (Art. 78), we get from Eq. (141),

$$V_\Delta=-\frac{EI\sin\phi_0\,\Delta_0}{r^2\left(1+\frac{i^2}{r^2}\right)(\phi_0-\cos\phi_0\sin\phi_0)} \quad (171a)$$

$$=-\frac{EI\,l\,\Delta\phi}{r\left(1+\frac{i^2}{r^2}\right)\{2r^2\phi_0-l(r-h)\}} \quad (171b)$$

$$H_\Delta = \frac{EI\,(\sin\phi_0 - \phi_0\cos\phi_0)\,\Delta\phi}{r^2\left\{\phi_0\left(1+\frac{i^2}{r^2}\right)(\phi_0 + \sin\phi_0\cos\phi_0 - 2\sin^2\phi_0)\right\}} \quad (172a)$$

$$= \frac{EI\,\{l - 2\,\phi_0\,(r-h)\}\,\Delta\phi}{r\left[\phi_0\,\{2\,\phi_0\,r^2 + l\,(r-h)\}\left(1+\frac{i^2}{r^2}\right) - l^2\right]} \quad (172b)$$

$$M_\Delta = H_\Delta\frac{r}{\phi_0}(\sin\phi_0 - \phi_0\cos\phi_0) - V_\Delta r\sin\phi_0 + \frac{EI\,\Delta\phi}{2\,\phi_0 r} \quad (173a)$$

$$= H_\Delta\frac{l - 2\,\phi_0\,(r-h)}{2\,\phi_0} - V_\Delta\frac{l}{2} + \frac{EI\,\Delta\phi}{2\,\phi_0\,r} \quad (173b)$$

For a change of Δl in span length (Art. 79), we obtain from Eq. (142),

$$H_\Delta = \frac{EI\,\phi_0\Delta\,l}{r^3\left[\phi_0\left(1+\frac{i^2}{r^2}\right)(\phi_0 + \sin\phi_0\cos\phi_0) - 2\sin^2\phi_0\right]} \quad (174a)$$

$$= \frac{2\,EI\,\phi_0\Delta l}{r\left[\phi_0\,\{2\,r^2\,\phi_0 + l\,(r-h)\}\left(1+\frac{i^2}{r^2}\right) - l^2\right]} \quad (174b)$$

$$M_\Delta = H_\Delta\frac{r\,(\sin\phi_0 - \phi_0\cos\phi_0)}{\phi_0} = H_\Delta\frac{l - 2\,\phi_0\,(r-h)}{2\,\phi_0} \quad (175)$$

$$V_\Delta = 0.$$

FLAT ARCH WITHOUT HINGES

86. In arches with comparatively small versed-sines, we may put, as before, without material error,

$$dc = dx,$$

so that

$$\int_0^l \sin\phi\,dx = 0,$$

$$\int_0^l \cos\phi\,dx = l,$$

$$\int \sin\phi\,dy = 0,$$

$$\int_0^l \cos\phi\,dy = 0.$$

Introducing these in Eqs. (133), (134), and (135), and assuming the cross-section of the rib to be uniform throughout, with

$$\frac{I}{A} = i^2,$$

we get,

$$M_1\int_0^l y\,dx + V_1\int_0^l xy\,dx - H\left(\int_0^l y^2dx + li^2\right)$$
$$- W\int_a^l (x-a)\,y\,dx = 0 \quad (176a)$$

$$M_1 l + V_1\frac{l^2}{2} - H\int_0^l y\,dx - W\frac{(l-a)^2}{2} = 0 \quad (176b)$$

$$M_1\frac{l^2}{2} + V_1\frac{l^3}{3} - H\int_0^l xy\,dx - W\frac{(2\,l+a)\,(l-a)^2}{6} = 0 \quad (176c)$$

Combining these equations, we get,

$$V_1 = \frac{(l+2\,a)\,(l-a)^2}{l^3}W \quad (177)$$

$$H = \frac{\int_a^l (x-a)\,y\,dx - \frac{(l-a)^2}{2}\int_0^l y\,dx}{\left(\int_0^l y\,dx\right)^2 - l\int_0^l y^2dx - l^2i^2}\,W \quad (178)$$

$$M_1 = H\,\frac{\int_0^l y\,dx}{l} - W\,\frac{a(l-a)^2}{l^2} \quad (179)$$

It is to be noted that $\int_0^l y\,dx$ is the area above the horizontal line joining the ends of the arch and bounded by the axis of the latter.

