CHAPTER IV

ARCHES, WITH TWO HINGES

42. IN an arch with hinges at both ends, since the moments cannot exist at these points, the reactions ought to pass through the latter.

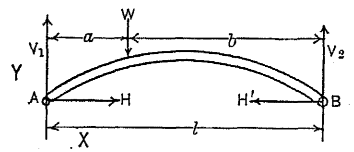

Fig. 38

Fig. 38 shows a symmetrical arch-rib with hinges at A and B.

The following designations will be used throughout the discussion:

V_1 and H . the vertical and horizontal components of the reaction at A.
V_2 and H' . ditto at B.
l . . . . . span length.
l' . . . . . length of arch measured along the axis of the rib.
x and y . . coördinates with origin at A.
c . . . . . distance from A measured along the axis of the rib.
ϕ . . . . . inclination of tangent at x, y to the horizontal.
a and b . . distances of a load from A and B respectively.
a' . . . . . the distance measured along the axis of the rib from A.
m . . . . . moment at any point.
N . . . . . normal stress at a section.
T . . . . . tangential stress at the section.
R . . . . . resultant force.
E . . . . . modulus of elasticity assumed to be constant.

I . . . . . moment of inertia of the normal section of the rib.
A . . . . . cross-sectional area of the rib.

Forces acting upward are taken as positive.

Moments producing compression at the extrados are positive.

Forces acting toward right are positive.

Tensions are positive and all vice versâ.

Since equilibrium requires that among external forces as well as between external and internal forces

Σ horiz. forces = o, Σ vert. forces = o, and Σ moments = o,

we have,

$$H - H' = 0,$$
$$V_1 + V_2 - W = 0;$$

and from moments taken with respect to B and A,

$$V_1 = \frac{b}{l} W,$$

$$V_2 = \frac{a}{l} W.$$

43. Since at any section of the rib, wherever the resultant of external forces does not pass through the centre of gravity, a moment will be caused at the section, and further, if the direction of the resultant does not coincide with that of the tangent to the axis of the rib, the latter, beside being axially compressed, will be subjected to tangential stress at the section.

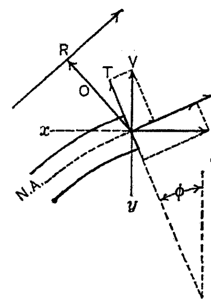

Fig. 39

At any point x, y of the neutral axis of the rib, then, referring to Figs. 38 and 39, we have,

$$m = Ro = V_1 x - Hy, \text{ for } x < a$$
$$= V_1 x - W(x - a) - Hy, \text{ for } x > a.$$

Decomposing R into V and H we get,

$$N + V \sin \phi + H \cos \phi = 0,$$
$$T + V \cos \phi - H \sin \phi = 0,$$

in which

$$V = \frac{b}{l} W \quad \text{for } x < a,$$
$$= \left(\frac{b}{l} - 1\right) W \quad \text{for } x > a;$$

and since the loading is vertical, H will be constant throughout the arch.

44. Neglecting the effect of tangential stress for the reason already stated (Art. 5), we have for the internal work in the rib due to W:

$$\omega = \int_0^{l'} \frac{m^2 dc}{2\, IE} + \int_0^{l'} \frac{N^2 dc}{2\, AE},$$

in which

$$dc = \frac{dx}{\cos \phi},$$

being the elementary length measured along the axis of the rib.

Substituting in this expression for work, the values of m and N already given, we get:

$$\omega = \int_0^{a'} \frac{(V_1 x - Hy)^2 dc}{2\, IE} + \int_{a'}^{l'} \frac{\{V_1 x - Hy - W(x - a)\}^2 dc}{2\, IE}$$
$$+ \int_0^{a'} \frac{(V \sin \phi + H \cos \phi)^2 dc}{2\, AE} + \int_{a'}^{l'} \frac{(V \sin \phi + H \cos \phi)^2 dc}{2\, AE}.$$

Since H must make ω a minimum, we obtain for

$$\frac{d\omega}{dH} = 0$$

the following:

$$\int_0^{a'} - \frac{Wbxy}{lI} dc + \int_0^{a'} \frac{Hy^2}{I} dc + \int_{a'}^{l'} - \frac{Wbxy}{lI} dc + \int_{a'}^{l'} \frac{W(x - a)y}{I} dc$$
$$+ \int_{a'}^{l'} \frac{Hy^2}{I} dc + \int_0^{a} \frac{Wb \sin \phi}{lA} dx + \int_0^{a} \frac{H \cos \phi}{A} dx$$
$$+ \int_a^{l} \frac{W(b - l) \sin \phi}{lA} dx + \int_a^{l} \frac{H \cos \phi}{A} dx = 0,$$

from which,

$$H = \frac{\int_0^{l'} \frac{bxy}{lI} dc - \int_{a'}^{l'} \frac{(x-a)y}{I} dc - \int_0^{l} \frac{b \sin \phi}{lA} dx + \int_a^{l} \frac{\sin \phi}{A} dx}{\int_0^{l'} \frac{y^2}{I} dc + \int_0^{l} \frac{\cos \phi}{A} dx} W \quad (85)$$

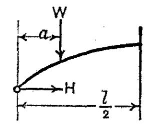

Fig. 40

45. This equation could be somewhat simplified by taking, instead of one one-sided loading, two symmetrical loads, which will evidently give H simply double that for single one. Referring to Fig. 40, and extending the integral over one-half the arch, we get the following expression for H due to one W:

$$H = \tfrac{1}{2} \frac{\int_0^{a'} \frac{xydc}{I} + a \int_{a'}^{\frac{l'}{2}} \frac{ydc}{I} - \int_0^{a} \frac{\sin \phi \, dx}{A}}{\int_0^{\frac{l'}{2}} \frac{y^2 dc}{I} + \int_0^{\frac{l}{2}} \frac{\cos \phi \, dx}{A}} W \quad . \quad . \quad (86)$$

Approximate results may be obtained by neglecting the effect of the normal or *axial stress*, which is generally inconsiderable when compared to that of the moment. For this it is simply necessary to leave out the terms containing N and A in the preceding equations, so that we get from Eq. (85),

$$H = \frac{\int_0^{l'} \frac{bxy}{lI} dc - \int_{a'}^{l'} \frac{(x-a)\,y}{I} dc}{\int_0^{l'} \frac{y^2 dc}{I}} W \quad . \quad . \quad . \quad (87)$$

or from Eq. (86),

$$H = \tfrac{1}{2} \frac{\int_0^{a'} \frac{xydc}{I} + a \int_{a'}^{\frac{l'}{2}} \frac{ydc}{I}}{\int_0^{\frac{l'}{2}} \frac{y^2 dc}{I}} W \quad . \quad . \quad . \quad . \quad (88)$$

46. Temperature Stresses.—A temperature change causes variation in the length of the rib, and were the arch-end free to move, a corresponding change would take place in span length; but as the supports are here assumed to be immovable, the rib is forced back, as it were, to its supports.

Let

t = temperature change in number of degrees,
θ = coefficient of expansion and contraction,
H_t = horizontal reaction at the left support due to the temperature change.

Imagine the arch to be free to move at the left end, then the force H_t exerted at the support must be sufficient to force the arch through a distance of

$$t\theta l,$$

reckoned in the direction of the force, i.e., *positive* for the

rising temperature. Then, according to the first theorem of Castigliano,

$$\frac{d\omega}{dH_t} = t\theta l.$$

Using the same designations as before,

$$\omega = \int_0^{l'} \frac{m^2 dc}{2\,IE} + \int_0^{l'} \frac{N^2 dc}{2\,AE},$$

m and N here representing moment and normal stress due to H_t.

Since

$$m = -H_t y,$$
$$N = -H_t \cos\phi,$$
$$\omega = H_t^2 \left(\int_0^{l'} \frac{y^2 dc}{2\,IE} + \int_0^{l'} \frac{\cos^2\phi\, dc}{2\,AE} \right),$$

whence

$$\frac{d\omega}{dH_t} = H_t \left(\int_0^{l'} \frac{y^2 dc}{IE} + \int_0^{l} \frac{\cos\phi\, dx}{AE} \right) = t\theta l,$$

from which

$$H_t = \frac{t\theta l E}{\int_0^{l'} \frac{y^2 dc}{I} + \int_0^{l} \frac{\cos\phi\, dx}{A}} \quad . \quad . \quad . \quad . \quad . \quad (89)$$

Neglecting the effect of axial stress, we get,

$$H_t = \frac{t\theta l E}{\int_0^{l'} \frac{y^2 dc}{I}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

Eqs. (85) to (90) will give the amount of horizontal reaction for vertical loadings and *uniform* changes of temperature when the form of the arch is known.

47. Displacement of Supports. — If, owing to yielding or settling of supports, a change in span length takes place, the effect on the arch would be similar to that due to temperature changes.

Let

Δl = change in span length measured at the left support in the direction of the force causing the same, i.e., *negative* for the *increase of span length*, and vice versâ.

H_Δ = horizontal reaction at the left support due to change in span length.

