CHAPTER III

CONTINUOUS GIRDERS

24. LET Fig. 22 represent two consecutive spans of a continuous beam, supposed to be resting on immovable supports of such heights that the girder would be unstrained were it completely unloaded.

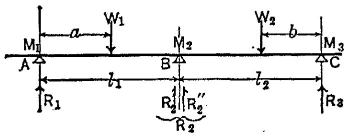

Fig. 22

The following designations will be used throughout the discussion:

M_1, M_2, M_3 . . . moments in the beam at the supports A, B, C respectively.

R_1, R_2' . . . . . reactions at A and B respectively, due to moments and loads in span l_1.

R_2'', R_3 . . . . . reactions at B and C respectively, due to moments and loads in span l_2.

a, b . . . . . . distances of loads from A and C respectively.

I_1, I_2 . . . . . moments of inertia of the beam at AB and BC respectively.

m_1, m_2, m_3, m_4 . . moments at any points between A and W_1, W_1 and B, B and W_2, W_2 and C respectively.

E . . . . modulus of elasticity assumed to be constant.

Forces acting upward are positive, and vice versâ. Moments causing compression in the upper fibres are positive, and vice versâ.

Tension is taken as positive, and compression, negative. Neglecting the effect of shear, we have for the internal work in the beam,

$$\omega = \frac{1}{2EI_1}\left(\int_0^a m_1^2 dx + \int_a^{l_1} m_2^2 dx\right) + \frac{1}{2EI_2}\left(\int_0^b m_4^2 dx + \int_b^{l_2} m_3^2 dx\right).$$

The reason for taking two consecutive spans as an element of indefinitely continuous girder and confining the summation of internal work to them, lies in the fact that in order to find the value of M_2 which will make ω a minimum, it is unnecessary to go beyond the two spans, since M_2 depends, as will be seen immediately in the following, on M_1 and M_3 and loadings on l_1 and l_2 only. Calling, as before, those moments producing compression in the upper flange positive, and vice versâ, we get the following equations:

$m_1 = M_1 + R_1x$, origin of x at A,

$m_2 = M_1 + R_1x - W_1(x-a)$, origin of x at A,

$m_3 = M_3 + R_3x - W_2(x-b)$, origin of x at C,

$m_4 = M_3 + R_3x$, origin of x at C,

$$R_1 = \frac{W_1(l_1-a)}{l_1} + \frac{M_2-M_1}{l_1},$$

$$R_3 = \frac{W_2(l_2-b)}{l_2} + \frac{M_2-M_3}{l_2}.$$

Substituting these equations in the expression for work, and setting the first derivative of the same with respect to M_2 equal to zero, we at once obtain the following equation:

$$\frac{d\omega}{dM_2} = \frac{l_1}{I}(M_1 + 2M_2) + \frac{l_2}{I_2}(2M_2 + M_3) + \frac{W_1 a}{l_1 I_1}(l_1^2 - a^2) + \frac{W_2 b}{l_2 I_2}(l_2^2 - b^2) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (37)$$

Whence for any number of loads, we get when $I_1 = I_2$ the following:

$$M_1l_1 + 2M_2(l_1+l_2) + M_3l_2 = -\frac{\sum^{l_1} Wa}{l_1}(l_1^2-a^2) - \frac{\sum^{l_2} Wb}{l_2}(l_2^2-b^2). \quad (38)$$

25. For *partial uniform load* w per unit length (Fig. 23), we have but to replace W_1 and W_2 with wda and

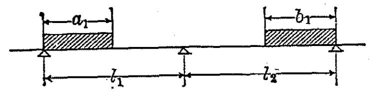

Fig. 23

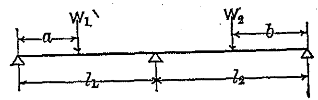

Fig. 24

wdb in (38) and integrate between given limits to obtain the following equation:

$$M_1l_1 + 2M_2(l_1+l_2) + M_3l_2 = -\frac{w}{4l_1}(2l_1^2a_1^2 - a_1^4) - \frac{w}{4l_2}(2l_2^2b_1^2 - b_1^4). \quad (39)$$

For *full uniform load*, Eq. (39) becomes

$$M_1l_1 + 2M_2(l_1+l_2) + M_3l_2 = -\frac{wl_1^3}{4} - \frac{wl_2^3}{4} \quad . \; . \quad (40)$$

26. In case both ends of the girder are free and simply supported (Fig. 24), M_1 and M_3 will be equal to zero, so that we get from (38),

$$2M_2(l_1+l_2) = -\frac{\sum^{l_1} Wa}{l_1}(l_1^2-a^2) - \frac{\sum^{l_2} Wb}{l_2}(l_2^2-b^2) \quad . \quad (41)$$

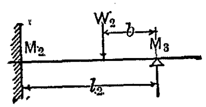

Fig. 25

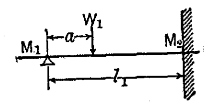

Fig. 26

27. If the left end of the girder were firmly fixed and continuous at the other end (Fig. 25), then it would be equivalent to making $I_1 = \infty$ in Eq. (37), and we get

$$2\,M_2 l_2 + M_3 l_2 = -\frac{\sum^{l_2} Wb}{l_2}(l_2^2 - b^2) \quad . \; . \; . \; . \quad (42)$$

28. Similarly if the right end were fixed and continuous at the other (Fig. 26), we would obtain

$$M_1 l_1 + 2\,M_2 l_1 = -\frac{\sum^{l_1} Wa}{l_1}(l_1^2 - a^2) \quad . \; . \; . \; . \quad (43)$$

