CHAPTER I

TRUSSED BEAMS

9. A trussed beam is often treated as a continuous girder resting on fixed supports, and sometimes as so many discontinuous beams as the number of panels into which the beam is divided. That neither treatment is correct hardly requires any explanation.

In the trussed beam of Fig. 4, it is evident that for any load W, as soon as the pressure in the post is made known, stresses in all other members will at once become determinate. Throughout the discussion of beams the following signs will be used:

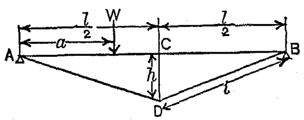

Fig. 4

Compression $-$.

Tension $+$.

Moment $+$, when producing compression on the upper fibres, and vice versâ.

Representing by P the unknown pressure in CD we have the following direct stresses in the several members:

$$\overline{CD} = - P,$$

$$\overline{AD} = \overline{BD} = + \frac{P}{2} \frac{i}{h},$$

$$\overline{AB} = - \frac{P}{2} \frac{l}{2h}.$$

Let A_1, A_2, A_3, represent the cross-sectional areas, and E_1, E_2, E_3, the moduli of elasticity of members AB, CD,

and AD respectively. Then referring to Eq. (1) we get for the work of resistance due to the direct stresses the following expressions:

$$\text{Work in } CD \quad \frac{P^2 h}{2 E_2 A_2} \quad (a)$$

$$\text{“ “ } AD \text{ and } BD \quad \frac{P^2 i^3}{4 E_3 A_3 h^2} \quad (b)$$

$$\text{“ “ } AB \quad \frac{P^2 l^3}{32 E_1 A_1 h^2} \quad (c)$$

The beam AB sustains beside the direct stress, the bending moment which at any point distant x from A is

$$A \text{ to } a \quad \left\{ \frac{W(l-a)}{l} - \frac{P}{2} \right\} x,$$

$$a \text{ to } C \quad Wa - \left(\frac{P}{2} + \frac{Wa}{l} \right) x,$$

$$C \text{ to } B \quad \left(\frac{Wa}{l} - \frac{P}{2} \right) (l - x),$$

so that for the internal work due to the same we get by referring to Eq. (3) the following expressions:

$$\frac{1}{2EI} \left[\int_0^a \left\{ \frac{W(l-a)}{l} - \frac{P}{2} \right\}^2 x^2 dx + \int_a^{\frac{l}{2}} \left\{ Wa - \left(\frac{P}{2} + \frac{Wa}{l} \right) x \right\}^2 dx + \int_{\frac{l}{2}}^{l} \left\{ \left(\frac{Wa}{l} - \frac{P}{2} \right) (l - x) \right\}^2 dx \right]. \quad (d)$$

in which I denotes the moment of inertia of the section of the beam assumed to be uniform throughout.

Summing up the several works, we get for the total internal work:

$$\omega = (a) + (b) + (c) + (d).$$

Since the value of P must be such as to make ω a minimum, we get for

$$\frac{d\omega}{dP} = 0$$

the following expression,

$$\frac{Ph}{E_2A_2} + \frac{Pi^3}{2E_3A_3h^2} + \frac{Pl^3}{16E_1A_1h^2} + \frac{1}{2E_1I}\Bigg[-\frac{Wa^3(l-a)}{3l} + \frac{Pa^3}{6} + \Bigg\{\frac{P}{2} - \frac{W(l-a)}{l}\Bigg\}\Bigg(\frac{l^3-8a^3}{24}\Bigg) + \frac{W(l^3-8a^3)}{24} - \frac{Wa(l^2-4a^2)}{8} - \frac{Wal^2}{24} + \frac{Pl^3}{48}\Bigg] = 0,$$

from which

$$P = \frac{\dfrac{3al^2 - 4a^3}{48E_1I}}{\dfrac{h}{E_2A_2} + \dfrac{i^3}{2h^2E_3A_3} + \dfrac{l^3}{16h^2E_1A_1} + \dfrac{l^3}{48E_1I}}W \qquad (5)$$

To obtain the stress in each member it is simply necessary to substitute this value of P in the expressions for stresses already given. It is evident that the beam AB is subjected to bending and direct stress combined.

Differentiating the second member of Eq. (5) with respect to a and setting the derivative equal to zero, it will be found that P will be maximum for $a = \frac{l}{2}$, as might be anticipated.