Then, since

$$\omega = H_\Delta^2\left(\int_0^{l'} \frac{y^2dc}{2\,IE} + \int_0^{l} \frac{\cos\phi\,dx}{2\,AE}\right),$$

by the same reasoning as before, we have

$$\frac{d\omega}{dH_\Delta} = H_\Delta\left(\int_0^{l'} \frac{y^2dc}{IE} + \int_0^{l} \frac{\cos\phi\,dx}{AE}\right) = \Delta l,$$

from which

$$H_\Delta = \frac{E\Delta l}{\int_0^{l'} \frac{y^2dc}{I} + \int_0^{l} \frac{\cos\phi\,dx}{A}}. \quad \text{(91)}$$

Neglecting the effect of axial stress, we get,

$$H_\Delta = \frac{E\Delta l}{\int_0^{l'} \frac{y^2dc}{I}} \quad \text{(92)}$$

The effect of slight *changes in the heights of supports* is generally so small in this kind of arches, that it is unnecessary to take them into consideration in the calculation of stress due to displacement of supports.

PARABOLIC ARCH WITH TWO HINGES

48. If we assume the cross-section of the rib to increase from the crown toward both ends in such a way that at any point

$$I = I_0 \sec\phi,$$
$$A = A_0 \sec\phi,$$

in which I_0 and A_0 denote the moment of inertia and the cross-sectional area of the rib at the crown, the calculation of stresses in a parabolic arch-rib becomes considerably simplified. Thus, introducing in Eq. (86) the equation of parabola with origin at A (Fig. 41),

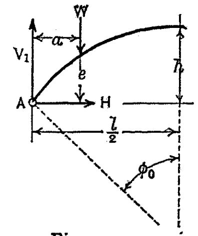

Fig. 41

$$y = \frac{4\,h}{l^2}\,x\,(l - x),$$

and remembering that

$$dx = \cos\phi\,dc,$$
$$\sin\phi\,dx = \cos\phi\,dy,$$

we get,

$$H = \frac{1}{2}\,\frac{\frac{4\,h}{l^2 I_0}\left\{\int_0^{a} x^2(l-x)dx + a\int_a^{\frac{l}{2}} x\,(l-x)\,dx\right\} - \frac{1}{A_0}\int_0^{a}\cos^2\phi\,dy}{\frac{16\,h^2}{l^4 I_0}\int_0^{\frac{l}{2}} x^2\,(l-x)^2\,dx + \frac{1}{A_0}\int_0^{\frac{l}{2}}\cos^2\phi\,dx}\,W.$$

Since

$$\int_0^{a} x^2\,(l-x)\,dx + a\int_a^{\frac{l}{2}} x\,(l-x)\,dx = \frac{a}{12}\,(a^3 - 2\,a^2\,l + l^3),$$

$$\int_0^{a}\cos^2\phi\,dy = \frac{l^2 e}{l^2 + 16\,h^2} \text{ nearly,*}$$

$$\int_0^{\frac{l}{2}} x^2(l-x)^2\,dx = \frac{l^5}{60},$$

$$\int_0^{\frac{l}{2}}\cos^2\phi\,dx = \frac{l^2}{8\,h}\,\phi_0,$$

* Howe, Treatise on Arches.

e denoting the value of y at a; and ϕ_0, the inclination of the tangent at A to the horizontal.

Putting

$$\frac{I_0}{A_0} = i^2,$$

we get,

$$H = \frac{1}{2}\,\frac{\dfrac{ha}{3\,l^2}(a^3 - 2\,a^2l + l^3) - \dfrac{l^2i^2e}{l^2 + 16\,h^2}}{\dfrac{4\,h^2l}{15} + \dfrac{i^2l^2}{8\,h}\phi_0.}\,W \quad . \quad . \quad (93)$$

Neglecting the effect of axial stress, the terms containing A disappear, and we get,

$$H = \frac{5\,a\,(a^3 - 2\,a^2l + l^3)}{8\,hl^3}\,W \quad . \quad . \quad . \quad . \quad (94)$$

49. For *uniform temperature* change t — positive for rise — by making similar substitutions as before in Eq. (89), since

$$\int_0^{l'} \frac{y^2dc}{I} = \frac{8}{15}\frac{h^2l}{I_0},$$

$$\int_0^{l'} \frac{\cos^2\phi\; dc}{A} = \frac{l^2\phi_0}{4\,hA_0},$$

we get,

$$H_t = \frac{t\theta EI_o}{\dfrac{8\,h^2}{15} + \dfrac{i^2l}{4\,h}\phi_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

Neglecting axial stress, similarly we get from Eq. (90),

$$H_t = \frac{15\,t\theta I_0E}{8\,h^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

50. For *change in span length* Δl — negative for increase — we get similarly from Eq. (91),

$$H_\Delta = \frac{EI_0\Delta l}{\dfrac{8\,h^2l}{15} + \dfrac{i^2l^2\phi_0}{4\,h}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

Neglecting axial stress,

$$H_\Delta = \frac{15\,EI_0\Delta l}{8\,h^2l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

CIRCULAR ARCH WITH TWO HINGES

51. Fig. 42 shows the axis of a symmetrical circular arch-rib with *uniform cross-section.*

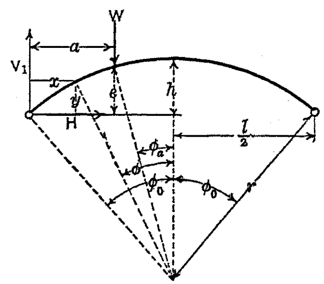

Fig. 42

In Eq. (86), making I and A constants, and putting

$$\frac{I}{A} = i^2,$$

we get

$$H = \frac{1}{2}\,\frac{\displaystyle\int_0^{a'} xydc + a\int_{a'}^{\frac{l'}{2}} ydc - i^2\int_0^{a} \sin\phi\; dx}{\displaystyle\int_0^{\frac{l'}{2}} y^2dc + i^2\int_0^{\frac{l}{2}} \cos\phi\; dx}\,W.$$

With designations as given in the figure, we have for circular arc,

$$x = r\,(\sin\phi_0 - \sin\phi),$$
$$dx = -\,r\cos\phi\; d\phi,$$
$$y = r\,(\cos\phi - \cos\phi_0),$$
$$dc = \frac{dx}{\cos\phi} = -\,rd\phi,$$
$$a = r\,(\sin\phi_0 - \sin\phi_a).$$

Substituting these in several terms of the expression for H, and integrating,

$$\int_0^{a'} xydc = r^3[\tfrac{1}{2}\sin^2\phi_0 - \sin\phi_0 \sin\phi_a + \tfrac{1}{2}\sin^2\phi_a - (\phi_0 - \phi_a)\sin\phi_0\cos\phi_0 - \cos^2\phi_0 + \cos\phi_0\cos\phi_a],$$

$$a\int_{a'}^{\frac{l'}{2}} ydc = r^3(\sin\phi_0 - \sin\phi_a)(\sin\phi_a - \phi_a\cos\phi_0),$$

$$\int_0^{a} \sin\phi\, dx = \frac{r}{2}(\sin^2\phi_0 - \sin^2\phi_a),$$

$$\int_0^{\frac{l'}{2}} y^2dc = r^3(\tfrac{1}{2}\phi_0 - \tfrac{3}{2}\sin\phi_0\cos\phi_0 + \phi_0\cos^2\phi_0),$$

$$\int_0^{\frac{l}{2}} \cos\phi\, dx = \frac{r}{2}(\sin\phi_0\cos\phi_0 + \phi_0),$$

we get,

$$H = \frac{1}{2}\,\frac{2\cos\phi_0(\cos\phi_a + \phi_a\sin\phi_a - \cos\phi_0 - \phi_0\sin\phi_0) + \left(1 - \frac{i^2}{r^2}\right)(\sin^2\phi_0 - \sin^2\phi_a)}{(\phi_0 - 3\sin\phi_0\cos\phi_0 + 2\phi_0\cos^2\phi_0) + \frac{i^2}{r^2}(\sin\phi_0\cos\phi_0 + \phi_0)}\,W. \quad (99a)$$

or, since

$$\sin\phi_0 = \frac{l}{2r}, \qquad \cos\phi_0 = \frac{r-h}{r};$$

$$\sin\phi_a = \frac{l-2a}{2r}, \qquad \cos\phi_a = \frac{r-h+e}{r};$$

$$\phi_0 = \frac{l'}{2r}, \qquad \phi_a = \frac{l'-2a'}{2r};$$

$$H = \frac{(r-h)\{2e + \phi_a(l-2a) - \phi_0 l\} + \left(1 - \frac{i^2}{r^2}\right)(l-a)a}{4(r-h)\{\phi_0(r-h) - l\} + \left(1 + \frac{i^2}{r^2}\right)\{2r^2\phi_0 + (r-h)l\}}\,W \quad . \ (99b)$$