29. When a beam with uniform cross-section is continuous over several supports (Fig. 27), apply Eq. (38)

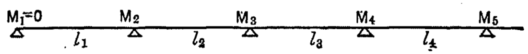

Fig. 27

successively to every two spaces (paying attention to suffixes), in the following manner:

l_1 and l_2, $\quad 0 + 2\,M_2(l_1 + l_2) + M_3 l_2 = -\dfrac{\sum^{l_1} Wa(l_1^2 - a^2)}{l_1} - \dfrac{\sum^{l_2} Wb(l_2^2 - b^2)}{l_2}.$

l_2 and l_3, $\quad M_2 l_2 + 2\,M_3(l_2 + l_3) + M_4 l_3 = -\dfrac{\sum^{l_2} Wa(l_2^2 - a^2)}{l_2} - \dfrac{\sum^{l_3} Wb(l_3^2 - b^2)}{l_3}.$

l_3 and l_4, $\quad M_3 l_3 + 2\,M_4(l_3 + l_4) + M_5 l_4 = -\dfrac{\sum^{l_3} Wa(l_3^2 - a^2)}{l_3} - \dfrac{\sum^{l_4} Wb(l_4^2 - b^2)}{l_4}.$

In this way as many equations as there are unknown moments could be obtained. The rest is a purely algebraic work.

30. In all the foregoing cases of continuous beam, the supports were supposed to be unyielding. If, however, the beam were made to rest on a comparatively yielding support or supports, such as tall metallic columns for instance, then the deformation of the latter would modify the bending moment in the beam by so much as the deflection produced by the sinking of the support makes the beam to take up a portion of the load. Fig. 28 shows a beam continuous over three supports, of which the intermediate one is a column of the same material as the beam.

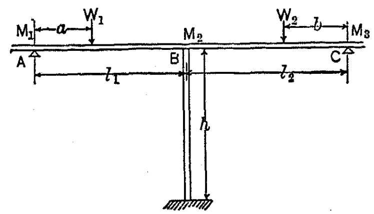

Fig. 28

Using the same designations as before, we have for ω due to W_1 only

$$\omega = \frac{1}{2\,EI}\left\{\int_0^a m_1^2 dx + \int_a^{l_1} m_2^2 dx + \int_0^{l_2} m_3^2 dx\right\} + \frac{hR_2^2}{2\,EA},$$

in which A represents the cross-sectional area of the column, and R_2 the pressure acting in the same.

Since

$$R_1 = \frac{W_1(l_1 - a)}{l_1} + \frac{M_2 - M_1}{l_1},$$

$$R_2 = \frac{M_1 - M_2}{l_1} + \frac{M_3 - M_2}{l_2} + \frac{W_1 a}{l_1},$$

$$R_3 = \frac{M_2 - M_3}{l_2},$$

$$R_1 + R_2 + R_3 = W_1,$$

and $m_1 = M_1 + R_1x$, origin of x at A,
$m_2 = M_1 + R_1x - W_1(x - a)$, origin of x at A,
$m_3 = M_3 + R_3x$, origin of x at C,

substituting these in the equation of work, we get for

$$\frac{d\omega}{dM_2} = 0,$$

$$\frac{1}{6I}\left\{M_1l_1 + 2M_2(l_1 + l_2) + M_3l_2 + \frac{W_1a}{l_1}(l_1^2 - a^2)\right\}$$
$$- \frac{h}{A}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)\left(\frac{M_1 - M_2}{l_1} + \frac{M_3 - M_2}{l_2} + \frac{W_1a}{l_1}\right) = 0.$$

In case $M_1 = 0$, and $M_3 = 0$, we get

$$M_2 = -\frac{\frac{a}{l_1}\left\{\frac{l_1^2 - a^2}{6I} - \frac{h}{A}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)\right\}}{\frac{l_1 + l_2}{3I} + \frac{h}{A}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)^2}W_1 \quad \ldots \ldots \ldots \quad (44)$$

For W_1 and W_2

$$M_2 = -\frac{\frac{W_1a}{l_1}\left\{\frac{l_1^2 - a^2}{6I} - \frac{h}{A}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)\right\} + \frac{W_2b}{l_2}\left\{\frac{l_2^2 - b^2}{6I} - \frac{h}{A}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)\right\}}{\frac{l_1 + l_2}{3I} + \frac{h}{A}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)^2} \quad (45)$$

If $l_1 = l_2 = l$, we get for W_1

$$M_2 = -\frac{\frac{a(l^2 - a^2)}{6Il} - \frac{2ah}{Al^2}}{\frac{2l}{3I} + \frac{4h}{Al^2}}W_1 \quad \ldots \ldots \ldots \ldots \quad (46)$$

whence

$$R_2 = \frac{3al^2 - a^3}{3I\left(\frac{4h}{A} + \frac{2l^3}{3I}\right)}W_1 \quad \ldots \ldots \ldots \ldots \quad (47)$$

31. If, owing to any cause, the central support were found, either to yield when loaded or to be so displaced that the beam has to deflect to bear on the supports, the force exerted simply to keep the beam on to the latter produces reactions and moments. That force is no other than R_2 which has for its displacement the deflection of the beam.