10. For a *uniform load* w per unit length we have but to substitute wda for W in Eq. (5) and integrate between given limits of loading for each half span. Thus for partial uniform load a_1w (Fig. 5) we get,

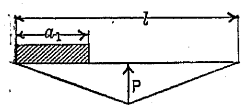

Fig. 5

$$P = \frac{\dfrac{3l^2 - 2a_1^2}{96E_1I}a_1^2w}{\dfrac{h}{E_2A_2} + \dfrac{i^3}{2h^2E_3A_3} + \dfrac{l^3}{16h^2E_1A_1} + \dfrac{l^3}{48E_1I}} \quad . \; . \; . \quad (6)$$

and for full uniform load wl,

$$P = \frac{\dfrac{5l^4w}{384E_1I}}{\dfrac{h}{E_2A_2} + \dfrac{i^3}{2h^2E_3A_3} + \dfrac{l^3}{16h^2E_1A_1} + \dfrac{l^3}{48E_1I}} \quad . \; . \; . \quad (7)$$

Example. — A wooden beam 12 in. × 10 in. × 20 feet long between supports, is reinforced by a steel rod 2 in. in diameter and a cast iron strut 3 in. sq. and 2 ft. high. To find the stress in each member due to a full uniform load of 1,200 lbs. per ft. run.

In this case

$$I = \tfrac{1}{12} \times 10 \times 12^3 = 1440$$
$$A_1 = 120 \quad A_2 = 9 \quad A_3 = 3.14 \quad h = 24 \quad l = 240$$
$$i = 122.4 \quad w = 100 \quad \text{(all in in. and lbs.)}$$

Assuming

E_3 = 30,000,000 lbs. per sq. in.
E_2 = 15,000,000 " "
E_1 = 1,500,000 " "

we get in Eq. (7),

$$\frac{5l^4w}{384E_1I} = 2, \qquad \frac{h}{E_2A_2} = .00000018,$$
$$\frac{i^3}{2h^2E_3A_3} = .000017, \qquad \frac{l^3}{16h^2E_1A_1} = .0000083,$$
$$\frac{l^3}{48E_1I} = .000133,$$

so that

$$P = 12{,}610 \text{ lbs.}$$

Denoting by m the moment at any point x of the beam, we have,

$$m = \frac{24{,}000 - 12{,}610}{2}x - \frac{100\,x^2}{2}.$$

The maximum moment will be found when $\frac{dm}{dx} = 0$; *i.e.*, for $x = 57$ in., so that

$$\text{max. } m = (5695 - 2850)\,57 = 162{,}165 \text{ in. lbs.}$$

The maximum fibre stress in the beam will therefore be

$$\frac{162{,}165}{1440} \times 6 + \frac{12{,}610}{2} \times \frac{240}{48} \times \frac{1}{120} = 938 \text{ lbs. per sq. in.}$$

The tension in the tie-rod is equal to

$$\frac{12{,}610}{2} \times \frac{122.4}{24} \times \frac{1}{3.14} = 10{,}240 \text{ lbs. per sq. in.}$$

The intensity of compression in the strut CD is simply

$$\frac{12{,}610}{9} = 1{,}401 \text{ lbs. per sq. in.}$$

The following table shows the comparison of stresses in the members as calculated above, with those obtained by assuming the beam first to be continuous over three fixed supports and then to consist of two discontinuous beams.

	P IN LBS.	Max. fib. stress in beam, lbs. per sq. in.	Dif.	Tension in tie-rod, lbs. per sq. in.	Dif.
Beam contin. on yield sup.	12,610	938		10,240	
Contin. on 3 fix. sup.	15,000	1,062	+13%	12,180	+19%
Discon. at centre . .	12,000	1,000	+6½%	9,740	− 5%

11. In the trussed beam of Fig. 6 the central panel, owing to its lack of diagonal, is incapable of transmitting

shear except through the beam itself. For this reason, so long as the relative positions of points C and E are vertically unchanged (which would practically be the case when the beam AF has sufficient rigidity to remain nearly straight when loaded) the stresses in BC and DE may be assumed to be equal. Let P denote the pressure in BC or DE, then by retaining the notations of the preceding case, we obtain the following works of resistance due to any load W, located between A and B:

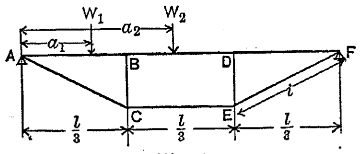

Fig. 6

Work in BC and DE . . . $\frac{P^2h}{E_2A_2}$,

" " AC, CE and EF . . $\frac{P^2}{E_3A_3h^2}\left(i^3 + \frac{l^3}{54}\right)$,

" " AF . . . . $\frac{P^2l^3}{18\,E_1A_1h^2}$.