Neglecting axial compression, we get,

$$H = \frac{\sin^2\phi_0 - \sin^2\phi_a + 2\cos\phi_0(\cos\phi_a + \phi_a\sin\phi_a - \cos\phi_0 - \phi_0\sin\phi_0)}{2(\phi_0 - 3\sin\phi_0\cos\phi_0 + 2\phi_0\cos^2\phi_0)}\,W \quad (100a)$$

or

$$H = \frac{(r-h)\{2e + \phi_a(l-2a) - \phi_0 l\} + a(l-a)}{2\phi_0\{r^2 + 2(r-h)^2\} - 3l(r-h)}\,W \quad . \ (100b)$$

52. Temperature Stress.—For uniform temperature change of t degrees, Eq. (89) may be written for constant I and A:

$$H_t = \frac{t\theta lEI}{2\int_0^{\frac{l'}{2}} y^2\,dc + 2i^2\int_0^{\frac{l}{2}}\cos\phi\,dx}.$$

Introducing in this the integrals already given, we obtain,

$$H_t = \frac{t\theta lEI}{r^3(\phi_0 - 3\sin\phi_0\cos\phi_0 + 2\phi_0\cos^2\phi_0) + i^2 r(\sin\phi_0\cos\phi_0 + \phi_0)} \quad (101a)$$

or

$$H_t = \frac{t\theta lEI}{r\left[\phi_0\{r^2 + 2(r-h)^2 + i^2\} - \frac{1}{2}\left(3 - \frac{i^2}{r^2}\right)(r-h)l\right]} \quad . \ . \ . \ (101b)$$

Neglecting axial stress, we get,

$$H_t = \frac{t\theta lEI}{r^3(\phi_0 - 3\sin\phi_0\cos\phi_0 + 2\phi_0\cos^2\phi_0)} \quad . \ . \ . \ . \ . \ . \ (102a)$$

or

$$H_t = \frac{t\theta lEI}{r[\phi_0\{r^2 + 2(r-h)^2\} - \frac{3}{2}l(r-h)]} \quad . \ . \ . \ . \ . \ . \ (102b)$$

53. Displacement Stress. — For change in span length by Δl — negative for increase of span length — we have from Eq. (91),

$$H_\Delta = \frac{EI\,\Delta l}{2\int_0^{\frac{l'}{2}} y^2\,dc + 2i^2\int_0^{\frac{l}{2}}\cos\phi\cdot dx}.$$

Making the same substitutions as before, we get,

$$H_\Delta = \frac{EI\,\Delta l}{(\text{denominators same as for } H_t)} \quad . \ . \ . \ (103)$$

SEMICIRCULAR ARCH

54. In this, since

$$\phi_0 = \frac{\pi}{2},$$

we obtain at once from Eq. (99*a*),

$$H = \frac{(r^2 - i^2)\cos^2\phi_a}{\pi\,(r^2 + i^2)} W \quad \cdots\cdots \quad (104)$$

Neglecting axial compression, we get from Eq. (100*a*),

$$H = \frac{\cos^2\phi_a}{\pi} W \quad \cdots\cdots\cdots \quad (105)$$

55. For *uniform temperature change t*, we have from Eq. (101*a*),

$$H_t = \frac{2\,t\theta l E I}{\pi r\,(r^2 + i^2)} \quad \cdots\cdots\cdots \quad (106)$$

and for the same, by neglecting axial stress,

$$H_t = \frac{2\,t\theta l E I}{\pi r^3} \quad \cdots\cdots\cdots \quad (107)$$

56. For *change in span length* Δl similarly from Eq. (103) referred to Eq. (102*a*), we get,

$$H_\Delta = \frac{2\,EI\Delta l}{\pi r\,(r^2 + i^2)} \quad \cdots\cdots\cdots \quad (108)$$

and for the same, with axial stress neglected,

$$H_\Delta = \frac{2\,EI\Delta l}{\pi r^3} \quad \cdots\cdots\cdots \quad (109)$$

FLAT ARCH WITH TWO HINGES

57. When the rise of an arch is very small compared with its span length, we may put without material error,

$$dc = dx.$$

Assuming the cross-section of the arch to be uniform throughout, and putting as before

$$\frac{I}{A} = i^2,$$

Eqs. (86), (89), and (91) may be written as follows:

$$H = \frac{1}{2}\,\frac{\int_0^a xydx + a\int_a^{\frac{l}{2}} ydx - i^2\int_0^{a'} \sin\phi\, dc}{\int_0^{\frac{l}{2}} y^2\,dx + i^2\int_0^{\frac{l'}{2}} \cos\phi\, dc} W \;.\; (110)$$

$$H_t = \frac{t\theta l E I}{\int_0^{l} y^2\,dx + i^2\int_0^{l'} \cos\phi\, dc} \quad \cdots \quad (111)$$

$$H_\Delta = \frac{EI\Delta l}{\int_0^{l} y^2\,dx + i^2\int_0^{l'} \cos\phi\, dc} \quad \cdots \quad (112)$$

FLAT PARABOLIC ARCH WITH TWO HINGES

(Uniform Cross-Section)

58. For this, we have but to introduce in Eq. (110) the equation of parabola

$$y = \frac{4\,h}{l^2}\,x\,(l - x),$$

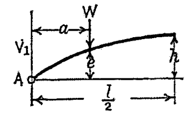

Fig. 43

to obtain expression for H due to any load W (Fig. 43). Integrating the terms of the equation severally, we have,

$$\int_0^a xydx + a\int_a^{\frac{l}{2}} ydx = \frac{a^3h}{3\,l^2}(4\,l - 3\,a) + \frac{ah}{3\,l^2}(l^3 - 6\,a^2 l + 4\,a^3),$$

$$\int_0^{a'} \sin\phi\, dc = e,$$

$$\int_0^{\frac{l}{2}} y^2\,dx = \frac{4\,h^2 l}{15},$$

$$\int_0^{\frac{l}{2}} \cos\phi\, dc = \frac{l}{2},$$

so that

$$H = \frac{1}{2} \frac{\frac{ah}{3\, l^2 i^2}(l - a)(l^2 + al - a^2) - e}{\frac{4\, h^2 l}{15\, i^2} + \frac{l}{2}} W \quad . \quad . \quad . \quad (113)$$

Neglecting e as being inconsiderable in comparison with other terms of the numerator, we obtain,

$$H = \frac{5\, ah\, (l - a)\, (l^2 + al - a^2)}{l^3\, (8\, h^2 + 15\, i^2)} W \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

Further neglecting the effect of axial stress, we get,

$$H = \frac{5\, a\, (l - a)\, (l^2 + al - a^2)}{8\, hl^3} W \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

59. For a *uniformly distributed load* of w per unit length of the span, we obtain, by substituting wda for W in the preceding equations, and integrating between given limits of loading, the equation for H. Thus, referring to Fig. 44, we have, from Eq. (115),

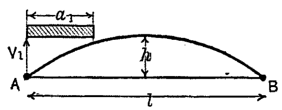

Fig. 44

$$H = \int_0^{a_1} \frac{5\, a\, (l-a)\, (l^2+al-a^2)}{8\, hl^3} wda = \frac{a_1^2\, (5\, l^3 - 5\, a_1^2 l + 2\, a_1^3)\, w}{16\, hl^3} \cdot (116)$$

By taking moment at B,

$$V_1 = \frac{a_1\, (2\, l - a_1)\, w}{2\, l}.$$

For full uniform load,

$$H = \frac{l^2 w}{8\, h} \text{ approximately.}$$

60. Temperature Stress. — Introducing in Eq. (111) the equation of parabola, and integrating as before, we obtain,

$$H_t = \frac{t\theta l EI}{\frac{8\, h^2 l}{15} + i^2 l} = \frac{15\, t\theta EI}{8\, h^2 + 15\, i^2} \quad . \quad . \quad (117)$$

Neglecting axial stress, we get,

$$H_t = \frac{15\, t\theta EI}{8\, h^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

61. Displacement Stress. — For a change of Δl in span length — negative for increase of the latter — similarly we get from Eq. (112),

$$H_\Delta = \frac{15\, EI\Delta l}{l\, (8\, h^2 + 15\, i^2)} \quad . \quad . \quad . \quad . \quad . \quad (119)$$

and for the same with axial stress neglected,

$$H_\Delta = \frac{15\, EI\Delta l}{8\, h^2 l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (120)$$

FLAT CIRCULAR ARCH WITH TWO HINGES

62. Since a circular arc with comparatively small versed sine closely follows parabolic curve, the formulas deduced for parabolic arches (Eqs. 113–120) may be used for this kind of arches without appreciable error.

SPANDREL-BRACED ARCH WITH TWO HINGES

63. The foregoing formulas are generally inapplicable to a spandrel-braced arch, owing to the lack in the latter of definite form in its axis and the irregular variation of the moment of inertia of its section. As the only statically-indeterminate force in this case is again H, in order to find the value of the latter which will make ω a minimum, it is simply necessary to find stresses in each member in terms of H and other external forces, and obtain,

$$\omega = \sum \frac{S^2 L}{2\, AE},$$

extending the second member over the whole structure.