Represent by Δh the deflection of the beam at the central support, reckoned in the direction of the force, acting through it, i.e., negative for sinking, and vice versâ. Let M_1, M_2, etc., be moments caused by the motion of the support. Then, according to the first theorem of Castigliano (Art. 6),

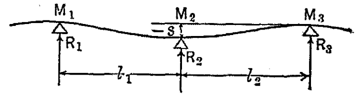

Fig. 29

$$\frac{d\omega}{dR_2} = \Delta h.$$

Since

$$R_2 = \frac{M_1 - M_2}{l_1} + \frac{M_3 - M_2}{l_2},$$

and for the internal work we have as before,

$$\omega = \frac{l_1}{6EI}(M_1^2 + M_1M_2 + M_2^2) + \frac{l_2}{6EI}(M_2^2 + M_2M_3 + M_3^2),$$

making M_2 the variable,

$$dR_2 = -\left(\frac{1}{l_1} + \frac{1}{l_2}\right)dM_2,$$

we get

$$\frac{d\omega}{dR_2} = -\frac{1}{6EI}\left\{l_1(M_1 + 2M_2) + l_2(M_3 + 2M_2)\right\}\frac{1}{\frac{1}{l_1} + \frac{1}{l_2}} = \Delta h,$$

from which

$$M_1l_1 + 2M_2(l_1 + l_2) + M_3l_2 = -6EI\Delta h\left(\frac{1}{l_1} + \frac{1}{l_2}\right). \quad (48)$$

In case the central support sinks by s, then $\Delta h = -s$, and if in that case the ends of the girder were free, we would have

$$2 M_2 (l_1 + l_2) = 6 EIs\left(\frac{1}{l_1} + \frac{1}{l_2}\right) \quad . \quad . \quad . \quad . \quad (49)$$

If, instead of the central support, the left support, for instance, deflect by Δh, then in this case, since

$$\omega = \frac{M_2^2}{6 EI}(l_1 + l_2),$$

$$R_1 = \frac{M_2}{l_1},$$

we get

$$\frac{d\omega}{dR_1} = \Delta h = \frac{M_2 l_1 (l_1 + l_2)}{3 EI},$$

whence

$$M_2 = \frac{3 EI\Delta h}{l_1 (l_1 + l_2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

EXAMPLE 1. — A continuous girder with a length of 200 ft., and a uniform section whose depth is 16 ft. and moment of inertia 552,960 in.^{4}, is supported at its centre by a metallic pier 50 ft. high and 50 sq. in. in section. To calculate the maximum stresses found in the chords and pier due to a full uniform load of 3600 lbs. per ft. run.

From Eq. (47),

$$R_2 = 2\frac{\int_0^l (3 al^2 - a^3) wda}{3 I\left(\frac{4 h}{A} + \frac{2 l^3}{3 I}\right)} = \frac{5 l^4 w}{6 I\left(\frac{4 h}{A} + \frac{2 l^3}{3 I}\right)} = 440,000 \text{ lbs.}$$

Since

$$2 R_1 + R_2 - 3600 \times 200 = 0,$$

$$R_1 = 3600 \times 100 - \frac{440,000}{2} = 140,000 \text{ lbs.}$$

Comparing + and − moments in the girder, the latter will be found to be greater in this case, being at the central support

$$140,000 \times 100 - 3600 \times \frac{100^2}{2} = -4,000,000 \text{ ft.-lbs.},$$

from which we obtain for maximum flange stress in the girder,

$$\frac{4,000,000 \times 12^2 \times 8}{552,960} = 8333 \text{ lbs. per sq. in.},$$

and for the stress in the pier,

$$\frac{440,000}{50} = 8800 \text{ lbs. per sq. in.}$$

EXAMPLE 2. — If in the foregoing example the central support were of masonry, so that it might be considered practically indeformable, but owing to yielding foundation, suppose it to settle by .176 in., what would be the moment and reaction at the centre, assuming $E = 30,000,000$ lbs. per sq. in.?

From Eqs. (41) and (49),

$$M_2 = -\frac{1}{2 l^2}\int_0^l a(l^2 - a^2) wda + \frac{3 EIs}{l^2} = -\frac{wl^2}{8} + \frac{3 EIs}{l^2}$$

$$= -\frac{3600 \times 100^2}{8} + \frac{3 \times 30,000,000 \times 552,960 \times .176}{100^2 \times 12^3}$$

$$= -3,993,120 \text{ ft.-lbs.},$$

from which

$$R_2 = 2\left(3600 \times 50 + \frac{3,993,120}{100}\right) = 440,000 \text{ lbs.}$$

32. When the truss forming a continuous girder is of considerable depth, the influence of deformations of web-members which has been neglected in the foregoing discussions becomes felt to some extent. A method of taking the same into consideration will be explained when deducing formulas for swing bridges. Trusses continuous over several supports are, owing to several drawbacks, so seldom constructed, that it will not be necessary to go farther into the subject in this place.

SWING BRIDGE, WITH THREE SUPPORTS

33. For a swing bridge with three supports, when it is of plate girders or trusses of comparatively small depth in which the effect of deformations of web-members is inconsiderable when compared to that of chords, Eq. (41) may be used with correctness sufficient for all practical purposes, and from it other necessary equations may be at once written.

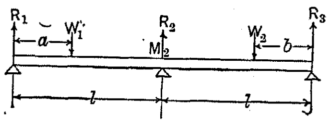

Fig. 30

In (41) making $l_1 = l_2 = l$,

$$M_2 = -\frac{\Sigma W_1 a(l^2 - a^2) + \Sigma W_2 b\,(l^2 - b^2)}{4\,l^2} \quad . \quad . \quad . \quad (51)$$

$$R_1 = \frac{1}{l}\{M_2 + \Sigma W_1 (l - a)\},$$

$$R_2 = \frac{1}{l}(-\,2\,M_2 + \Sigma W_1 a + \Sigma W_2 b),$$

$$R_3 = \frac{1}{l}\{M_2 + \Sigma W_2 (l - b)\},$$

$$R_1 + R_2 + R_3 = \Sigma W_1 + \Sigma W_2.$$

34. When both ends of the bridge are *simply supported* without being raised, the dead-loads act as on two overhanging arms, either when the bridge is closed or open, the live-load alone acting as on a continuous girder on three supports when both arms are loaded partially or fully.