The bending moment in the beam causes the following work:

$$\frac{1}{2\,E_1I}\left[\int_0^a\left(\frac{l-a_1}{l}W_1 - P\right)^2 x^2dx + \int_a^{\frac{l}{3}}\left\{\frac{W_1a_1(l-x)}{l} - Px\right\}^2 dx + \int_{\frac{l}{3}}^{\frac{2l}{3}}\left\{W_1a_1\left(\frac{l-x}{l}\right) - \frac{Pl}{3}\right\}^2 dx + \int_0^{\frac{l}{3}}\left(\frac{W_1a_1}{l} - P\right)^2 x^2dx\right].$$

Summing up these expressions for work and setting the first derivative of the sum with respect to P equal to zero, we get,

$$P\left\{\frac{2\,h}{E_2A_2}+\frac{2\,i^3}{E_3A_3h^2}+\frac{l^3}{27\,E_3A_3h^2}+\frac{l^3}{9\,E_1A_1h^2}\right\}$$
$$+\frac{1}{2\,E_1I}\left[\left\{2\,P-\frac{2\,(l-a_1)}{l}W_1\right\}\frac{a_1^{\,3}}{3}+\frac{2\,P}{3}\left(\frac{l^3}{27}-a_1^{\,3}\right)\right.$$
$$-\frac{2\,W_1a_1}{l}\left(\frac{7\,l^3}{162}-\frac{a^2l}{2}+\frac{a^3}{3}\right)+\frac{2\,Pl^3}{27}-\frac{W_1a_1l^2}{9}$$
$$\left.+\frac{2\,Pl^3}{81}-\frac{2\,W_1a_1l^2}{81}\right]=0,$$

from which

$$P=\frac{\dfrac{(2\,l^2-3\,a_1^{\,2})\,a_1}{18\,E_1I}}{\dfrac{2\,h}{E_2A_2}+\dfrac{2\,i^3}{E_3A_3h^2}+\dfrac{l^3}{27\,E_3A_3h^2}+\dfrac{l^3}{9\,E_1A_1h^2}+\dfrac{5\,l^3}{81\,E_1I}}W_1. \quad (8)$$

For any load W_2 between B and D we obtain in a similar manner as for W_1 the following expression for P:

$$P=\frac{\dfrac{27\,a_2\,(l-a_2)-l^2}{162\,E_1I}\,l}{\dfrac{2\,h}{E_2A_2}+\dfrac{2\,i^3}{E_3A_3h^2}+\dfrac{l^3}{27\,E_3A_3h^2}+\dfrac{l^3}{9\,E_1A_1h^2}+\dfrac{5\,l^3}{81\,E_1I}}W_2. \quad (9)$$

12. For *full uniform load* w per unit length, by substituting wda for W_1 and W_2 and integrating, we get,

$$P=\frac{\dfrac{11\,l^4w}{486\,E_1I}}{\dfrac{2\,h}{E_2A_2}+\dfrac{2\,i^3}{E_3A_3h^2}+\dfrac{l^3}{27\,E_3A_3h^2}+\dfrac{l^3}{9\,E_1A_1h^2}+\dfrac{5\,l^3}{81\,E_1I}}. \quad (10)$$

EXAMPLE. — A beam with dimensions, materials and loading, as in the preceding example, is trussed as in Fig. 6, with $h = 24$ in. $i = 83.5$ in. Then in (10) we have

$$\frac{11\,l^4w}{486\,E_1I}=3.4765, \qquad \frac{2\,i^3}{E_3A_3h^2}=.00002145,$$
$$\frac{2\,h}{E_2A_2}=.00000035, \qquad \frac{l^4}{27\,E_3A_3h^2}=.00000922,$$
$$\frac{l^3}{9\,E_1A_1h^2}=.00001485, \qquad \frac{5\,l^3}{81\,E_1l}=.00039506,$$

so that

$$P = 7{,}900 \text{ lbs.}$$

Since the moment at any point x between A and B is

$$\frac{wl-2\,P}{2}x-\frac{wx^2}{2},$$

the moment will be maximum for $x = 41$, and will be equal to 252,150 in.-lbs.

