主対角線係数及びこれに直接する係数のみを有する多元一次方程式の厳正解法

AN EXACT METHOD OF SOLVING THE LINEAR SIMULTANEOUS EQUATIONS WITH THE PRINCIPAL DIAGONAL COEFFICIENTS AND THOSE ADJACENT TO THEM ONLY

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Synopsis By utilizing the preestimated multipliers, the ordinary determinantal solutions of the linear simultaneous equations, gotten by applying the theory of three moments to analyse structures, such as Rohse girders[11], Vierendeel girders[4) 7)], etc.,[7)] are expanded into the very rapidly convergent series and simultaneously the recurrent retroactions of corrections are nulified.

Introduction

Up to the present, though the methods of rationally solving the linear simultaneous equations with the principal diagonal coefficients and those adjacent to them only have been publicated in some special cases by the other authorities[2) 4) 7) 11)], they can not be used generally. One of the effective methods, found by the author since 1940, is made public in this paper, in which the ordinary determinantal solutions are expanded into the most rapidly convergent series by applying the matrix-premultipliers to from the first and last columns of the determinants in succession, quite similar to the cases, given in Bibliography 15). It needs no more correction, because the recurrent retroactions of the corrections can completely be eliminated by multiplying all the coefficients by the preestimated multipliers, from whose ground it is called Method of Multipliers.

Theory of Method of Multipliers

If the linear simultaneous equations:

$$\left.\begin{array}{l} a_{1,1}x_1+a_{1,2}x_2 = h_1 \\ a_{2,1}x_1+a_{2,2}x_2+a_{2,3}x_3 = h_2 \\ \quad a_{3,2}x_2+a_{3,3}x_3+a_{3,4}x_4 = h_3 \\ \qquad \cdots\cdots\cdots\cdots \\ \qquad \cdots\cdots\cdots\cdots \\ \qquad a_{n,n-1}x_{n-1}+a_{n,n}x_n = h_n \end{array}\right\} \qquad \cdots\cdots\cdots (1)$$

or in the matrix

$$ax = h \qquad \cdots\cdots\cdots (2)$$

where

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1), 2), ……, 16) Refer to Bibliography given at the end of this paper.

$$a=\begin{pmatrix} a_{1,1}, & a_{1,2}, & & & \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & & \\ & a_{3,2}, & a_{3,3}, & a_{3,4}, & \\ & & \cdots\cdots & & \\ & & & \cdots\cdots & \\ & & & & \cdots\cdots \\ & & & a_{n,n-1}, & a_{n,n} \end{pmatrix}, \quad x=\{x_i\}, \quad h=\{h_i\}, \qquad \cdots\cdots(3)$$

are given, then we get

$$x_r=D_r/D, \quad D=|a|, \quad D_r=|a|_{a_{i,r}\to h_i}, \quad r=1,2,\cdots\cdots,n. \qquad \cdots\cdots(4)$$

Taking the principal diagonal elements, the matrices, and their determinant such that

$$a_{i,i}=1; \qquad \cdots\cdots(5)$$

for $s<r$, $s=1,2,\cdots\cdots,r$,

$$I^s_{s-1}=\begin{pmatrix} 1. & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & b^{s-1}_{s+1,s} & \ddots & \\ & & & & 1 \end{pmatrix}, \quad b^{s-1}_{s+1,s}=-a_{s+1,s}\,m'_s; \qquad \cdots(6)$$

for $t>r$, $t=1,2,\cdots\cdots,n-r$,

$$E^{t-1}_{n-t+1}=\begin{pmatrix} 1. & & & & \\ & \ddots & b^{t-1}_{n-t,n-t+1} & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad b^{t-1}_{n-t,n-t+1}=-a_{n-t,n-t+1}\,m_{n-t+1}; \qquad \cdots\cdots(7)$$

$$D=\left|\prod_{s=1}^{r-1} I^s_{s-1}\left(\prod_{t-1}^{n-r} E^{t-1}_{n-t+1}\,a\right)\right|; \qquad \cdots\cdots(8)$$

then we obtain

$$D=1/m'_1, m'_2, \cdots\cdots, m'_{r-1}, z_r, m_{r+1}, m_{r+2}, \cdots\cdots, m_n; \qquad \cdots\cdots(9)$$

$$D_r=|D|_{a_{i,r}\to h_i}, \quad i=1,2,\cdots\cdots,n; \qquad \cdots\cdots(10)$$