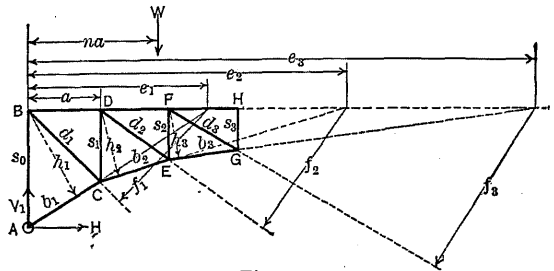

Fig. 45

Fig. 45 shows the left half of a symmetrical spandrel-braced arch. The following designations will be used:

s_0, s_1, etc. . . the lengths of vertical members.
d_1 d_2, etc. . . the lengths of diagonal members.
a . . . . . . the horizontal panel length.
b_1 b_2, etc. . . . the lengths of lower chord-members.
A . . . . . . the sectional area of a member, with suffix corresponding to the members to which it pertains.

If we assume, in the first place, the arch to be loaded with two equal loads of W each, distant na from each end, we obtain the following stresses for the case $n=1$ by taking moments at successive sections, the arm-lengths being designated as shown in the figure:

WEB-MEMBERS

$$\overline{AB} = -W + \frac{s_0 - s_1}{a} H. \qquad \overline{BC} = \frac{We_1 - Hs_0}{f_1}.$$

$$\overline{CD} = -\frac{We_2 - Hs_0}{e_2 - a}. \qquad \overline{DE} = \frac{Wa - Hs_0}{f_2}.$$

$$\overline{EF} = -\frac{Wa - Hs_0}{e_3 - 2a}. \qquad \overline{FG} = \frac{Wa - Hs_0}{f_3}.$$

CHORD-MEMBERS

$$\overline{BD} = -\frac{Wa - H(s_0 - s_1)}{s_1}. \qquad \overline{AC} = -\frac{Hs_0}{h_1}.$$

$$\overline{DF} = -\frac{Wa - H(s_0 - s_2)}{s_2}. \qquad \overline{CE} = \frac{Wa - Hs_0}{h_2}.$$

$$\overline{FH} = -\frac{Wa - H(s_0 - s_3)}{s_3}. \qquad \overline{EG} = \frac{Wa - Hs_0}{h_3}.$$

Introducing these in the expression for total internal work, we get,

$$\frac{\Sigma S^2 L}{2AE} = \frac{2}{E}\left\{\left(-W + \frac{s_0 - s_1}{a} H\right)^2 \frac{s_0}{2A_{ab}} + \frac{(We_2 - Hs_0)^2}{(e_2 - a)^2} \frac{s_1}{2A_{cd}}\right.$$
$$+ \left(\frac{Wa - Hs_0}{e_3 - 2a}\right)^2 \frac{s_2}{2A_{ef}} + \left(\frac{We_1 - Hs_0}{f_1}\right)^2 \frac{d_1}{2A_{bc}} + \left(\frac{Wa - Hs_0}{f_2}\right)^2 \frac{d_2}{2A_{de}}$$
$$+ \left(\frac{Wa - Hs_0}{f_3}\right)^2 \frac{d_3}{2A_{fg}} + \left(\frac{Hs_0}{h_1}\right)^2 \frac{b_1}{2A_{ac}} + \left(\frac{Wa - Hs_0}{h_2}\right) \frac{b_2}{2A_{ce}}$$
$$+ \left(\frac{Wa - Hs_0}{h_3}\right)^2 \frac{b_3}{2A_{eg}} + \left(\frac{Wa - H(s_0 - s_1)}{s_1}\right)^2 \frac{a}{2A_{bd}}$$
$$\left. + \left(\frac{Wa - H(s_0 - s_2)}{s_2}\right)^2 \frac{a}{2A_{df}} + \left(\frac{Wa - H(s_0 - s_3)}{s_3}\right)^2 \frac{a}{2A_{fh}}\right\}.$$

Differentiating this with respect to H, and setting the differential coefficient equal to zero, we at once get,

$$H = \frac{\dfrac{(s_0 - s_1)s_0}{aA_{ab}} + \dfrac{e_2 s_0 s_1}{(e_2 - a)^2 A_{cd}} + \dfrac{a s_0 s_2}{(e_3 - 2a)^2 A_{ef}} + \dfrac{e_1 s_0 d_1}{f_1^2 A_{bc}} + \dfrac{a^2(s_0 - s_1)}{s_1^2 A_{bd}}}{\dfrac{(s_0 - s_1)^2 s_0}{a^2 A_{ab}} + \dfrac{s_0^2 s_1}{(e_2 - a)^2 A_{cd}} + \dfrac{s_0^2 s_2}{(e_3 - 2a)^2 A_{ef}} + \dfrac{b_1 s_0^2}{h_1^2 A_{ac}} + \dfrac{d_1 s_0^2}{f_1^2 A_{bc}}}$$
$$\frac{+ \dfrac{a s_0 b_2}{h_2^2 A_{ce}} + \dfrac{a s_0 d_2}{f_2^2 A_{de}} + \dfrac{a^2(s_0 - s_2)}{s_2^2 A_{df}} + \dfrac{a s_0 b_3}{h_3^2 A_{eg}} + \dfrac{a s_0 d_3}{f_3^2 A_{fg}} + \dfrac{a^2(s_0 - s_3)}{s_3^2 A_{fh}}}{+ \dfrac{a(s_0 - s_1)^2}{s_1^2 A_{bd}} + \dfrac{b_2^2 s_0^2}{h_2^2 A_{ce}} + \dfrac{s_0^2 d_2}{f_2^2 A_{de}} + \dfrac{a(s_0 - s_2)^2}{s_2^2 A_{df}} + \dfrac{b^3 s_0^2}{h_3^2 A_{eg}} + \dfrac{d_3 s_0^2}{f_3^2 A_{fg}} + \dfrac{a(s_0 - s_3)^2}{s_3^2 A_{fh}}} W,$$

which is the value of H for 2 W, so that H for 1 W will be one-half the amount given by this equation.

In a similar manner we obtain in the arch of this type with *any number of panels*, for 1 W the following expression for H:

$$H = \frac{\sum_{0}^{na}\left\{\frac{pa^2(s_0-s)}{s^2A_u}+\frac{pas_0b}{h^2A_l}+\frac{es_0s}{(e-pa)^2A_v}+\frac{es_0d}{f^2A_d}\right\}+\frac{(s_0-s_1)s_0}{aA_{ab}}+\sum_{na}^{\frac{l}{2}}\left\{\frac{na^2(s_0-s)}{s^2A_u}+\frac{nas_0b}{h^2A_l}+\frac{nas_0s}{(e-pa)^2A_v}+\frac{nas_0d}{f^2A_d}\right\}}{2\sum_{0}^{\frac{l}{2}}\left\{\frac{(s_0-s)^2a}{s^2A_u}+\frac{s_0^2b}{h^2A_l}+\frac{s_0^2s}{(e-pa)^2A_v}+\frac{s_0^2d}{f^2A_d}\right\}+2\,\frac{(s_0-s_1)^2s_0}{a^2A_{ab}}}\,W. \quad (121)$$

in which p represents, in case of chord-members, the distance — in number of panels — of the panel point opposite the member under consideration, and in case of web-members the ordinal number — from the nearest support — of the upper chord opposite the web-member in question. Thus, referring to Fig. 45, we find for DF, EG, EF, and DE, $p = 2$.

A_u, A_l, A_v, and A_d = cross-sectional areas of the upper and lower chords, verticals and diagonals respectively.

e . . . . . = distance from the support — nearest to the member — of the intersection of the lower chord-member opposite the web under consideration with the upper chord.

It is further understood that Σ is to be extended over the loaded half of the arch only.

64. Eq. (121) gives mathematically correct results so long as the supports are perfectly immovable, and is applicable when dimensions of all the members of the arch are given. For designing an arch of this kind, the calculation may be started with following approximations:

Assume each chord to be of uniform cross-section throughout its length, and let

$$m = \frac{A_l}{A_u}.$$

Further, neglect the effect of web-stresses, which is generally inconsiderable when compared with that of chord-stresses. Then we get from (121) the following approximate expression for H, freed of all the cross-sectional areas of members:

$$H = \frac{\sum_{0}^{na}\left\{\frac{pa^2(s_0-s)}{s^2}+\frac{pas_0b}{mh^2}\right\}+\sum_{na}^{\frac{l}{2}}\left\{\frac{na^2(s_0-s)}{s^2}+\frac{nas_0b}{mh^2}\right\}}{2\sum_{0}^{\frac{l}{2}}\left\{\frac{s_0^2b}{mh^2}+\frac{(s_0-s)^2a}{s^2}\right\}}W. \quad (122)$$

With H obtained with this equation, the dimensions of all the members may be calculated, and then corrected, if desired, by the use of Eq. (121). Generally, Eq. (122) by itself gives results sufficiently correct for all practical purposes.