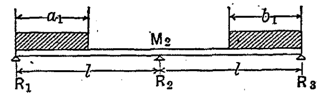

Fig. 31

When the moving load is a uniformly distributed one of w per unit length (Fig. 31), then from Eq. (39),

$$M_2 = -\frac{a_1^2\,(2\,l^2 - a_1^2) + b_1^2\,(2\,l^2 - b_1^2)}{16\,l^2}\,w \quad . \quad . \quad (52)$$

$$R_1 = \frac{M_2}{l} + \frac{w a_1\,(2\,l - a_1)}{2\,l},$$

$$R_2 = -\,2\,\frac{M_2}{l} + \frac{w\,(a_1^2 + b_1^2)}{2\,l},$$

$$R_3 = \frac{M_2}{l} + \frac{w b_1\,(2\,l - b_1)}{2\,l},$$

$$R_1 + R_2 + R_3 = w\,(a_1 + b_1).$$

If in this case, one arm only be loaded, then the end of the other arm would be lifted clear of its support, and the loaded arm would be a simple girder with span length l. This mode of loading generally gives maximum positive moment and shear, which are for the left arm at any point distant x from the left end,

$$m = \frac{wx}{2}\,(l - x) \text{ for full load,}$$

$$s = \frac{w\,(l - x)^2}{2\,l} \text{ for load covering } (l - x),$$

s denoting shear taken as positive when it tends to move the left side upward past the right side of the section. A little consideration will show that the greatest negative moment and shear at any point of the left arm will be produced by the greatest negative amount of R_1 combined with load between the point and the left end of the arm. Now from the expression for R_1 it will be seen that all loads on the right arm make R_1 negative, while those on the left arm positive. Consequently the maximum negative moment and shear at any point x (Fig. 32) will be caused by the load covering the right arm and the portion of the left arm between the point and the left end of the arm. They are,

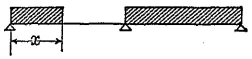
Fig. 32

$$m = R_1 x - \frac{wx^2}{2} = -\left\{\frac{x^2(2l^2 - x^2) + l^4}{16 l^3} - \frac{x(l-x)}{2l}\right\} wx,$$

$$s = R_1 - wx = -\left\{\frac{x^2(2l^2 - x^2) + l^4}{16 l^3} + \frac{x^2}{2l}\right\} w.$$

The absolute maximum negative moment will, for the same reason, be found at the central support when both arms are fully loaded.

These considerations are all that will be necessary in determining maximum stresses in different members of the truss.

35. In case both ends of the bridge are *fully lifted*, the dead-load will be supported on three supports when the bridge is closed, and the central moment due to the same is to be calculated with Eq. (51) or (52).

36. When the girder, instead of being a beam as in the preceding case, is a truss with considerable depth, the deformations of web-members may sometimes be so great that it would be necessary to take them into consideration in accurate calculations. To do this, however, since the dimensions of each member of the truss should be known, it is the general practice to make preliminary calculation of stresses in all the members with the external forces as found by the equations already given for the case of uniform cross-section, with the effect of web-stresses neglected, and afterward to make such tentative corrections as are necessary on the dimensions according to the more accurate computations based on them. The following is an accurate method of determining the external forces.

Let

A = the cross-section of any member of the truss,
E = the modulus of elasticity of the material, assumed to be constant,
S = the stress in the member,
L = the length of the member.

Then for the total internal work in the truss in which the members are subjected to direct stresses only, we have,

$$\omega = \Sigma \frac{S^2 L}{2 AE} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

In the swing-bridge truss of Fig. 33, since M_2 must always be such as to make the total internal work a minimum, if we now express S in each member due to any given loading in terms of M_2 and substitute it in (53), then from

$$\frac{d\omega}{dM_2} = 0$$

we can at once obtain the required value of M_2. For simplicity, assume two symmetrical loads W, W and distinguish A, S and L of each member with corresponding

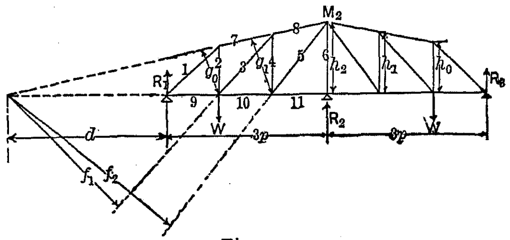

Fig. 33

suffixes and the arm-lengths of several members as shown in the figure. Then taking moments at the successive sections, we get the following values of S;

$$S_1 = -R_1 \frac{L}{h_0},$$

$$S_2 = R_1 \frac{d}{d+p},$$

$$S_3 = \frac{W(d+p) - R_1 d}{f_1},$$

$$S_4 = -\frac{W(d+p) - R_1 d}{d+2p},$$

$$S_5 = \frac{W(d+p) - R_1 d}{f_2},$$

$$S_6 = -R_2,$$

$$S_7 = -\frac{R_1 p}{g_0},$$

$$S_8 = -\frac{(2R_1 - W)p}{g_1},$$

$$S_9 = R_1 \frac{p}{h_0},$$

$$S_{10} = \frac{(2R_1 - W)p}{h_1},$$

$$S_{11} = \frac{(3R_1 - 2W)p}{h_2};$$

and since

$$R_1 = \frac{M_2}{3p} + \frac{2W}{3},$$

$$R_2 = -\frac{2M_2}{3p} + \frac{2W}{3},$$

$$R_3 = \frac{M_2}{3p} + \frac{2W}{3},$$

$$R_1 + R_2 + R_3 = 2W.$$

Substituting, we get,

$$\omega = \frac{1}{2E}\left\{2R_1^2 \frac{L_1^3}{h_0^2 A_1} + 2R_1^2 \frac{d^2 h_0}{(d+p)^2 A_2} + \cdots + R_2^2 \frac{h_2}{A_6} + 2R_1^2 \frac{p^2 L_7}{g_0^2 A_7} + \cdots\right\}.$$