referring to Eq. (4),

$$x_r=z_r\,h^{n-1}_r, \quad h^{n-1}_r=(1/z_r)_{a_{i,r}\to h_i}, \quad i=1,2,\cdots\cdots,n; \qquad \cdots\cdots(11)$$

$$h^{n-1}_r=h_r-\bar{a}_{r,r-1}\,h_{r-1}+\bar{a}_{r,r-1}\,\bar{a}_{r-1,r-2}\,h_{r-2}-\bar{a}_{r,r-1}\,\bar{a}_{r-1,r-2}\,\bar{a}_{r-2,r-3}\,h_{r-3}+\cdots\cdots$$
$$-\bar{a}_{r,r+1}\,h_{r+1}+\bar{a}_{r,r+1}\,\bar{a}_{r+1,r+2}\,h_{r+2}-\bar{a}_{r,r+1}\,\bar{a}_{r+1,r+2}\,\bar{a}_{r+2,r+3}\,h_{r+3}+\cdots\cdots; \qquad \cdots\cdots(12)$$

and

$$x=ZRh, \quad Z=z_i\,\delta_{i,k}, \quad \delta_{i,k}=\text{Kronecker's sign}; \qquad \cdots\cdots(13)$$

where

for $i>r$, $i=r+1, r+2, \cdots\cdots, n$,

$$\bar{a}_{i,i+1}=a_{i,i+1}\,m_{i+1}, \qquad \cdots\cdots(14)$$

$$1/\mathrm{m}_i=1-a_{i,i+1}\,a_{i+1,i}\,m_{i+1}=1-\bar{a}_{i,i+1}\,a_{i+1,i} \qquad \cdots\cdots(15)$$

$$h_i^{n-i}=h_i-\bar{a}_{i,i+1}h_{i+1}^{n-i-1}$$
$$=h_i-\bar{a}_{i,i+1}h_{i+1}+\bar{a}_{i,i+1}\bar{a}_{i+1,i+2}h_{i+2}-\bar{a}_{i,i+1}\bar{a}_{i+1,i+2}\bar{a}_{i+2,i+3}h_{i+3}+\cdots\cdots; \quad (16)$$

for $i<r,\quad i=1,2,\cdots\cdots,n,$

$$\bar{a}_{i,i-1}=a_{i,i-1}m'_{i-1}, \quad \cdots\cdots(17)$$
$$1/m'_i=1-a_{i,i-1}a_{i-1,i}m'_{i-1}=1-\bar{a}_{i,i-1}a_{i-1,i}, \quad \cdots\cdots(18)$$
$$h_i^{i-1}=h_i-\bar{a}_{i,i-1}h_{i-1}^{i-2}$$
$$=h_i-\bar{a}_{i,i-1}h_{i-1}+\bar{a}_{i,i-1}\bar{a}_{i-1,i-2}h_{i-2}-\bar{a}_{i,i-1}\bar{a}_{i-1,i-2}\bar{a}_{i-2,i-3}h_{i-3}+\cdots\cdots; \quad \cdots(19)$$
$$1/z_r=1-\bar{a}_{r,r-1}a_{r-1,r}-\bar{a}_{r,r+1}a_{r+1,r}; \quad \cdots\cdots(20)$$

and ZR is given by Eq. (28).

Numerical Calculations

The numerical calculations must be performed in order such as follows:

1. Divide with $a_{i,i}$ both the sides of the given equations (1) respectively, and according to Eq. (5), let $a_{i,i}=1$.

2. From Eqs. from (14) to (20), we obtain

for $i\geqq r,\quad i=n,n-1,\cdots\cdots,2,1,$

$$m_i=1/(1-a_{i,i+1}a_{i+1,i}m_{i+1}), \quad \cdots\cdots(21)$$

from which reads

$$\left.\begin{aligned}&m_n=1,\quad m_{n-1}=1/(1-a_{n-1,n}a_{n,n-1}),\\&m_{n-2}=1/(1-a_{n-2,n-1}a_{n-1,n-2}m_{n-1}),\\&m_{n-3}=1/(1-a_{n-3,n-2}a_{n-2,n-3}m_{n-2}),\\&\cdots\cdots\cdots\cdots;\end{aligned}\right\} \quad \cdots\cdots(22)$$