65. Temperature Stress. — The internal work caused by H_t in the arch of Fig. 45, neglecting the effect of web-stresses, will be,

$$\omega = \frac{H_t^2}{EA}\left\{\frac{s_0^2b_1}{mh_1^2}+\frac{s_0^2b_2}{mh_2^2}+\frac{s_0^2b_8}{mh_8^2}+\frac{(s_0-s_1)^2a}{s_1^2}+\frac{(s_0-s_2)^2a}{s_2^2}+\frac{(s_0-s_3)^2a}{s_8^2}\right\}$$

in which A represents the section of upper chord, and mA that of the lower.

Since
$$\frac{d\omega}{dH_t} = t\theta l,$$
we get,
$$H_t = \frac{t\theta lEA}{2\left\{\frac{s_0^2 b_1}{mh_1^2} + \frac{s_0^2 b_2}{mh_2^2} + \frac{s_0^2 b_3}{mh_3^2} + \frac{(s_0 - s_1)^2 a}{s_1^2} + \frac{(s_0 - s_2)^2 a}{s_2^2} + \frac{(s_0 - s_3)^2 a}{s_3^2}\right\}},$$

t being as before positive for rising temperature. Generally, for any number of panels we get in a similar manner,

$$H_t = \frac{t\theta lEA}{2\sum_0^{\frac{l}{2}}\left\{\frac{s_0^2 b}{mh^2} + \frac{(s_0 - s)^2 a}{s^2}\right\}} \quad . \quad . \quad . \quad (123)$$

66. Displacement Stress. — From the preceding discussions, it will at once be seen that for a change in span length, of Δl, we have but to substitute Δl — negative for increase of span length — for $t\theta l$ to obtain an expression for H_Δ, so that we get,

$$H_\Delta = \frac{EA\Delta l}{2\sum_0^{\frac{1}{2}l}\left\{\frac{s_0^2 b}{mh^2} + \frac{(s_0 - s)^2 a}{s^2}\right\}} \quad . \quad . \quad . \quad (124)$$

THE STRESSES IN FLANGES AND WEBS OF A RIB

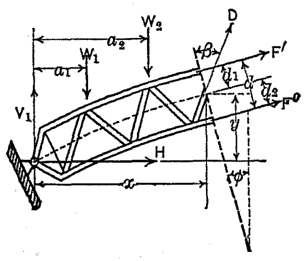

Fig. 46

67. Knowing V_1 and H for a given loading, the stresses in the flanges and web of a parallel rib may at once be obtained statically. Thus, at a radial section through any point x, y (Fig. 46) of an arch-rib, let

F' = the upper flange stress.
F'' = the lower flange stress.
D = web-stress.
d = $d_1 + d_2$ = distance between centres of gravity of upper and lower flanges.

Taking moment with respect to the point x, y in the axis of the rib, we have,

$$V_1 x - Hy - \sum_0^x W(x - a) + F'd_1 - F''d_2 = 0,$$

from which

$$F''d_2 - F'd_1 = V_1 x - Hy - \sum_0^x W(x - a).$$

But

$$F'' + F' = N = -(V \sin\phi + H \cos\phi) \text{ (Art. 43)}.$$

Combining these two equations, we get,

$$F'' = \frac{1}{d}\left\{V_1 x - Hy - \sum_0^x W(x - a) - (V \sin\phi + H \cos\phi)d_1\right\} \quad (125)$$

$$F' = -\frac{1}{d}\left\{V_1 x - Hy - \sum_0^x W(x - a) + (V \sin\phi + H \cos\phi)d_2\right\} \quad (126)$$

If
$$d_1 = d_2,$$
i.e., if the section of the rib is symmetrical about the neutral axis,

$$F'' = \frac{1}{d}\left\{V_1 x - Hy - \sum_0^x W(x - a)\right\} - \frac{d}{2}(V \sin\phi + H \cos\phi)\Big\} \quad (127)$$

$$F' = -\frac{1}{d}\left\{V_1 x - Hy - \sum_0^x W(x - a)\right\} + \frac{d}{2}(V \sin\phi + H \cos\phi)\Big\} \quad (128)$$

In case the chord-members are *curved between two panel points*, the direct stress thus found must be combined with the moment equal to the direct stress multiplied by the versed sine of the panel arc.

Again, since at the section

$$D \cos \beta \doteq T = -(V \cos \phi - H \sin \phi) \text{ (Art. 43)},$$
$$D = -(V \cos \phi - H \sin \phi) \sec \beta \quad . \quad . \quad . \quad . \quad (129)$$

in which β represents the inclination of the web-member to the radius of the arc at the section.

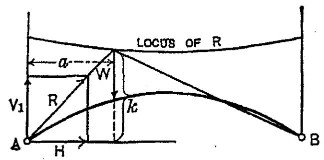

Fig. 47

In case the chords are *not parallel*, the stress in any member is best obtained by taking moment with respect to the intersection of the other two belonging to the panel, as in the case of spandrel-braced arch.

POSITION OF LOADS FOR MAXIMUM STRESS

68. For finding the position of loads to give maximum stress at any point of the arch, *reaction locus* may be made use of with advantage.

Let

R = reaction at A due to any load W.
V_1 = vertical component of R.
H = horizontal component of R.
k = ordinate to the locus of R.

Referring then to Fig. 47, it will be seen that,

$$k = \frac{V_1}{H} a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

which makes the locus at once determinate.

69. Having the locus drawn, the mode of loading giving maximum moment with respect to any given point of

the rib may be laid off, remembering that R passing above the point produces + moment, and that below, − moment. Thus in Fig. 48 it will be seen that the stress in any member EF will be maximum when the moment with respect to point O reaches its greatest amount. The curve $CKGD$ being the reaction locus, a load at G will produce no stress in EF, for then the reaction passes through O. For loads to the right of G, by considering the portion of the rib left of O, it will be seen that the moment of R being −, the stress in EF will be tension, while for loads to the left of G, by considering successively the left and right portions of the rib, the moment with respect to O being positive, EF will be in compression. For loads beyond K, considering the right portion of the arch, the moment of right reaction with respect to O being negative, EF will once more be in tension.

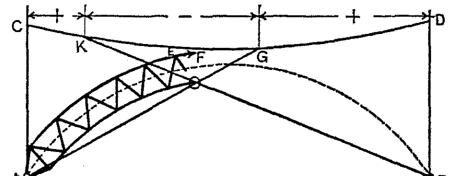

Fig. 48

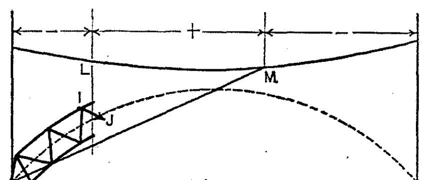

Fig. 49

Similarly, for maximum stress in a lower chord-member, the reaction line drawn through the panel point opposite the member will give the mode of loading.

70. For maximum stress in a web-member generally, the reaction line is to be passed through the intersection of chord-members belonging to the panel to determine the limits of loading. In case the chords are parallel at the panel, since the web-member will not then be strained when the direction of the resultant force coincides with that of the chords, the position of load, for no stress in the web-member, is given by drawing the reaction line parallel to the chords, or to the tangent to the curve. Thus in Fig. 49, point M is the position of load producing

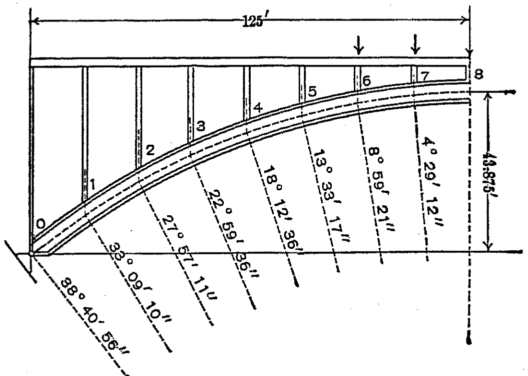

Fig. 50

no stress in IJ. Another position of load for no stress in IJ is at L directly over the section, — a point which will be evident by considering alternately the right and left of the section with its respective reactions. The signs of shear, and, with them, those of the stress in the web-member due to loads between the points of no stress, will at once be known by referring to Art. 43.

Example. — In the circular plate-webbed arch with two hinges, of which Fig. 50 shows its left half, to calculate the

maximum stress in the rib at panel point 3, due to following panel loads:

Dead load = 20 tons per panel,
Live load = 10 tons per panel.

The following dimensions are given:

$l = 250$ ft.
$r = 200$ ft.
$\phi_0 = 38° - 40' - 56'' = .67514.$
Panel length = 15.625 ft.
Cross-section uniform and symmetrical throughout,
$\frac{i^2}{r^2} = .00019.$
Effective depth (dist. of c. g. of flanges) = 6 ft.