Again, since at x from A between B and D the moment is

$$\frac{wl-2\,P}{2}x+P\left(x-\frac{l}{3}\right)-\frac{wx^2}{2},$$

the maximum moment will be found at $x = 119$ in., and will be equal in amount to 87,950 in.-lbs. Taking, then, the first maximum, we get for the maximum fibre stress the following:

$$\frac{252{,}150}{1440}\times 6+7900\times\frac{l}{3\,h}\times\frac{1}{A_2}=1{,}270 \text{ lbs. per sq. in.}$$

The tension in tie-rods AC and EF equals

$$7900\times\frac{83.5}{24}\times\frac{1}{3.14}=8{,}750 \text{ lbs. per sq. in.}$$

while that in tie-rod CE equals

$$7900\times\frac{80}{24}\times\frac{1}{3.14}=8{,}060 \text{ lbs. per sq. in.}$$

13. In the two preceding cases of trussed beams, when the depth h of the truss is considerable the influence of the beam will become lost in comparison to the truss, and such a structure approximates itself to a King or Queen post truss whichever it happens to be.

14. A beam reinforced by sloping struts and a straining beam as shown in Fig. 7, is often met with in wooden constructions. The case is somewhat similar to that of Fig. 6. The reinforcing frame *CEFD* could retain its form only when the forces acting at *E* and *F* are equal; and since the beam *AB* will remain practically straight under all circumstances, the reactions produced at *E* and *F* may be assumed to be equal. For any load W_1 between *A* and *E*, then, since Σ vert. forces = 0,

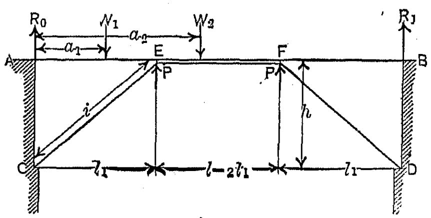

Fig. 7

$$R_0 + R_1 + 2P_1 - W_1 = 0,$$

in which P_1 denotes that part of P due to W_1. Taking moments successively at B and A, we get,

$$R_0 = \frac{W_1(l-a_1)}{l} - P_1$$

$$R_1 = \frac{W_1 a_1}{l} - P_1.$$

It will be seen from these equations that the reactions at *A* and *B* will, according to modes of loading, be + or −, which latter is to be met by anchoring the beam down to the supports.

The internal work of resistance may now be written,

$$\omega = \frac{P_1^2}{EA}\frac{i^3}{h^2} + \frac{P_1^2 l_1^2(l-2l_1)}{2EAh^2} + \int_0^l \frac{m^2dx}{2EI},$$

in which A represents the sectional area of individual members of the frame, I the moment of inertia of the beam AB, and m the moment at any point x of the beam. Substituting in this the following expressions for m (the origin of x being taken at A),

A to a_1 . . $\left\{\frac{W_1(l-a_1)}{l} - P_1\right\}x,$

a_1 to E . . $\frac{W_1 a_1(l-x)}{l} - P_1 x,$

E to F . . $\frac{W_1 a_1(l-x)}{l} - P_1 l_1,$

B to F . . $\left(\frac{W_1 a_1}{l} - P_1\right)(l-x),$

and differentiating ω with respect to P_1 and setting the differential coefficient equal to zero, we get,

$$P_1 = \frac{\frac{a_1}{6I}(3l_1 l - 3l_1^2 - a_1^2)}{\frac{2i^3}{Ah^2} + \frac{l_1^2(l-2l_1)}{Ah^2} + \frac{l_1^2(3l-4l_1)}{3I}} W_1 \quad . \quad . \quad (11)$$

For any load W_2 between E and F similarly we get

$$P_2 = \frac{\frac{l_1}{6I}(3a_2 l - l_1^2 - 3a_2^2)}{\frac{2i^3}{Ah^2} + \frac{l_1^2(l-2l_1)}{Ah^2} + \frac{l_1^2(3l-4l_1)}{3I}} W_2 \quad . \quad . \quad (12)$$

It is evident that in this kind of construction, the beam *AB* may be considered to be free of direct stress.