Setting the first derivative of ω with respect to M_2 equal to zero, remembering that,

$$\frac{dR_1}{dM_2} = \frac{dR_3}{dM_2} = \frac{1}{3p},$$

$$\frac{dR_2}{dM_2} = -\frac{2}{3p},$$

we get,

$$R_1 = \frac{d(d+p)\left\{\frac{L_3}{f_1^2 A_3} + \frac{L_4}{(d+2p)^2 A_4} + \frac{L_5}{f_2^2 A_5}\right\} + \frac{2h_2}{A_6} + p^2\left(\frac{2L_8}{g_1^2 A_8} + \frac{2p}{h_1^2 A_{10}} + \frac{6p}{h_2^2 A_{11}}\right)}{\frac{L_1^3}{h_0^2 A_1} + d^2\left\{\frac{h_0}{(d+p)^2 A_2} + \frac{L_3}{f_1^2 A_3} + \frac{L_4}{(d+2p)^2 A_4} + \frac{L_5}{f_2^2 A_5}\right\} + \frac{2h_2}{A_6} + p^2\left(\frac{L_7}{g_0^2 A_7} + \frac{4L_8}{g_1^2 A_8} + \frac{p}{h_0^2 A_9} + \frac{4p}{h_1^2 A_{10}} + \frac{9p}{h_2^2 A_{11}}\right)}$$

whence M_2 and R_2 may be obtained by substitution in the foregoing equations.

In this way R_1, R_2, and M_2 are to be calculated for different modes of loading in order to obtain stresses due to them.

A comparison of approximate and correct methods of computation shows that the difference in results obtained by the two methods is generally inconsiderable, as will be shown in the case of a swing bridge with four supports (Art. 37).

SWING BRIDGE, WITH FOUR SUPPORTS AND PARTIALLY CONTINUOUS

37. A swing bridge *fully continuous* over four supports has probably never been constructed, owing to practical difficulties in construction arising from the great difference in amounts between central reactions when the bridge is partially loaded, which may necessitate special provisions for holding down the central supports on to the masonry. For this reason, such bridge is made either partially continuous or entirely discontinuous. Partial continuity in such case is effected by omitting the web-members between central supports, thus cutting off the means of transmitting shear from one span to the other. Fig. 34 shows this kind of construction. As there can be no shear in the panel BC, it is evident that M_2 will always be equal to M_2'.

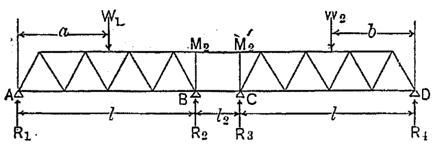

Fig. 34

Represent by

m_1 = moment at any point x from A between A and W_1.
m_2 = moment at any point x from A between W_1 and B.
m_3 = moment at any point x from D between C and W_2.
m_4 = moment at any point x from D between W_2 and D.

Assuming the truss to have uniform moment of inertia I, and neglecting the influence of deformations of web-members, we get for the internal work due to moments:

$$\omega = \frac{1}{2EI}\left\{\int_0^a m_1^2 dx + \int_a^l m_2^2 dx + \int_0^{l_2} M_2^2 dx + \int_b^l m_3^2 dx + \int_0^b m_4^2 dx\right\}.$$

But

$$m_1 = R_1 x = \frac{1}{l}\{M_2 + W_1(l-a)\}x,$$

$$m_2 = R_1 x - W_1(x-a) = \frac{M_2}{l}x + \frac{W_1 a(l-x)}{l},$$

$$m_3 = R_4 x - W_2(x-b) = \frac{M_2}{l}x + \frac{W_2 b(l-x)}{l},$$

$$m_4 = R_4 x = \frac{1}{l}\{M_2 + W_2(l-b)\}x.$$

Substituting these values of m in the above expression for work, we get for

$$\frac{d\omega}{dM_2} = 0,$$

the following equation:

$$2\{M_2 + W_1(l-a)\}\frac{l}{3} + 2M_2 l_2 + 2\{M_2 + W_2(l-b)\}\frac{l}{3}$$

$$- W_1\left(\frac{2l^2}{3} - al + \frac{a^3}{3l}\right) - \left(\frac{2l^3}{3} - bl + \frac{b^3}{3l}\right)W_2 = 0,$$

from which generally for any number of W_1 and W_2 we get

$$M_2 = -\frac{\Sigma W_1 a\,(l^2 - a^2)}{l\,(6l_2 + 4\,l)} - \frac{\Sigma W_2 b\,(l^2 - b^2)}{l\,(6\,l_2 + 4\,l)} \quad . \quad . \quad (54)$$

Knowing M_2, all the external forces become at once known, thus:

$$R_1 = \{M_2 + \Sigma W_1 (l - a)\}\,\frac{1}{l},$$

$$R_2 = (-M_2 + \Sigma W_1 a)\,\frac{1}{l},$$

$$R_3 = (-M_2 + \Sigma W_2 b)\,\frac{1}{l},$$

$$R_4 = \{M_2 + \Sigma W_2 (l - b)\}\,\frac{1}{l},$$

$$R_1 + R_2 + R_3 + R_4 = \Sigma W_1 + \Sigma W_2.$$

These equations give approximate results for most kinds of trusses; a more accurate result is obtained by taking the deformations of the web-members into consideration, and forming

$$\omega = \frac{\Sigma S^2 L}{2\,EA},$$

extended over all the members of the truss, as explained in the case of three supports, the necessary A and S being provisionally obtained by means of the approximate equations above given. The mode of loading to give maximum moment or shear at any point in the truss is essentially the same as in the case of three supports.