The distances of the points of application of loads, etc., are as follows:

a (ft.).	e (ft.).	ϕ_a (circ. meas.).	$\phi_a\,(l-2a)$.	$a\,(l-a)$.	$\left(1-\frac{i^2}{r^2}\right) a(l-a)$.
15.63	11.32	.57863	126.58	3662	3661
31.25	20.54	.48788	91.48	6836	6835
46.88	27.99	.40114	62.68	9517	9520
62.50	33.86	.31782	39.73	11719	11716
78.13	38.31	.23658	22.18	13428	13425
93.75	41.42	.15689	9.81	14648	14646
109.38	43.26	.07831	2.45	15381	15378
125.00	43.88	.00000	0.00	15625	15622

In Eq. (99b),

$$H = \frac{(r-h)\{2e + \phi_a(l-2a) - \phi_0 l\} + \left(1 - \frac{i^2}{r^2}\right)(l-a)\,a}{4\,(r-h)\{\phi_0\,(r-h) - l\} + \left(1 + \frac{i^2}{r^2}\right)\{2r^2\phi_0 + (r-h)\,l\}}\,W.$$

Introducing the numerical values in the numerator, the denominator being

$$4(r-h)\{\phi_0(r-h)-l\}+\left(1+\frac{i^2}{r^2}\right)\{2r^2\phi_0+(r-h)l\}=2766,$$

we get for $W = 1$ the following values of H:

Load at	1	2	3	4	5	6	7	8
$H=$	.2189	.4263	.6118	.7737	.9027	.9971	1.0546	1.0739

For the dead load we have then,

$$V_1 = 7\tfrac{1}{2} \times 20 = 150 \text{ tons},$$
$$H = 2 \times 20\,(.2189 + .4263 + .6118 + .7737 + .9027 + .9971 + 1.0546 + .5370) = 220.88 \text{ tons.}$$

Drawing the reaction locus, and passing reaction lines through E and F,—the centres of gravity of upper and lower

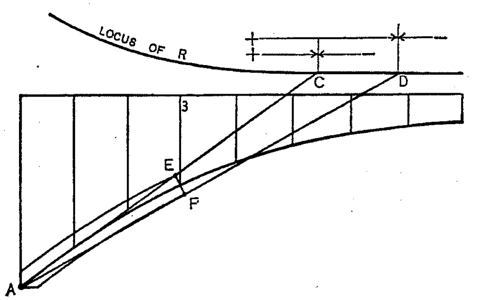

Fig. 51

flanges at 3,—we find (Fig. 51) that all loads lying to the right of C produce negative moment with respect to E and hence compression in the lower flange; while for similar reason all loads to the left of D produce compression in the upper flange.

By drawing AM parallel to the tangent to the axis of the rib at 3, and passing a vertical through E, we find that the loads between L and M produce positive shear in the section EF, and those outside of LM, the negative shear.

The amounts of H for the positions of the live load found above are:

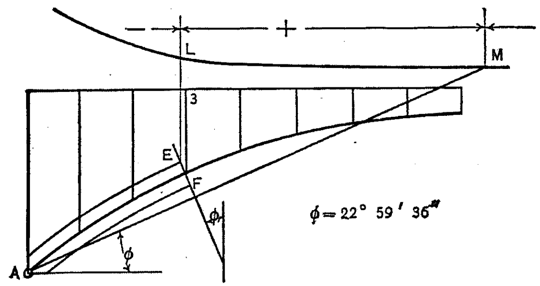

Fig. 52

For the live load covering $A - D$,

$$V_1 = \tfrac{10}{16}(15 + 14 + 13 + 12 + 11 + 10) = 46.88 \text{ tons.}$$
$$H = 10\,(.2189 + \cdots .9027 + 1.0546) = 39.88 \text{ tons.}$$

For the live load covering $C - B$,

$$V_1 = \tfrac{10}{16}(1 + \cdots 10) = 34.38 \text{ tons.}$$
$$H = 10\{2\,(1.0739 + 1.0546 + .9971) + .9027 + \cdots .2189\}$$
$$= 91.85 \text{ tons.}$$

For the live load covering $A - L$ and $M - B$,

$$V_1 = \tfrac{10}{16}(1 + \cdots 7 + 14 + 15) = 35.62 \text{ tons.}$$
$$H = 10\{2(.2189 + .4263) + .6118 + \cdots 1.0546\} = 56.30 \text{ tons.}$$

For the live load covering $L - M$,

$$V_1 = \tfrac{10}{16}(8 + 9 + 10 + 11 + 12 + 13) = 39.38 \text{ tons.}$$
$$H = 10\,(.6118 + \cdots 1.0739) = 54.14 \text{ tons.}$$

At the neutral axis under the panel point 3, we have

$$x = 46.88 \text{ ft.} \qquad y = 27.99 \text{ ft.} \qquad \phi = 22°59'36''.$$
$$\sin\phi = .3906. \qquad \cos\phi = .9336.$$

Substituting these values in Eqs. (127) and (128), we get for the total stress in flanges at 3:

$$F'' = \tfrac{1}{6}\{184.38 \times 46.88 - 312.73 \times 27.99 - 20\,(31.25 + 15.63)$$
$$- 3\,(184.38 \times .3906 + 312.73 \times .9336)\} = -\,356.38 \text{ tons.}$$
$$F' = -\tfrac{1}{6}\{196.88 \times 46.88 - 260.76 \times 27.99 - 30\,(31.25 + 15.63)$$
$$+ 3\,(196.88 \times .3906 + 260.76 \times .9336)\} = -\,247.62 \text{ tons.}$$

As to shear acting in the normal section at 3, we have (Art. 43),

$$V\cos\phi - H\sin\phi$$
$$= (150 + 39.38 - 2 \times 20)\,.9336 - (220.88 + 54.14)\,.3906$$
$$= 32.04 \text{ tons for maximum.}$$
$$= (150 + 35.62 - 2 \times 30)\,.9336 - (220.88 + 56.30)\,.3906$$
$$= 9.01 \text{ tons for minimum.}$$

Had we neglected the effect of axial stress in the preceding calculation, Eq. (100*b*) would have given for H,

Load at	1	2	3	4	5	6	7	8
$H=$	.2206	.4295	.6164	.7794	.9095	1.0046	1.0625	1.0811

So that we get for dead load,

$$V_1 = 150 \text{ tons,}$$
$$H = 20 \times 2 \times 5.5630 = 222.52 \text{ tons;}$$

and for live load covering A -- D,

$$V_1 = 46.88 \text{ tons,}$$
$$H = 39.60.$$

Substituting these values in Eq. (128), we get for maximum compression in the upper flange at 3,

$$F' = -\tfrac{1}{6}\{196.88 \times 46.88 - 262.12 \times 27.99 - 30\,(31.25 + 15.63)$$
$$+ 3\,(196.88 \times .3906 + 262.12 \times .9336)\} = -\,241.91 \text{ tons.}$$

It will be seen from these calculations, that the effect of neglecting axial stress, while producing a difference of less than 0.8 per cent in the amounts of H, is more strongly felt in chord-stresses, in which the difference, in the case taken, amounts to more than 2 per cent.

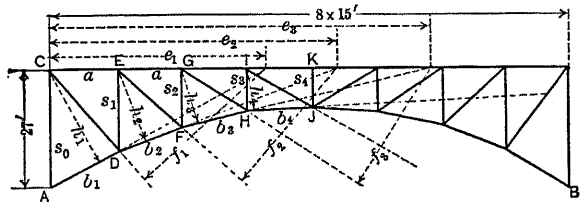

Fig. 53

EXAMPLE. — In the spandrel-braced arch of Fig. 53 with dimensions as tabulated below, to calculate the maximum stress in a member FH due to the following panel loads:

Dead load . . 16 tons per panel,
Live load . . 20 tons per panel.

Lengths of members and arms in feet:

	s.	*d.*	*b.*	*e.*	*h.*	*f.*
1	18.7	23.9	17.1	48.8	23.6	38.2
2	13.2	20.0	16.0	66.0	17.6	33.7
3	10.0	18.0	15.3	91.9	12.9	34.4
4	9.0	17.5	15.0	195.0	10.0	77.1

Assuming $m = 2$ (Art. 64), we have in Eq. (122):