87. Temperature Stresses. — For a uniform temperature change of t (Art. 75), similarly we get from Eqs. (138) and (139),

$$H_t = \frac{t\,\theta\, l^2 EI}{l\int_0^l y^2dx + l^2i^2 - \left(\int_0^l ydx\right)^2} \quad . \quad . \quad . \quad . \qquad (180)$$

$$M_t = H_t\frac{\int_0^l ydx}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (181)$$

88. Displacement Stresses. — Similarly from Eqs. (140), (141), and (142), we get, for Δy (Art. 77),

$$V_\Delta = \frac{12\,EI\,\Delta\,y}{l^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (182)$$

$$M_\Delta = -V_\Delta\frac{l}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (183)$$

for $\Delta\phi$ (Art. 78),

$$H_\Delta = \frac{EI\,\Delta\phi\int_0^l ydx}{l\int_0^l y^2dx - \left(\int_0^l ydx\right)^2 + l^2i^2} \quad . \quad . \quad . \qquad (184)$$

$$V_\Delta = -\frac{6\,EI\,\Delta\phi}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (185)$$

$$M_\Delta = -V_\Delta\frac{2}{3}l + H_\Delta\frac{\int_0^l ydx}{l} \quad . \quad . \quad . \quad . \qquad (186)$$

and for Δl (Art. 79),

$$H_\Delta = \frac{l EI\,\Delta\, l}{l\int_0^l y^2dx - \left(\int_0^l ydx\right)^2 + l^2i^2} \quad . \quad . \quad . \qquad (187)$$

$$M_\Delta = H_\Delta\,\frac{\int_0^l ydx}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (188)$$

FLAT PARABOLIC ARCH WITHOUT HINGES

(*Uniform Cross-Section.*)

89. Introducing in Eqs. (177), (178), and (179) the equation of parabola

$$y = \frac{4\,h}{l^2}x\,(l - x),$$

we get for one load W (Fig. 64) the following equations:

$$H = \frac{15\,a^2\,(l-a)^2}{4\,hl^3\left(1 + \frac{45\,i^2}{4\,h^2}\right)}W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (189)$$

$$M_1 = -\frac{a\,(l-a)^2}{2\,l^3}\left(2\,l - \frac{5\,a}{1 + \frac{45\,i^2}{4\,h^2}}\right)W \quad . \quad . \qquad (190)$$

$$V_1 = \frac{(l + 2\,a)\,(l - a)^2}{l^3}W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (191)$$

Neglecting the effect of axial compression,

$$H = \frac{15\,a^2\,(l-a)^2}{4\,hl^3}W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (192)$$

$$M_1 = -\frac{a\,(l-a)^2\,(2\,l - 5\,a)}{2\,l^3}W \quad . \quad . \quad . \quad . \quad . \quad . \qquad (193)$$

$$V_1 = \frac{(l + 2\,a)\,(l-a)^2}{l^3}W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (194)$$

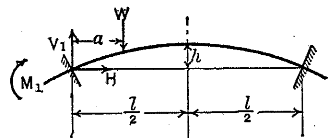

Fig. 64

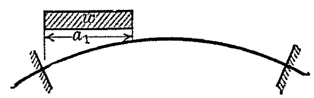

Fig. 65

90. For a uniform load w per unit length of the span (Fig. 65), substituting wda for W, and integrating between the given limits of loading, we get,

$$H = \frac{(10l^2 - 15a_1l + 6a_1^2)a_1^3w}{8hl^3\left(1 + \frac{45i^2}{4h^2}\right)} \quad . \quad . \quad . \quad . \quad (195)$$

$$M_1 = -\left\{6l^2 - 8a_1l + 3a_1^2 - \frac{a_1(10l^2 - 15a_1l + 6a_1^2)}{l\left(1 + \frac{45i^2}{4h}\right)}\right\}\frac{a_1^2w}{12l^2} \quad (196)$$

$$V_1 = \frac{(2l^3 - 2a_1^2l + a_1^3)a_1w}{2l^3} \quad . \quad . \quad . \quad . \quad . \quad (197)$$

Neglecting axial stress, we get,

$$H = \frac{(10l^2 - 15a_1l + 6a_1^2)a_1^3w}{8hl^3} \quad . \quad . \quad . \quad . \quad (198)$$

$$M_1 = -\frac{(l - a_1)^3a_1^2w}{2l^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (199)$$

$$V_1 = \frac{(2l^3 - 2a_1^2l + a_1^3)a_1w}{2l^3} \quad . \quad . \quad . \quad . \quad (200)$$

