THE WAVE VELOCITIES OF TRANSVERSE VIBRATIONS OF RECTANGULAR THIN PLATES, CONSIDERING RAYLEIGH'S ROTATORY MOMENT OF INERTIA AND TIMOSHENKO'S EFFECT OF SHEAR

——ADDENDUM TO THE STUDIES ON RECTANGULAR PLATES※——

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Synopsis In order to make clear the high-frequency vibrations of rectangular plates, it was neccessitated to appreciate the accuracies of the computed values of the wave velocities. It was satisfactorily verified that, if Rayleigh's rotatory moment of inertia and Timoshenko's effect of shear were considered, the exact results were gotten in practice.

Introduction

For studying the dynamical behaviors of rectangular thin plates, the customary differential equation can not be used, because it gives the extremely erroneous results of transverse vibrations in the high-frequency interval. However, if Rayleigh's rotatory moment of inertia and Timoshenko's effect of shear are supplemented, it gives the exact ones practically. Bibliography from 1) to 8) also confirm it for the case of the infinite thin plate. The present work was already completed before Bibliography 8) was published, whereas the results of the latter completely agree with those of the author.

The Differential Equation and Wave Velocities

Taking the notations such as the x- and y-coordinates along the edge-wise directions of the rectangular plates, h the thickness, a and b the length and breadth, $r=h/\sqrt{12}$ the radius of gyration of the section, F the sectional area of unit breadth,

Fig. 1

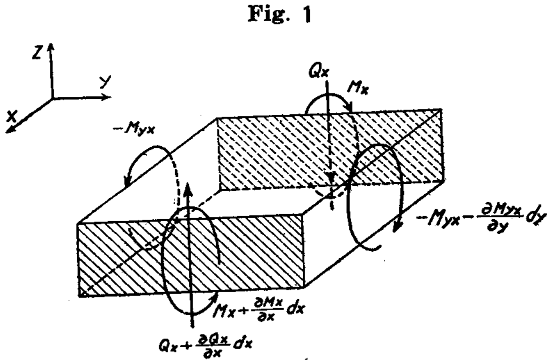
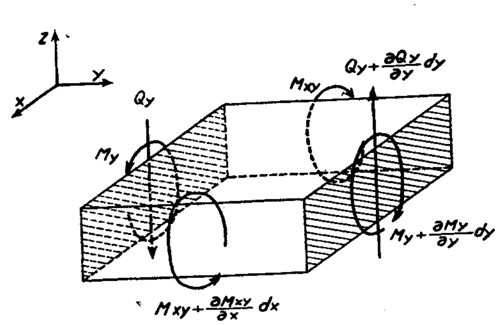


Fig. 2



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1), 2), ……, 9) Bibliography given at the end.

I the moment of inertia of the section of unit breadth, E Joung's modulus, G the modulus of rigity, ν Poisson's ratio, κ the constant of the section relating to shear, D the flexural rigidity, ρ the mass of the unit volume, p the circular frequency, v the wave velocity of transverse vibration, λ the half wave length, w and ζ the total deflection and that caused by the moments only, M_x, M_y, M_{xy} the moments about the y- and x-axes and the twisting, and Q_x and Q_y the vertical shear in the x- and y-sections respectively, then gives

$$M_x = -D\left(\frac{\partial^2\zeta}{\partial x^2}+\nu\frac{\partial^2\zeta}{\partial y^2}\right); \qquad M_y = -D\left(\frac{\partial^2\zeta}{\partial y^2}+\nu\frac{\partial^2\zeta}{\partial x^2}\right);$$

$$M_{xy} = D(1-\nu)\frac{\partial^2\zeta}{\partial x\partial y}; \qquad \cdots\cdots(1)$$

$$Q_x = GF\frac{\partial}{\partial x}(w-\zeta), \qquad Q_y = GF\frac{\partial}{\partial y}(w-\zeta); \qquad \cdots\cdots(2)$$

$$D = \frac{EI}{(1-\nu^2)} = \rho I v_E{}^2, \qquad G = (1-\nu)\frac{\rho v_E{}^2}{2} = \rho v_G{}^2; \qquad \cdots\cdots(3)$$

$$v_E = \sqrt{\frac{E}{\rho(1-\nu^2)}}, \qquad v_G = \sqrt{\frac{G}{\rho}}, \qquad v_R = \sqrt{\kappa}\, v_G; \qquad \cdots\cdots(4)$$

Referring to **Fig. 1** and **2**, we get

$$-\frac{\partial M_x}{\partial x}dx\,dy + Q_x\,dx\,dy + \frac{\partial M_{xy}}{\partial y}dy\,dx = \rho I\frac{\partial^2}{\partial t^2}\frac{\partial\zeta}{\partial x}dx\,dy$$

$$-\frac{\partial M_x}{\partial y}dy\,dx + Q_y\,dy\,dx + \frac{\partial M_{xy}}{\partial x}dx\,dy = \rho I\frac{\partial^2}{\partial t^2}\frac{\partial\zeta}{\partial y}dx\,dy$$

$$\frac{\partial Q_x}{\partial x}dx\,dy + \frac{\partial Q_y}{\partial y}dy\,dx = \rho F\frac{\partial^2 w}{\partial t^2}\frac{\partial\zeta}{\partial y}dy\,dx,$$

from which we reduce

$$\left[\nabla^4 - \frac{2+\kappa(1-\nu)}{2\kappa}\frac{\rho}{G}\nabla^2\frac{\partial^2}{\partial t^2} + \frac{1-\nu}{2}\frac{\rho}{Gr^2}\frac{\partial^2}{\partial t^2} + \frac{1-\nu}{2\kappa}\left(\frac{\rho}{G}\right)^2\frac{\partial^4}{\partial t^4}\right]w,\zeta = 0 \qquad \cdots\cdots(5)$$

or

$$\left[\nabla^4 - \left(\frac{1}{v_R{}^2}+\frac{1}{v_E{}^2}\right)\nabla^2\frac{\partial^2}{\partial t^2} + \frac{1}{r^2 v_E{}^2}\frac{\partial^2}{\partial t^2} + \frac{1}{v_R{}^2 v_E{}^2}\frac{\partial^4}{\partial t^4}\right]w,\zeta = 0 \quad \cdots(6)$$