Load	Panel.	$\sum_0^{na} \frac{pa^2(s_0 - s)}{s^2}$	$\sum_{na}^{\frac{l}{2}} \frac{na^2(s_0 - s)}{s^2}$	$\sum_0^{na} \frac{pa\,s_0 b}{2h^2}$	$\sum_{na}^{\frac{l}{2}} \frac{na\,s_0 b}{mh^2}$	$\sum_0^{\frac{l}{2}} \frac{s_0^2 b}{mh^2}$	$\sum_0^{\frac{l}{2}} \frac{(s_0 - s)^2 a}{s^2}$
Load 1 at E $(n = 1)$	CE	5.34	. . .	. . .	. . .	11.19	2.96
	EG	. . .	17.82	. . .	10.46	19.37	16.40
	GI	. . .	38.25	. . .	18.62	33.51	43.35
	IK	. . .	50.00	. . .	30.38	54.68	60.00
	Σ	5.34	106.07	. . .	59.46	118.75	122.71
	$H_1 = \frac{5.34 + 106.07 + 59.46}{2(118.75 + 122.71)} = .356$						
Load 1 at G $(n = 2)$	CE	5.34	. . .	. . .	. . .	. . .	. . .
	EG	35.64	. . .	10.46	. . .	. . .	. . .
	GI	. . .	76.50	. . .	37.24	. . .	. . .
	IK	. . .	100.00	. . .	60.76	. . .	. . .
	Σ	40.98	176.50	10.46	98.00	118.75	122.71
	$H_2 = \frac{325.95}{482.92} = .675$						
Load 1 at I $(n = 3)$	CE	5.34	. . .	. . .	. . .	. . .	. . .
	EG	35.64	. . .	10.46	. . .	. . .	. . .
	GI	114.75	. . .	37.24	. . .	. . .	. . .
	IK	. . .	150.00	. . .	91.14	. . .	. . .
	Σ	155.73	150.00	47.70	91.14	118.75	122.71
	$H_3 = \frac{444.57}{482.92} = .921$						
Load 1 at K $(n = 4)$	CE	5.34	. . .	. . .	. . .	. . .	. . .
	EG	35.64	. . .	10.46	. . .	. . .	. . .
	GI	114.75	. . .	37.24	. . .	. . .	. . .
	IK	200.00	. . .	91.14	. . .	. . .	. . .
	Σ	355.73	. . .	138.84	. . .	118.75	122.71
	$H_4 = \frac{494.57}{482.92} = 1.024$						

For the dead load we then have,

$$V_1 = 16 \times 3\tfrac{1}{2} = 56 \text{ tons},$$
$$H = 16 \times 2(.356 + .675 + .921 + .512) = 78.85 \text{ tons}.$$

Drawing the reaction locus and then the reaction line through G (Fig. 54) to the locus, we see that all loads to the right of L will produce compression in FH, while those to the left, tension.

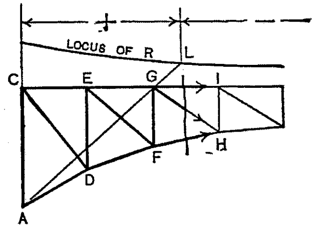

Fig. 54

For these positions of the live load we have,

Live load covering $C - G$,

$$V_1 = \tfrac{20}{8}(7 + 6) = 32.50 \text{ tons},$$
$$H = 20(.356 + .675) = 20.62 \text{ tons}.$$

Live load covering $I - B$,

$$V_1 = \tfrac{20}{8}(1 + 2 + 3 + 4 + 5) = 37.5 \text{ tons},$$
$$H = 20(.356 + .675 + .921 + 1.024 + .921) = 77.94 \text{ tons}.$$

Taking moment with respect to G, we obtain the following extreme stresses in FH:

$$\text{Max. } FH = \frac{1}{12.9}\{(56 + 37.5)\,30 - (78.85 + 77.94)\,27 - 16 \times 15\}$$
$$= -\,129.33 \text{ tons}.$$

$$\text{Min. } FH = \frac{1}{12.9}\{(56 + 32.5)\,30 - (78.85 + 20.62)\,27 - 16 \times 15\}$$
$$= 3 - 43.47 \text{ tons}.$$

Suppose now that the following cross-sections (in sq. in.) of the members are given:

Panel.	Upper Chord.	Lower Chord.	Diagonals.	Verticals.
CE	6	28	7	6
EG	9	23	7	10
GI	15	17	7	9
IK	17	17	5	7

Then in Eq. (121),

$$H = \frac{\sum_0^{na}\left\{\frac{pa^2(s_0-s)}{s^2A_u}+\frac{pas_0b}{h^2A_l}+\frac{es_0s}{(e-pa)^2A_v}+\frac{es_0d}{f^2A_d}\right\}+\frac{(s_0-s_1)s_0}{aA_{ab}}+\sum_{na}^{\frac{l}{2}}\left\{\frac{na^2(s_0-s)}{s^2A_u}+\frac{nas_0b}{h^2A_l}+\frac{nas_0s}{(e-pa)^2A_v}+\frac{nas_0d}{f^2A_d}\right\}}{2\sum_0^{\frac{l}{2}}\left\{\frac{(s_0-s)^2a}{s^2A_u}+\frac{s_0^2b}{h^2A_l}+\frac{s_0^2s}{(e-pa)^2A_v}+\frac{s_0^2d}{f^2A_d}\right\}+2\frac{(s_0-s_1)^2s_0}{a^2A_{ab}}}W$$

we have,

For Denominator

Panel	$\sum_0^{\frac{l}{2}} \frac{a\,(s_0-s)^2}{s^2A_u}$	$\sum_0^{\frac{l}{2}} \frac{s_0^2b}{h\cdot A_l}$	$\sum_0^{\frac{l}{2}} \frac{s_0^2s}{(e-pa)^2A_v}$	$\sum_0^{\frac{l}{2}} \frac{s_0^2d}{f^2A_d}$	$\frac{(s_0-s_1)^2\,s_0}{a^2A_{ac}}$
CE	.49	.80	1.19	1.70	. . .
EG	1.82	1.64	.83	1.83	. . .
GI	2.89	3.94	.47	1.58	. . .
IK	3.53	6.43	. . .	.43	. . .
Σ	8.73	12.81	2.49	5.54	1.38

Denominator = 2 (8.73 + 12.81 + 2.49 + 5.54 + 1.38) = 61.90.

For Numerator

	Panel.	$\sum_0^{na}\frac{pa^2(s_0-s)}{s^2A_u}$	$\sum_{na}^{\frac{l}{2}}\frac{na^2(s_0-s)}{s^2A_u}$	$\sum_0^{na}\frac{pas_0b}{h^2A_l}$	$\sum_{na}^{\frac{l}{2}}\frac{nas_0b}{h^2A_l}$	$\sum_0^{na}\frac{es_0s}{(e-pa)^2A_v}$	$\sum_{na}^{\frac{l}{2}}\frac{nas_0s}{(e-pa)^2A_v}$	$\sum_0^{na}\frac{es_0d}{f^2A_d}$	$\sum_{na}^{\frac{l}{2}}\frac{nas_0d}{f^2A_d}$	$\frac{(s_0-s_1)s_0}{aA_{ac}}$
$n = 1$	*CE*	.89	. . .	. . .	. . .	2.16	. . .	3.10	. . .	. . .
	EG	. . .	1.98	. . .	.91	. . .	.46	. . .	1.02	. . .
	GI	. . .	2.55	. . .	2.19	. . .	.26	. . .	.88	. . .
	KI	. . .	2.94	. . .	3.57	. . .	. . .	. . .	.24	. . .
	Σ	.89	7.47	. . .	6.67	2.16	.72	3.10	2.14	2.49
	$H_1 = \frac{25.64}{61.90} = .414$									
$n = 2$	*CE*	.89	. . .	. . .	. . .	2.16	. . .	3.10	. . .	. . .
	EG	3.96	. . .	.91	. . .	2.02	. . .	4.48	. . .	. . .
	GI	. . .	5.10	. . .	4.38	. . .	.52	. . .	1.76	. . .
	IK	. . .	5.88	. . .	7.14	. . .	. . .	. . .	.48	. . .
	Σ	4.85	10.98	.91	11.52	4.18	.52	7.58	2.24	2.49
	$H_2 = \frac{45.27}{61.90} = .732$									
$n = 3$	*CE*	.89	. . .	. . .	. . .	2.16	. . .	3.10	. . .	. . .
	EG	3.96	. . .	.91	. . .	2.02	. . .	4.48	. . .	. . .
	GI	7.65	. . .	4.38	. . .	1.61	. . .	5.40	. . .	. . .
	IK	. . .	8.82	. . .	10.71	. . .	. . .	. . .	.72	. . .
	Σ	12.50	8.82	5.29	10.71	5.79	. . .	12.98	.72	2.49
	$H_3 = \frac{59.30}{61.90} = .958$									
$n = 4$	*CE*	.89	. . .	. . .	. . .	1.28	. . .	3.10	. . .	. . .
	EG	3.96	. . .	.91	. . .	.94	. . .	4.48	. . .	. . .
	GI	7.65	. . .	4.38	. . .	.35	. . .	5.40	. . .	. . .
	IK	11.76	. . .	10.71	. . .	. . .	. . .	3.11	. . .	. . .
	Σ	24.26	. . .	16.00	. . .	2.57	. . .	16.09	. . .	2.49
	$H_4 = \frac{61.41}{61.90} = .992$									

With these values of H, we get for dead load,

$$H = 16 \times 2\,(.414 + .732 + .958 + .496) = 83.20 \text{ tons},$$
$$V_1 = 3\tfrac{1}{2} \times 16 = 56 \text{ tons}.$$

For live load covering $I - B$,

$$H = 20\,(.414 + .732 + .958 + .992 + .958) = 81.08 \text{ tons},$$
$$V_1 = \tfrac{20}{8} \times 15 = 37.5 \text{ tons}.$$

Taking moment at G as before, we get for the maximum stress in FH,

$$\frac{1}{12.9}\{(56 + 37.5)\,30 - (83.20 + 81.08)\,27 - 16 \times 15\} = -144.96 \text{ tons}.$$

Comparing the values of H obtained by the use of Eqs. (121) and (122), it will be seen that the neglect of web-stresses and the assumption of uniform chord sections have led to no appreciable error, the difference being about $4\tfrac{1}{2}$ per cent; but its effect is more strongly felt by individual members, as shown by a comparison of maximum stresses in EF, the difference of the latter amounting to more than 10 per cent.