91. Temperature Stresses. — Similarly from Eqs. (180) and (181) we get,

$$H_t = \frac{45t\theta EI}{4h^2 + 45i^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201)$$

$$M_t = H_t\tfrac{2}{3}h = \frac{30t\theta EIh}{4h^2 + 45i^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (202)$$

Neglecting axial stress,

$$H_t = \frac{45t\theta EI}{4h^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (203)$$

$$M_t = \frac{15t\theta EI}{2h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (204)$$

92. Displacement Stresses. — From Eqs. (182) to (188) we obtain in a similar manner,

for Δy (Art. 77),

$$V_\Delta = \frac{12EI\Delta y}{l^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (205)$$

$$M_\Delta = -V_\Delta\frac{l}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (206)$$

for $\Delta\phi$ (Art. 78),

$$H_\Delta = \frac{30EIh\Delta\phi}{l(4h^2 + 45i^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (207)$$

$$V_\Delta = -\frac{6EI\Delta\phi}{l^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (208)$$

$$M_\Delta = -V_\Delta\tfrac{2}{3}l + H_\Delta\tfrac{2}{3}h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (209)$$

and for Δl (Art. 79),

$$H_\Delta = \frac{45EI\Delta l}{l(4h^2 + 45i^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (210)$$

$$M_\Delta = H_\Delta\tfrac{2}{3}h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (211)$$

93. For *flat circular arches* without hinges, the foregoing formulas deduced for parabolic arches may be used without sensible error, for the reason already stated in the case of arches with two hinges (Art. 62).

REACTION LOCUS AND ENVELOPE

94. For showing reactions in an arch without hinges in amount and direction, reaction locus and envelope are required.

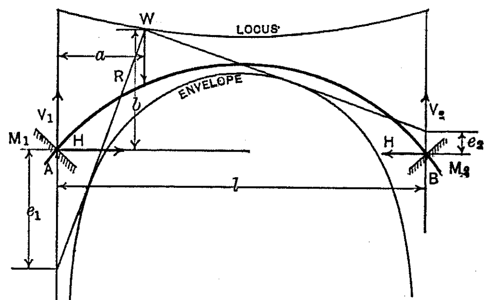

Fig. 66

Since end moments are due to the deviation of reactions from the axis, if we represent by e the vertical distance taken as positive above and negative below the horizontal line connecting the ends of the arch, we have in Fig. 66 at the left end,

$$e_1 = \frac{M_1}{H},$$

and at the right end,

$$e_2 = \frac{M_2}{H}.$$

Since

$$\frac{V_1}{H} = \frac{b - e_1}{a}, \qquad b = e_1 + \frac{V_1 a}{H} = \frac{M_1 + V_1 a}{H};$$

$$\frac{V_2}{H} = \frac{b - e_2}{l - a}, \qquad b = e_2 - \frac{V_2}{H}(l - a) = \frac{M_2 - V_2(l-a)}{H};$$

which are the equations of the locus.

Next, let $x_1 y_1$ be coördinates — with origin at A — of the point in the line of reaction R, assumed to be one of contact with the envelope. Then we have,

$$y_1 = e_1 + \frac{V_1}{H} x_1 = e_1 + \frac{b - e_1}{a} x_1.$$

In order to find the relation between x_1 and y_1 for variable a, differentiate this equation with respect to a and eliminate a from the same. The equation thus obtained will be that of the envelope.

It is evident that locus and envelope could be drawn by simple plotting of reaction lines for different positions of loads, instead of by deducing their equations.

95. Taking the case of a *flat parabolic arch*, since we have by neglecting the effect of axial stress (Art. 89),

$$V_1 = \frac{(l + 2a)(l - a)^2}{l^3} W,$$

$$H = \frac{15 a^2 (l - a)^2}{4 h l^3} W,$$

$$M_1 = -\frac{a (l - a)^2 (2l - 5a)}{2 l^3} W,$$

we get,

$$e_1 = -\frac{2h(2l - 5a)}{15a},$$

$$b = e_1 + \frac{V_1}{H} a = \frac{6h}{5},$$

showing that the reaction locus is a horizontal line.