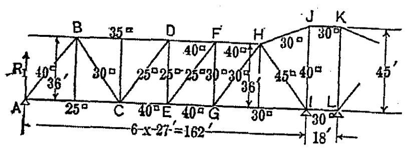

Fig. 35

EXAMPLE. — In the swing bridge of Fig. 35 to calculate the reactions due to uniform moving load of 10 tons per panel, when both arms are fully loaded.

For the assumed position of moving load,

$$R_2 = 50 - R_1.$$

The following stresses and internal works may now be written:

Members.		S.	L (ft.).	A (sq. in.).	$\frac{S^2L}{2EA}$.
Web-members.	AB	$1.25\,R_1$	45	40	$1.76\,R_1^2\left(\frac{1}{E}\right)$
	BC	$1.25\,(R_1 - 10)$	45	30	$2.34\,(R_1 - 10)^2$ (")
	CD	$1.25\,(R_1 - 20)$	45	25	$2.81\,(R_1 - 20)^2$ (")
	DE	$R_1 - 20$	36	25	$1.44\,(R_1 - 20)^2$ (")
	EF	$1.25\,(R_1 - 30)$	45	25	$2.81\,(R_1 - 30)^2$ (")
	FG	$R_1 - 30$	36	30	$1.2\,(R_1 - 30)^2$ (")
	GH	$1.25\,(R_1 - 40)$	45	30	$2.34\,(R_1 - 40)^2$ (")
	HI	$2.2\,(R_1 - 36.4)$	45	45	$4.84\,(R_1 - 36.4)^2$ (")
	JI	$1.2\,(R_1 - 25)$	45	40	$1.62\,(R_1 - 25)^2$ (")
Up. Chd.	BD	$1.5\,(R_1 - 5)$	54	35	$3.47\,(R_1 - 5)^2$ (")
	DF	$2.25\,(R_1 - 10)$	27	40	$3.42\,(R_1 - 10)^2$ (")
	FH	$3\,(R_1 - 15)$	27	40	$6.08\,(R_1 - 15)^2$ (")
	HJ	$3.79\,(R_1 - 25)$	28.5	30	$13.68\,(R_1 - 25)^2$ (")
	JK	$3.6\,(R_1 - 25)$	18	30	$7.78\,(R_1 - 25)^2\left(\frac{1}{2E}\right)$
Low. Chd.	AC	$0.75\,R_1$	54	25	$1.21\,R_1^2\left(\frac{1}{E}\right)$
	CE	$2.25\,(R_1 - 10)$	27	40	$3.42\,(R_1 - 10)^2$ (")
	EG	$3\,(R_1 - 15)$	27	40	$6.08\,(R_1 - 15)^2$ (")
	GI	$3.75\,(R_1 - 20)$	54	30	$25.31\,(R_1 - 20)^2$ (")
	IL	$3.6\,(R_1 - 25)$	18	30	$7.78\,(R_1 - 25)^2\left(\frac{1}{2E}\right)$

Summing up the works, and putting the first derivative of the sum with respect to R_1 equal to zero, we at once get,

$$R_1 = 20.19 \text{ tons},$$
$$R_2 = 50 - R_1 = 29.81 \text{ tons}.$$

38. Had the moment of inertia been assumed to be uniform throughout the girder and at the same time the deformation of web-members neglected, we would have obtained from Eq. (54),

$$M_2 = -2\frac{\Sigma\, 10\, a\,(26{,}244 - a^2)}{162\,(108 + 648)}.$$

Substituting in this, $a = 27$, 54, 81, 108, and 135, we obtain

$$M_2 = -1012.5 \text{ ft.-tons},$$

so that

$$R_1 = \{M_2 + \Sigma 10\,(162 - a)\}\,\tfrac{1}{162} = 18.75 \text{ tons}.$$

Comparing this with the preceding result, it will be seen that the assumption of uniform moment of inertia and the neglect of web-member deformations give R_1 smaller by about 7 per cent in this case than given by the more correct calculation. In practice, however, all this nicety in calculation becomes almost valueless, owing to the overwhelming disturbance brought about by unequal temperature changes, which constantly tend to throw out of adjustment the end supports on which the stresses of all the members solely depend.

DOUBLE-SWING BRIDGE

39. Double-swing bridges are latched at the centre when closed, thus transmitting shear, but no moment from one span to another.

Fig. 36 shows a double-swing bridge with four supports. The point C serves for both trusses as a common yielding support.

To simplify the discussion, all the spans will be made

alike. Then for any load W_1 we get the following moments in the several spans:

$R_1 x$	between A and W_1,	distant x from A.
$R_1 x - W_1(x - a)$	between W_1 and B,	distant x from A.
$M_1 + R_2'' x$ . . .	between B and C,	distant x from B.
$M_2 + R_3' x$ . . .	between D and C,	distant x from D.
$R_4 x$	between E and D,	distant x from E.