Putting

$$w = \sum_{m=0}^{\infty}{}^{\cos}_{\sin}\,\overline{m}_0 x(\overline{A}_m\cos\overline{m}_1 y + \overline{B}_m\sin\overline{m}_1 y + \overline{C}_m\cosh\overline{m}_2 y + \overline{D}_m\sinh\overline{m}_2 y)$$

$$+\sum_{n=0}^{\infty}(A_n\cos\overline{n}_1 x + B_n\sin\overline{n}_1 x + C_n\cosh\overline{n}_2 x + D_n\sinh\overline{n}_2 x)\,{}^{\cos}_{\sin}\,\overline{n}_0 y \quad \cdots(7)$$

into (6), we get

$$\chi^4 - \chi^2\left(\frac{1}{v_R{}^2}+\frac{1}{v_E{}^2}\right)p^2 - \frac{p^2}{r^2 v_E{}^2} + \frac{p^4}{v_R{}^2 v_E{}^2} = 0 \qquad \cdots\cdots(8)$$

$$\chi^2 = \overline{m}_0{}^2 + \overline{m}_i{}^2 = \overline{n}_0{}^2 + \overline{n}_j{}^2, \qquad \overline{m}_0 a = m\pi, \qquad \overline{n}_0 b = n\pi, \qquad i,j = 1,2,$$

from which gives

$$a\chi = \frac{(a/r)v\,v_r}{\sqrt{(v_R{}^2-v^2)(v_E{}^2-v^2)}}, \qquad v = \frac{p}{\chi} \qquad \cdots\cdots(9)$$

$$p = \frac{v^2 v_R}{r\sqrt{(v_R{}^2-v^2)(v_E{}^2-v^2)}} \qquad \cdots\cdots(10)$$

$$\lambda = \frac{r\sqrt{(v_R{}^2-v^2)(v_E{}^2-v^2)}}{v\,v_R} \qquad \cdots\cdots(11)$$

and

for $p \ll v_R/r$: $v=\sqrt{prv_E}=\chi r v_E$ ....(12)

for $p=v_R/r$: $v=\dfrac{v_R v_E}{\sqrt{v_R^2+v_E^2}}=v_G\sqrt{\kappa/1+\kappa(1-\nu)/2}$

$$=\frac{v_R}{\sqrt{1+\kappa(1-\nu)/2}}=v_E\sqrt{\frac{\kappa(1-\nu)}{[2+\kappa(1-\nu)]}} \quad \cdots\cdots(13)$$

for $p \gg v_R/r$: $v_1=v_R \quad v_2=v_E$ ....(14)

Eq. from (10) to (14) perfectly coincide with those given in Bibliography 5) and 8).

On the contray, from the customary differential equation:

$$\nabla^4 w-\rho\frac{1-\nu^2}{r^2 E^2}\frac{\partial^2 w}{\partial t^2}=\nabla^4 w-\frac{1}{r^2 v_E^2}\frac{\partial^2 w}{\partial t^2}=0 \quad \cdots\cdots(15)$$

the wave velocity:

$$v'=\sqrt{prv_E}=r\chi v_E \quad \cdots\cdots(16)$$

is obtained. (16) is clearly erroneous, compared with (12), (13), and (14); the higher the frequency of the transverse vibration becomes, the more the error increases.♀

Ex. If, for the simply-supported quadratic plate: $a=b$, $a/b=100$, $m=23$, and $n=1$ are taken, gives from (9) and (16)

$$v'/v=1.692 \quad \cdots\cdots(17)$$

that is, the error of the velocity of the twenty-third vibration mounts up to +69.2 per cent.♀

Conclusion

1. The velocities of high-frequency transverse vibrations, quite independently studied, thoroughly coincide with those given in Bibliography 3), 5) and 8). It clearly certifies that the obtained results are practically exact, if Rayleigh's rotatory moment of inertia and Timoshenko's effect of shear are supplemented in problems, relating to rectangular thin plates as well as infinite ones, or the results, derived from the customary differential equation[9], are extremely erroneous.♀

2. The wave velocities of high-frequency transverse vibrations are not subjected to the boundary conditions of rectangular plates substantially, until they reach the boundaries.

3. They also have v_R and v_E, that is, the velocities of Rayleigh's surface wave and the dilatational one at the extremity $p\to\infty$; while the customary differential equation (17) gives only one and that the infinite.

Acknowledgement

This is the synopsis, devotedly dedicated to Prof. Emeritus Yutaka Tanaka, Tokyō University, M.J.A., for the sixty-first anniversary of his birthday.

The author also avails himself to express his grateful thanks to Dr. Eng. Nobuji Yamaga, formerly President of Keijō University, Prof. Emeritus Tokujirō Yoshida,

♀ The error of the twenty-third transverse vibration of the quadratic thin plate, calculated by W. Ritz, with his famous method may mount up to a fairly large value, because he used the customary differential equation.

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