BALANCED ARCH WITH TWO HINGES

71. In a balanced arch, such as shown in Fig. 55, with independent span at each end, the method of calculating reactions does not differ in general from that explained in the case of spandrel-braced arch, the main difference being that in the present case V_1 and H will be + or − according to modes of loading.

EXAMPLE. — In the symmetrical balanced arch of Fig. 55, to calculate H, V_1 and V_2 due to a uniform load of 1.5 tons per ft. run covering the whole of the left arm.

Since the left cantilever arm is loaded at its end with

$$45 + 22.5 = 67.5^t,$$

by taking moments at A and B we get,

$$-67.5 \times 60 - 45 \times 30 - V_2 \times 240 = 0. \qquad V_2 = -22.5^t.$$
$$-67.5 \times 300 - 45 \times 270 + V_1 \times 240 = 0. \qquad V_1 = +135.0^t.$$

Neglecting the stresses in web-members, we have — by taking moments at successive sections — the following stresses, which, with given lengths and cross-sectional areas of members, give the corresponding works of resistance in chord-members :

	Panel.	S. (tons.)	A. (sq. in.)	L. (ft.)	$\frac{S^2L}{2AE}$.
Upper Chord	I	$102.7 + \frac{9}{23}H$	20	30	$\left(102.7 + \frac{9}{23}H\right)^2 \frac{30}{40\,E}$
	II	$119.1 + \frac{15}{17}H$	25	30	$\left(119.1 + \frac{15}{17}H\right)^2 \frac{30}{50\,E}$
	III	$130.0 + \frac{19}{13}H$	28	30	$\left(130.0 + \frac{19}{13}H\right)^2 \frac{30}{56\,E}$
	IV, V	$112.5 + \frac{5}{3}H$	30	60	$\left(112.5 + \frac{5}{3}H\right)^2 \frac{60}{60\,E}$
	VI	$78.0 + \frac{19}{13}H$	28	30	$\left(78.0 + \frac{19}{13}H\right)^2 \frac{30}{56\,E}$
	VII	$40.0 + \frac{15}{17}H$	25	30	$\left(40.0 + \frac{15}{17}H\right)^2 \frac{30}{50\,E}$
	$VIII$	$14.7 + \frac{9}{23}H$	20	30	$\left(14.7 + \frac{9}{23}H\right)^2 \frac{30}{40\,E}$
Lower Chord	I	$-\left(98.2 + \frac{64}{55}H\right)$	55	35	$\left(98.2 + \frac{64}{55}H\right)^2 \frac{35}{110E}$
	II	$-\left(112.5 + \frac{32}{21}H\right)$	55	33	$\left(112.5 + \frac{32}{21}H\right)^2 \frac{33}{110E}$
	III	$-(126.6 + 2H)$	52	31	$\left(126.6 + 2H\right)^2 \frac{31}{104E}$
	IV	$-\left(130.0 + \frac{32}{13}H\right)$	50	30	$\left(130.0 + \frac{32}{13}H\right)^2 \frac{30}{100E}$
	V	$-\left(77.9 + \frac{32}{13}H\right)$	50	30	$\left(77.9 + \frac{32}{13}H\right)^2 \frac{30}{100E}$
	VI	$-(42.2 + 2H)$	52	31	$\left(42.2 + 2H\right)^2 \frac{31}{104E}$
	VII	$-\left(16.1 + \frac{32}{21}H\right)$	55	33	$\left(16.1 + \frac{32}{21}H\right)^2 \frac{33}{110E}$
	$VIII$	$-\frac{64}{55}H$	55	35	$\left(\frac{64}{55}H\right)^2 \frac{35}{110E}$

Summing up all the terms of the fifth column, and differentiating the sum with respect to H, and setting the differential coefficient equal to zero, we get,

$$H = -\ 56.5^t,$$

showing that the horizontal reaction is directed opposite to that shown by the arrow at A.

For loads in the central span, considering the side

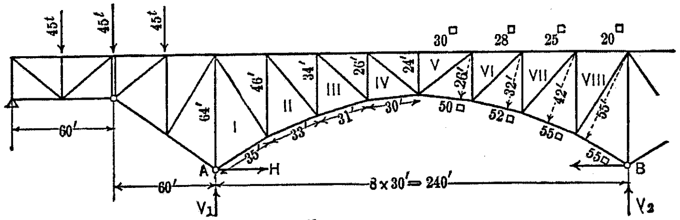

Fig. 55

spans to be weightless, the calculation of H and V is exactly the same as explained in Art. 63.

H will, therefore, be positive — i.e., acting toward right — or negative according as the central or side span is loaded.

TIED ARCH WITH TWO HINGES

72. In the tied arch the horizontal thrust is taken up by the resistance offered by the tie. It has an advantage of the absence of stresses due either to changes of temperature or displacement of supports. Figs. 56 and 57 show arches of this kind. Representing by A_t the cross-section of the tie, we have for the work of resistance in the same due to H,

$$\frac{H^2 l}{2\,A_t E}.$$

This, then, is to be included in the expression for the work of resistance.

In the case of arch-rib (Fig. 56), referring to Art. 45, we get, after differentiating ω with respect to H, the following equation of the latter for one load W:

$$H = \frac{1}{2}\,\frac{\int_0^{a'} \frac{xydc}{I} + a\int_{a'}^{\frac{l'}{2}} \frac{ydc}{I} - \int_0^{a} \frac{\sin\,\phi dx}{A}}{\int_0^{\frac{l'}{2}} \frac{y^2dc}{I} + \int_0^{\frac{l}{2}} \frac{\cos\,\phi\;dx}{A} + \frac{l}{2\,A_t}}\,W \quad . \quad (131)$$

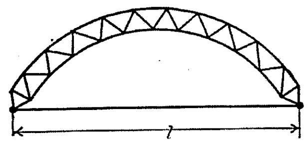

Fig. 56

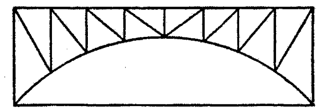

Fig. 57

And in the case of spandrel-braced arch (Fig. 58), from Art. 66, we obtain similarly,

$$H = \frac{\sum_0^{na}\left\{\frac{pa^2(s_0-s)}{s^2Au}+\frac{pas_0b}{h^2Al}+\frac{es_0s}{(e-pa)^2Av}+\frac{es_0d}{f^2Ad}\right\}+\frac{(s_0-s_1)s_0}{aA_{ab}}+\sum_{na}^{\frac{l}{2}}\left\{\frac{na^2(s_0-s)}{s^2Au}+\frac{nas_0b}{h^2Al}+\frac{nas_0s}{(e-pa)^2Av}+\frac{nas_0d}{f^2Ad}\right\}}{2\sum_0^{\frac{l}{2}}\left\{\frac{(s_0-s)^2a}{s^2Au}+\frac{s_0^2b}{h^2Al}+\frac{s_0^2s}{(e-pa)^2Av}+\frac{s_0^2d}{f^2Ad}\right\}+2\frac{(s_0-s_1)^2s_0}{a^2A_{ab}}+\frac{l}{A_t}}\,W$$

(132)

The effect of introducing the tie, on the amount of H, becomes conspicuous with diminished rise of the arch and increased moment of inertia, as will be seen in the case of flat parabolic arch, for which we have, from Art. 58, the following approximate equation for H due to one W:

$$H = \frac{5\,a\,(l-a)\,(l^2+al-a^2)}{l^3\left(8\,h+\frac{15\,I}{A_t h}\right)}W.$$

EXAMPLE. — Taking the circular arch of Fig. 50, given as an example in Art. 70, and adding to the same the horizontal tie with a cross-section of 50 square inches (Fig. 58), to calculate the stress in the tie, due to a uniform load of 20 tons per panel.

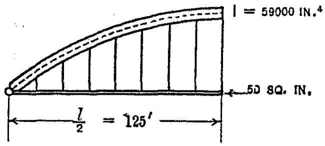

Fig. 58

Transforming Eq. (131) according to Art. 51, we have for the circular arch, by neglecting axial stress,

$$H = \frac{(r-h)\{2e + \phi_a(l-2a) - \phi_0 l\} + a(l-a)}{2\phi_0\{r^2 + 2(r-h)^2\} - 3l(r-h) + \dfrac{2Il}{A_t r}} W.$$

Referring to the data of the previous case, the denominator in this case equals

$$2748 + \frac{2 \times 59{,}000 \times 250}{50 \times 200 \times 144} = 2768;$$

and since we have for the numerator as before,

$$20 \times 30{,}578 = 611{,}560,$$

we get

$$H = \frac{611{,}560}{2768} = 220.94 \text{ tons},$$

which is the stress in the tie.