Assuming the cross-section of the trusses to be uni-

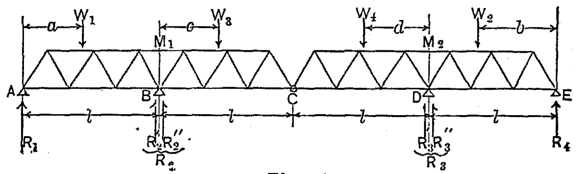

Fig. 36

form throughout, and considering moments only, we get for the total internal work due to W_1:

$$\omega = \frac{1}{2EI}\Big[\int_0^a (R_1 x)^2\,dx + \int_a^l \{R_1 x - W_1(x - a)\}^2\,dx + \int_0^l (M_1 + R_2'' x)^2\,dx + \int_0^l (M_2 + R_3' x)^2\,dx + \int_0^l (R_4 x)^2 dx\Big].$$

Taking moments successively at B, A, C, E, and D. we get,

$$R_1 = \frac{W_1(l - a)}{l} + \frac{M_1}{l},$$

$$R_2' = \frac{W_1 a}{l} - \frac{M_1}{l},$$

$$R_2'' = -\frac{M_1}{l},$$

$$R_2 = R_2' + R_2'' = -\frac{2M_1}{l} + \frac{W_1 a}{l},$$

$$R_3' = -\frac{M_2}{l},$$
$$R_3'' = -\frac{M_2}{l},$$
$$R_4 = \frac{M_2}{l}.$$

Taking moment at D,

$$3\,R_1 l - W_1 (3\,l - a) + 2\,R_2 l = M_2.$$

Substituting the values of R_1 and R_2 in this, we get

$$M_2 = -\,M_1.$$

Introducing these equations in the expression for the internal work, and setting the first derivative of it with respect to M_1 equal to zero, we get

$$M_1 = -\frac{a\,(l^2 - a^2)}{8\,l^2}\,W_1 \quad \ldots\ldots \quad (55)$$

and consequently,

$$M_2 = \frac{a\,(l^2 - a^2)}{8\,l^2}\,W_1 \quad \ldots\ldots \quad (56)$$

Similarly for load W_2 in the right-end span we get,

$$M_1 = \frac{b\,(l^2 - b^2)}{8\,l^2}\,W_2 \quad \ldots\ldots \quad (57)$$

$$M_2 = -\frac{b\,(l^2 - b^2)}{8\,l^2}\,W_2 \quad \ldots\ldots \quad (58)$$

For any load W_3 in the second span from the left, we have as before the following moments:

$R_1 x$ . . . . between A and B, origin of x at A,
$M_1 + R_2'' x$ . . . . . between B and W_3, origin of x at B,
$M_1 + R_2'' x - W_3 (x - c)$, between W_3 and C, origin of x at B,
$M_2 + R_3' x$ . . . . . between D and C, origin of x at D,
$R_4 x$ . . . . between E and D, origin of x at E,

from which we get the following internal work:

$$\omega = \frac{1}{2\,EI}\Big[\int_0^l (R_1 x)^2\,dx + \int_0^c (M_1 + R_2'' x)^2\,dx$$
$$+ \int_c^l \{M_1 + R_2'' x - W_3 (x - c)\}^2 dx + \int_0^l (M_2 + R_3' x)^2\,dx$$
$$+ \int_0^l (R_4 x)^2\,dx\Big].$$

Taking moments at B, C, and D successively, we have:

$$R_1 = \frac{M_1}{l},$$
$$R_2'' = -\frac{M_1}{l} + \frac{W_3 (l - c)}{l},$$
$$R_3' = -\frac{M_2}{l},$$
$$R_4 = \frac{M_2}{l},$$
$$M_2 = -\,(M_1 + W_3 C).$$

Substituting these values in the above expression for work, and putting as before the first differential coefficient of ω with respect to M_1 equal to zero, we at once obtain:

$$M_1 = -\frac{c\,(6\,l^2 - 3\,lc + c^2)}{8\,l^2}\,W_3 \quad \ldots \quad (59)$$

$$M_2 = -\frac{c\,(2\,l^2 + 3\,lc - c^2)}{8\,l^2}\,W_3 \quad \ldots \quad (60)$$

Similarly for any load W_4 in the third span from the left, we get,

$$M_1 = -\frac{d\,(2\,l^2 + 3\,ld - d^2)}{8\,l^2}\,W_4 \quad \ldots \quad (61)$$

$$M_2 = -\frac{d\,(6\,l^2 - 3\,ld + d^2)}{8\,l^2}\,W_4 \quad \ldots \quad (62)$$

Finally, we get for *any number of loads*,

$$M_1 = \frac{1}{8l^2}\{-\Sigma W_1 a(l^2-a^2) + \Sigma W_2 b(l^2-b^2) - \Sigma W_3 c(6l^2-3lc+c^2)$$
$$-\Sigma W_4 d(2l^2+3ld-d^2)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

$$M_2 = \frac{1}{8l^2}\{\Sigma W_1 a(l^2-a^2) - \Sigma W_2 b(l^2-b^2) - \Sigma W_3 c(2l^2+3lc-c^2)$$
$$-\Sigma W_4 d(6l^2-3ld+d^2)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

$$R_1 = \frac{\Sigma W_1(l-a)}{l} + \frac{M_2}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

$$R_2 = \frac{\Sigma W_1 a}{l} - \frac{2M_1}{l} + \frac{\Sigma W_3(l-c)}{l} \quad . \quad . \quad . \quad . \quad (66)$$

$$R_3 = \frac{\Sigma W_2 b}{l} - \frac{2M_2}{l} + \frac{\Sigma W_4(l-d)}{l} \quad . \quad . \quad . \quad . \quad (67)$$

$$R_4 = \frac{\Sigma W_2(l-b)}{l} + \frac{M_2}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

$$R_1 + R_2 + R_3 + R_4 = \Sigma W.$$

It has been assumed in the foregoing discussions that the ends A and E of the trusses are not lifted off the supports under all conditions of loading.

40. Fig. 37 shows a double-swing bridge with six supports, made partially continuous by the omission of diagonals in the central panel of each truss, for reasons already stated in the case of a common swing bridge.

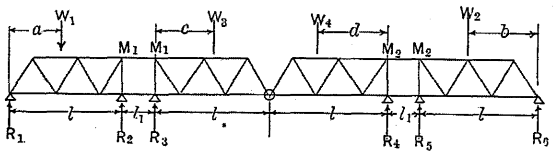

Fig. 37

Then, since by construction there can be no shear in the central panels, the moments over supports belonging to them must be equal to each other in both spans.