Since

$$y_1 = e_1 + \frac{b - e_1}{a} x_1,$$

substituting the values of e_1 and b_1,

$$y_1 = -\frac{2h(2l - 5a)}{15a} + \frac{4h(2a + l)}{15a^2} x_1.$$

Differentiating this with respect to a, we get,

$$a = \frac{2h(l - 2x_1)}{5(2h - 3y_1)}.$$

Substituting this value of a in the preceding equation, and at the same time transferring the origin of coördinates to the centre of the span and in level of the crown of the envelope where

$$y_1 = \frac{2}{3} h \text{ and } x_1 = \frac{l}{2},$$

we get,

$$8hx^2 + 15 l^2 y + 30 lxy = 0,$$

which is the equation of hyperbola.

POSITION OF LOADS FOR MAXIMUM STRESS

96. For finding the mode of loading to produce maximum stress in any part of the arch, reaction locus and envelope may be made use of in a similar manner as explained in the case of two-hinged arches (Art. 68).

In Fig. 67, let the outside lines of the rib represent the positions of centres of gravity of the flanges or chords. Since at any normal section CD of the rib the stress in the upper flange C is equal to the moment with respect to D divided by d, the reaction line passed through D and produced to the locus will indicate the position of load producing no stress in C, and that all loads to the left of O produce compression in C, while those to the right, tension. For the same reason the reaction line drawn through C determines the position for load to produce no stress in D, so that all loads to the left of O' produce tension in D, and those to the right, compression.

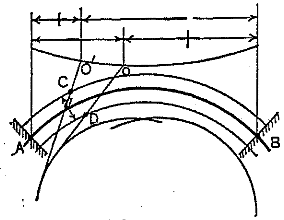

Fig. 67

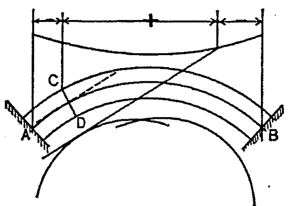

Fig. 68

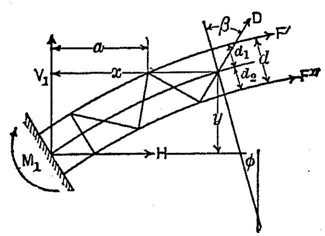

Fig. 69

For similar reasons, the position of load producing no shear at the normal section is given by drawing the reaction line parallel to the tangent to the axis of the arch at the section (Fig. 68) and by erecting a vertical over the section. The loads within these limits evidently produce (Art. 68) positive shear, while those outside the same, the negative.

In case the chords or flanges are not parallel, the reaction line should be passed through the intersection of chord-members of the panel — in which the shear is to be determined — instead of drawing parallel to the tangent, to find the position of load producing no shear.

THE STRESSES IN INDIVIDUAL MEMBERS

97. Referring to Art. 67, it will at once be seen that, in the arch-rib of Fig. 69, we have but to add M_1 to the moment of external forces.

Using the same designations as in Art. 70, we then have,

$$F'' = \frac{1}{d}\left\{ M_1 + V_1 x - Hy - \sum_0^x W(x-a) - (V \sin\phi + H\cos\phi)\, d_1 \right\} \quad . \quad . \quad . \quad (212)$$

$$F' = -\frac{1}{d}\left\{ M_1 + V_1 x - Hy - \sum_0^x W(x-a) + (V \sin\phi + H\cos\phi)\, d_2 \right\} \quad . \quad . \quad . \quad (213)$$

$$D = -(V\cos\phi - H\sin\phi)\sec\beta \quad . \quad . \quad . \quad . \quad (214)$$

in which
$$V = V_1 - \sum_0^x W.$$

For a *non-parallel rib*, the stress in each member is best obtained by taking moment at the intersection of the other two members cut by a section.

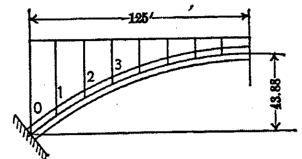

Fig. 70

Example. — In a full-webbed circular arch with the same general dimensions as given in the case of two-hinged arch on page 94, to find the maximum stress in the lower flange at 3 (Fig. 70).

Loads and dimensions :

Dead load = 20 tons per panel.
Live load = 10 tons per panel.
$l = 250$ ft.
$r = 200$.
$\phi_0 = 38° - 40' - 56'' = .67514$.
Panel length = 15.625 ft.
Cross-section uniform with $\frac{i^2}{r^2} = .00019$.
Effective depth = 6 ft.