EXAMPLE. — A wooden beam 12 in. × 8 in. × 30 ft. is reinforced by sloping struts and a straining beam 8 in. × 8 in., with $l_1 = 10$ ft. and $h = 8$ ft. To find the maximum stress in each member due to a full uniform load of 1800 lbs. per ft. run.

Substituting in Eqs. (11) and (12) the following values,

$$i = 12.8 \text{ ft.}, \qquad A = 64 \text{ sq. in.}, \qquad I = 1152 \text{ in.}^4,$$
$$\frac{2\,i^3}{Ah^2} = 12, \qquad \frac{l_1^2\,(l - 2\,l_1)}{Ah^2} = 3, \qquad \frac{l_1^2\,(3\,l - 4\,l_1)}{3\,I} = 2500,$$
$$6\,I = 6912,$$

we get,

$$P_1 = \frac{2\int_0^{l_1} a\,(3\,l_1 l - 3\,l_1^2 - a^2)\,w da}{6912 \times 2515} = \frac{2 \times \frac{11}{324}\,l^4 w}{6912 \times 2515},$$

$$P_2 = \frac{\int_{l_1}^{2\,l_1} l_1\,(3\,al - l_1^2 - 3\,a^2)\,w da}{6912 \times 2515} = \frac{\frac{11}{162}\,l^4 w}{6912 \times 2515},$$

and consequently

$$P = P_1 + P_2 = \frac{(2 \times 27{,}500 + 55{,}000)\,12^4 \times 150}{6912 \times 2515} = 19{,}680 \text{ lbs.}$$

Comparing the maximum moments in the side and central panels, it will be seen that in this case the greatest moment is found at E and F and is equal in amount to

$$\frac{30 \times 1800 - 19{,}680 \times 2}{2} \times 10 - \frac{1800 \times 10^2}{2} = -\,16{,}800 \text{ ft.-lbs.},$$

so that the maximum fibre stress in the beam will be

$$\frac{16{,}800 \times 12}{1152} \times 6 = 1050 \text{ lbs. per sq. in.}$$

Other stresses are as follows :

$$\text{Sloping strut,} \quad 19{,}680 \times \frac{12.8}{8} \times \frac{1}{64} = 492 \text{ lbs. per sq. in.}$$

$$\text{Straining beam,} \quad 19{,}680 \times \frac{10}{8} \times \frac{1}{64} = 345 \text{ lbs. per sq. in.}$$

15. In wooden beam-bridges, the supports, instead of being of masonry, often consist of pile-works such as shown in Fig. 8. In such a construction the struts move laterally, owing to the bending of the piles. It is necessary, therefore, in the summation of internal works, to take in the work in the posts, each of which having one end firmly fixed is simply supported at the other with a horizontal load of $\frac{Pl_1}{h}$ acting at C and D. Calling the moment of inertia of the pile I_0 and neglecting the influence of direct stress as being inconsiderable when compared with that of moment, we get the following internal work in the two posts due to $\frac{Pl_1}{h}$, the dotted portion of the structure not being considered,

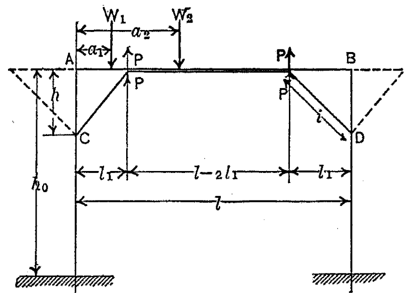

Fig. 8

$$\frac{1}{EI_0}\left[\int_0^h \left(\frac{M}{h_0} - \frac{Pl_1}{h}\,\frac{h_0 - h}{h_0}\right)^2 x^2 dx + \int_h^{h_0} \left(\frac{M}{h_0}x - Pl_1\frac{h_0 - x}{h_0}\right)^2 dx\right],$$

in which M represents the moment at the base of the post, and x the distance from A downward. Since the value of M must be such as to make this work a minimum, setting the first derivative of the above expression with respect to M equal to zero, we get,

$$M = \frac{(h_0^2 - h^2)}{2\,h_0^2} Pl_1.$$

Substituting this value of M in the above expression, the latter becomes,

$$\frac{P^2 l_1^2}{EI_0}\left\{\frac{(h_0 - h)^3 (3\,h_0 + h)}{12\,h_0^3}\right\}.$$