Assuming the cross-section of the trusses to be uniform throughout, and considering moments only, we have for total internal work due to any one load W_1 in the left-end span:

$$\omega = \frac{1}{2EI}\Big[\int_0^a (R_1x)^2\,dx + \int_a^l \{R_1x - W_1(x-a)\}^2 dx + \int_0^{l_1} M_1^2 dx$$
$$+ \int_0^l (M_1 + R_3x)^2\,dx + \int_0^l (M_2 + R_4x)^2 dx + \int_0^{l_1} M_2^2 dx$$
$$+ \int_0^l (R_6x)^2\,dx\Big].$$

Since

$$R_1 = \frac{W_1(l-a)}{l} + \frac{M_1}{l},$$
$$R_3 = -\frac{M_1}{l},$$
$$R_4 = -\frac{M_2}{l},$$
$$R_6 = \frac{M_2}{l},$$
$$M_2 = M_1 + 2R_3 l = -M_1.$$

Substituting these values in the expression for work, and setting the first derivative of the latter with respect to M_1 equal to zero, we get

$$M_1 = -\frac{a(l^2-a^2)}{l(8l+12l_1)}W_1 \quad . \quad . \quad . \quad . \quad . \quad (69)$$

whence, also,

$$M_2 = \frac{a(l^2-a^2)}{l(8l+12l_1)}W_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

Similarly for any load W_2 on the right-end span, we get,

$$M_1 = \frac{b(l^2-b^2)}{l(8l+12l_1)}W_2 \quad . \quad . \quad . \quad . \quad . \quad (71)$$

$$M_2 = -\frac{b(l^2-b^2)}{l(8l+12l_1)}W_2 \quad . \quad . \quad . \quad . \quad (72)$$

For any one load W_3 in the second span from the left, the internal work due to moments caused by the same may be expressed as follows:

$$\omega = \frac{1}{2EI}\Big[\int_0^l (R_1 x)^2 dx + \int_0^{l_1} M_1^2 dx + \int_0^c (M_1 + R_3 x)^2 dx$$
$$+ \int_c^l \{M_1 + R_3 x - W_3 (x-c)\}^2 dx$$
$$+ \int_0^l (M_2 + R_4 x)^2 dx + \int_0^{l_1} M_2^2 dx + \int_0^l (R_6 x)^2 dx\Big].$$

Since in this case,

$$R_1 = \frac{M_1}{l},$$
$$R_3 = \frac{W_3 (l-c)}{l} - \frac{M_1}{l},$$
$$R_4 = -\frac{M_2}{l},$$
$$R_6 = \frac{M_2}{l},$$
$$M_2 = -(M_1 + W_3 c).$$

Substituting them in the expression for internal work, and setting the differential coefficient with respect to M_1 equal to zero, we get,

$$M_1 = -\frac{6cl(l+l_1) - c^2(3l-c)}{l(8l+12l_1)} W_3 \quad . \; . \; . \quad (73)$$
$$M_2 = -\frac{2cl(l+3l_1) + c^2(3l-c)}{l(8l+12l_1)} W_3 \quad . \; . \quad (74)$$

Similarly for any load W_4 in the third span from the left, we get,

$$M_1 = -\frac{2dl(l+3l_1) + d^2(3l-d)}{l(8l+12l_1)} W_4 \quad . \; . \quad (75)$$
$$M_2 = -\frac{6dl(l+l_1) - d^2(3l-d)}{l(8l+12l_1)} W_4 \quad . \; . \quad (76)$$

Finally, for any number of loads we get,

$$M_1 = \frac{1}{l(8l+12l_1)}[-\Sigma W_1 a(l^2-a^2) + \Sigma W_2 b(l^2-b^2)$$
$$-\Sigma W_3 c\{6l(l+l_1) - c(3l-c)\} - \Sigma W_4 d\{2l(l+3l_1)$$
$$+ d(3l-d)\}] \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (77)$$

$$M_2 = \frac{1}{l(8l+12l_1)}[\Sigma W_1 a(l^2-a^2) - \Sigma W_2 b(l^2-b^2)$$
$$-\Sigma W_3 c\{2l(l+3l_1) + c(3l-c)\} - \Sigma W_4 d\{6l(l+l_1)$$
$$- d(3l-d)\}] \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (78)$$

$$R_1 = \frac{\Sigma W_1 (l-a)}{l} + \frac{M_1}{l} \quad . \; . \; . \; . \; . \quad (79)$$
$$R_2 = \frac{\Sigma W_1 a}{l} - \frac{M_1}{l} \quad . \; . \; . \; . \; . \; . \; . \quad (80)$$
$$R_3 = \frac{\Sigma W_3 (l-c)}{l} - \frac{M_1}{l} \quad . \; . \; . \; . \; . \quad (81)$$
$$R_4 = \frac{\Sigma W_4 (l-d)}{l} - \frac{M_2}{l} \quad . \; . \; . \; . \; . \quad (82)$$
$$R_5 = \frac{\Sigma W_2 b}{l} - \frac{M_2}{l} \quad . \; . \; . \; . \; . \; . \; . \quad (83)$$
$$R_6 = \frac{\Sigma W_2 (l-b)}{l} + \frac{M_2}{l} \quad . \; . \; . \; . \; . \quad (84)$$
$$\Sigma R = \Sigma W.$$

41. The foregoing equations for double-swing bridge give but approximate results for reasons already explained. To obtain more correct results, resort must be had to

$$\omega = \frac{\Sigma S_2 L}{2EA}$$

for expressing the internal work, extending the expression over all the members of the truss, based on the approximate values of S and A provisionally found by the foregoing equations, exactly as explained in the case of common swing bridges.