Panel.	a (ft.).	e (ft.).	ϕa.	ϕa (circ. meas.).
1	15.63	11.32	33° 09′ 10″	.57863
2	31.25	20.54	27 57 11	.48788
3	46.88	27.99	22 59 36	.40114
4	62.50	33.86	18 12 36	.31782
5	78.13	38.31	13 33 17	.23658
6	93.75	41.42	8 59 21	.15689
7	109.38	43.26	4 29 12	.07831
8	125.00	43.88	0	0

The horizontal reaction is given by Eq. (160*b*).

$$H = \frac{le + \left(\frac{l}{2} - a\right)\{\phi_a l - \phi_0 (l - 2a)\} - \phi_0 \left(1 + \frac{i^2}{r^2}\right) a (l - a)}{\phi_0 \left(1 + \frac{i^2}{r^2}\right)\{2 r^2 \phi_0 + l (r - h)\} - l^2} W.$$

Tabulating the terms of the numerator severally, we have,

Panel Pt.	le	$\phi_0 (l - 2a)$	$\phi a l$	$\left(\frac{l}{2} - a\right)\{\phi_0 (l-2a) - \phi a l\}$	$\left(1 + \frac{i^2}{r^2}\right)\phi_0 a (l-a)$	Numerator.
1	2829.46	147.69	144.66	331.21	2472.88	25.37
2	5135.29	126.59	121.97	432.98	4616.03	86.28
3	6996.25	105.49	100.28	406.66	6429.48	160.11
4	8464.64	84.39	79.46	308.46	7918.20	237.98
5	9576.05	63.29	59.14	194.51	9067.20	314.34
6	10354.62	42.20	39.22	92.89	9891.50	370.23
7	10815.54	21.10	19.58	23.76	10386.07	405.70
8	10968.75	0	0	0	10550.93	417.82

and for the denominator,

$$\phi_0 \left(1 + \frac{i^2}{r^2}\right)\{2 r^2 \phi_0 + l (r - h)\} - l^2 = 327.15,$$

whence we get the following value of H for $W = 1$:

Load at.	1	2	3	4	5	6	7	8
$H =$	.0775	.2637	.4893	.7275	.9605	1.1314	1.2397	1.2768

For M_1 we have Eq. (161*b*).

$$M_1 = \left(\frac{l}{2\phi_0} - r + h\right) H + \frac{1}{4\phi_0 \left(1 + \frac{i^2}{r^2}\right)\{2 r^2 \phi_0 - l (r - h)\}} \times$$

$$\Big[\phi_0 (l - 2a)(2 r^2 \phi_0 - le) - 2 \phi_0 \phi_a r^2 l - \left(1 + \frac{i^2}{r^2}\right)\{2 r^2 \phi_0 - l (r - h)\}\{2 \phi_a a + l (\phi_0 - \phi_a) - 2 e\} + \frac{i^2}{r^2} \phi_0 \{(l - 2a)(2 r^2 \phi_0 + le) - 2 r^2 \phi_a l\}\Big] W.$$

Tabulating the terms severally,

Panel Pt.	$\left(\frac{l}{2\phi_0} - r + h\right) H$	First Term of Numerator.	Second Term.	Third Term.	Fourth Term.	Numerator.
1	2.25	7,558,673	−7,812,985	− 278,225	11	− 532,526
2	7.65	6,186,976	−6,587,596	− 542,713	17	− 943,316
3	14.20	4,959,501	−5,416,391	− 751,134	19	−1,208,005
4	20.97	3,843,680	−4,291,456	− 918,971	19	−1,366,728
5	27.88	2,811,769	−3,194,392	−1,048,682	16	−1,431,288
6	32.84	1,842,092	−2,118,465	−1,140,741	11	−1,417,102
7	35.98	911,322	−1,057,323	−1,195,747	6	−1,341,743
8	37.06	0	0	−1,214,044	0	−1,214,044

The denominator being

$$4 \phi_0 \left(1 + \frac{i^2}{r^2}\right)\{2 r^2 \phi_0 - l (r - h)\} = 40{,}459.6,$$

we get the following values of M_1 in ft.-tons; and from $M_2 = M_1 + V_1 l - W (l - a)$ the values of M_2:

Load at	1	2	3	4	5	6	7	8
$M_1 =$	−10.91	−15.66	−15.65	−12.81	− 7.50	− 2.19	+ 2.82	+7.05
$M_2 =$	+ 1.02	+ 4.09	+ 6.98	+10.05	+11.38	+11.31	+ 9.95	+7.05

The values of V_1 are obtained from Eq. (162*b*).