Adding this to the total work of the preceding case (Art. 14), and differentiating with respect to P and setting the differential coefficient equal to zero, we obtain the following equations for loads W_1 and W_2 corresponding to Eqs. (11) and (12):

$$P_1 = \frac{\frac{a_1}{6\,I}(3\,l_1 l - 3\,l_1^2 - a_1^2)}{\frac{2\,i^3}{Ah^2} + \frac{l_1^2(l-2\,l_1)}{Ah^2} + \frac{l_1^2(3\,l-4\,l_1)}{3\,I} + \frac{l_1^2(h_0-h)^3(3\,h_0+h)}{6\,h_0^3 I_0}} W_1. \quad (13)$$

$$P_2 = \frac{\frac{l_1}{6\,I}(3\,a_2 l - l_1^2 - 3\,a_2^2)}{\frac{2\,i^3}{Ah^2} + \frac{l_1^2(l-2\,l_1)}{Ah^2} + \frac{l_1^2(3\,l-4\,l_1)}{3\,I} + \frac{l_1^2(h_0-h)^3(3\,h_0+h)}{6\,h_0^3 I_0}} W_2. \quad (14)$$

The actual measure of h_0 will always be a matter of judgment according to the nature of the ground into which the piles are driven. The heads of piles may in most cases be assumed to be laterally fixed in position.

When there are several consecutive spans with sloping struts as shown dotted and in full, the thrusts at C or D due to full uniform load will balance each other, and the case will be that of Fig. 7, while load on one span only will produce action intermediate between the cases of Figs. 7 and 8.

EXAMPLE. — Using the same dimensions and load as in the preceding example (Art. 14) and further with

$$h_0 = 18 \text{ ft.} \qquad I_0 = \tfrac{1}{12} \times 15^3 \times 15 = 4220 \text{ in.}^4$$

to find the stresses in different members of the frame.

Here we have as before,

$$\frac{2\,i^3}{Ah^2} = 12. \qquad \frac{l_1^2(l - 2\,l_1)}{Ah^2} = 3, \qquad \frac{l_1^2(3\,l - 4\,l_1)}{3\,I} = 2500,$$

$$\frac{l_1^2(h_0 - h)^3(3\,h_0 + h)}{6\,h_0^3 I_0} = 72, \qquad 6\,I = 6912,$$

so that

$$P = P_1 + P_2 = \frac{2 \times 27{,}500 + 55{,}000}{6912\,(2515 + 72)} \times 12^4 \times 150 = 19{,}130 \text{ lbs.}$$

Since the moment at any point x from the end of the beam in the side span is, in this case,

$$\left(\frac{1800 \times 30}{2} - 19{,}130\right)x - \frac{1800\,x^2}{2},$$

it will be maximum for $x = 4.36$ ft., and as it is found to be greater than the maximum moment in the central span, the maximum moment in the beam will be

$$4.36\,(7870 - 900 \times 4.36) = 17{,}200 \text{ ft.-lbs.}$$

The beam acting as a tie for post-heads will have to resist a pull of

$$\frac{Pl_1}{h}\left(\frac{h_0 - h}{h_0} - \frac{h\,(h_0^2 - h^2)}{2\,h_0^3}\right) = 9020 \text{ lbs.}$$

We now have the following intensities of stress :

Beam . . . $\frac{17{,}200 \times 12}{1152} \times 6 + \frac{9020}{96} = 1169$ lbs. per sq. in.

Sloping strut . $19{,}130 \times \frac{12.8}{8} \times \frac{1}{64} = 478$ lbs. per sq. in.

Straining beam, $19{,}130 \times \frac{10}{8} \times \frac{1}{64} = 373$ lbs. per sq. in.

In the post, comparing the moments at C with M, the latter is found to be the greater, and we obtain for the maximum fibre stress,

$$\frac{27{,}000}{15 \times 15} + \frac{76{,}756 \times 12}{4220} \times 6 = 1429 \text{ lbs. per sq. in.,}$$

showing that a considerable stress is thrown into the post in such a construction.