$$V_1 = \left[\frac{1}{2} + \frac{\left(1+\frac{i^2}{r^2}\right)\{2\,\phi_a r^2 - (l-2\,a)\,(r-h+e)\} + 2\,e\,(l-2\,a)}{2\left(1+\frac{i^2}{r^2}\right)\{2\,r^2\,\phi_0 - l\,(r-h)\}}\right] W.$$

Substituting the numerical values in all the terms, we get the following values of V_1:

Load at	1	2	3	4	5	6	7	8
V_1	.985	.954	.903	.839	.763	.679	.591	.500

Drawing the reaction locus and envelope (Fig. 71), and passing reaction line through *C* in the upper flange at 3, we

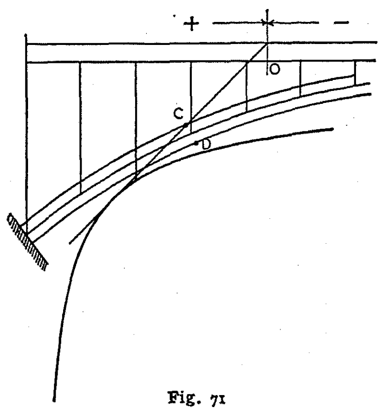

Fig. 71

at once see that all loads to the right of O produce compression in the lower flange at *D*.

For this position of loads, we have the following end-moment and reactions:

Due to dead load, —

$H = 2\,(.0775 + .2637 + .4893 + .7275 + .9605 + 1.1314 + 1.2397 + .6384)\,20 = 221.12$ tons.

$M_1 = (-10.91 - 15.66 - 15.65 - 12.81 - 7.50 - 2.19 + 2.82 + 7.05 + 1.02 + 4.09 + 6.98 + 10.05 + 11.38 + 11.31 + 9.95)\,20 = -2.80$ ft.-tons.

$V_1 = 7\tfrac{1}{2} \times 20 = 150$ tons.

Due to live load covering the right of O, —

$H = \{2\,(.9605 + 1.1314 + 1.2397 + .6384) + .0775 + .2637 + .4893 + .7275\}\,10 = 94.93$ tons.

$M_1 = (-7.50 - 2.19 + 2.82 + 7.05 + 1.02 + 4.09 + 6.98 + 10.05 + 11.38 + 11.31 + 9.95)\,10 = +549.60$ ft.-tons.

$V_1 = 3\tfrac{1}{2} \times 10 + (.015 + .046 + .097 + .161)\,10 = 38.19$ tons.

From Eq. (212) we then get for flange stress at *D*,

$F'' = \tfrac{1}{6}\,[(549.60 - 2.80) + (150 + 38.19)\,46.88 - (221.12 + 94.93)\,27.99 - 20\,(15.63 + 31.25) - \{(188.19 - 40)\,.3906 + 316.05 \times .9205\}\,3] = -243.50$ tons.

CONCLUDING REMARKS ON ARCHES

98. It must be borne in mind that all the formulas that have so far been deduced for arch-ribs apply with correctness only to ribs whose radii of curvature are considerably greater than the depths of the ribs themselves. Were this not the case, the fibre length would differ sensibly with its distance from the centre of curvature, and as a consequence the change in its length will not be proportional merely to the stress acting in the same, but will also depend on its distance from the centre of curvature. Thus in Fig. 72, let

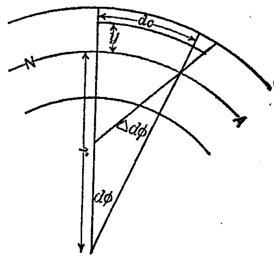

Fig. 72

r = radius of curvature of the rib,
dc = elementary fibre length at a distance of y from the neutral axis,
$d\phi$ = elementary central angle,

so that

$$dc = (r + y)\,d\phi.$$

Further, let

f = fibre stress at y,
$\Delta\, dc$ = deformation of dc due to f,
$\Delta\, d\phi$ = change of $d\phi$,

so that

$$\Delta dc = y\Delta d\phi.$$

Then we have,

$$f = E\frac{\Delta dc}{dc} = E\frac{y\Delta\, d\phi}{(r + y)\,d\phi}.$$

Since

$$M = \int fydA,$$

substituting the value of f,

$$M = E\frac{\Delta\, d\phi}{d\phi}\int\frac{y^2dA}{r + y},$$

or

$$f = \frac{My}{(r + y)\int\frac{y^2dA}{r + y}}.$$

From the last equation it will be seen that

$$f = \frac{My}{\int y^2dA} = \frac{My}{I}$$

only when y may be neglected.