for $i\leqq r,\quad i=1,2,\cdots\cdots,n,$

$$m'_i=1/(1-a_{i,i-1}a_{i-1,i}m'_i), \quad \cdots\cdots(23)$$

from which reads

$$\left.\begin{aligned}&m'_1=1,\quad m'_2=1/(1-a_{2,1}a_{1,2}),\quad m'_3=1/(1-a_{3,2}a_{2,3}m'_2),\\&m'_4=1/(1-a_{4,3}a_{3,4}m'_3),\quad m'_5=1/(1-a_{5,4}a_{4,5}m'_4),\\&\cdots\cdots\cdots\cdots;\end{aligned}\right\} \quad \cdots\cdots(24)$$

and

$$z_r=1/(1-a_{r,r-1}a_{r-1,r}m'_{r-1}-a_{r,r+1}a_{r+1,r}m_{r+1}). \quad \cdots\cdots(25)$$

3. Prepare the auxiliary matrix:

$$\begin{pmatrix} z_1, & -\bar{a}_{1,2}, & & & & \\ -\bar{a}_{2,1}, & z_2, & -\bar{a}_{2,3}, & & & \\ & -\bar{a}_{3,2}, & z_3, & -\bar{a}_{3,4}, & & \\ & & \cdots\cdots & & & \\ & & & \cdots\cdots & & \\ & & & -\bar{a}_{n-1,n-2}, & z_{n-1}, & \bar{a}_{n-1,n} \\ & & & & -\bar{a}_{n,n-1}, & z_{n,n}\end{pmatrix}, \quad \cdots\cdots(26)$$

where, referring to Eqs. (14) and (17),

$$\bar{a}_{i,i+1}=a_{i,i+1}m_{i+1}, \quad \bar{a}_{i,i-1}=a_{i,i-1}m'_{i-1}. \quad \cdots\cdots(27)$$

4. By utilizing the matrix (26), construct the matrix:

$$ZR=\begin{pmatrix} +z_1, & -z_1\bar{a}_{1,2}, & +z_1\bar{a}_{1,2}\bar{a}_{2,3}, & -z_1\bar{a}_{1,2}\bar{a}_{2,3}\bar{a}_{3,4}, & +z_1\bar{a}_{1,2}\bar{a}_{2,3}\bar{a}_{3,4}\bar{a}_{4,5}, & \cdots \\ -z_2\bar{a}_{2,1}, & +z_2, & -z_2\bar{a}_{2,3}, & +z_2\bar{a}_{2,3}\bar{a}_{3,4}, & -z_2\bar{a}_{2,3}\bar{a}_{3,4}\bar{a}_{4,5}, & \cdots \\ +z_3\bar{a}_{3,2}\bar{a}_{2,1}, & -z_3\bar{a}_{3,2}, & +z_3, & -z_3\bar{a}_{3,4}, & +z_3\bar{a}_{3,4}\bar{a}_{4,5}, & \cdots \\ -z_4\bar{a}_{4,3}\bar{a}_{3,2}\bar{a}_{2,1}, & +z_4\bar{a}_{4,3}\bar{a}_{3,2}, & -z_4\bar{a}_{4,3}, & +z_4, & -z_4\bar{a}_{4,5}, & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad \cdots\cdots(28)$$

By minutely comparing the right side of Eq. (12) and the matrix (26), we can readily see the preparing process of the matrix (28) from the latter. For examples, the third-row elements of (28) : $-z_3\bar{a}_{3,2}$, $+z_3\bar{a}_{3,2}\bar{a}_{2,1}$, $-z_3\bar{a}_{3,4}$, $+z_3\bar{a}_{3,4}\bar{a}_{4,5}$, $-z_3\bar{a}_{3,4}\bar{a}_{4,5}\bar{a}_{5,6}$, …… are respectively gotten by multiplying z_3 by $(-\bar{a}_{3,2})$ and $-z_3\bar{a}_{3,2}$ by $(-\bar{a}_{2,1})$ in the left direction from the principal diagonal, and z_3 by $(-\bar{a}_{3,4})$, $-z_3\bar{a}_{3,4}$ by $(-\bar{a}_{4,5})$, $+z_3\bar{a}_{3,4}\bar{a}_{4,5}$ by $(-\bar{a}_{5,6})$, …… in the right.

5. Post multiplying (28) by the column matrix h, given by Eq. (3), that is to say, multiplying respectively each column element by h_1, h_2, h_3, ……, and h_n in order and summing up all the elements in the same rows, we lastly obtain the solutions (13).

The Solutions of the Special Case: $a_{1,2}=a_{n,n-1}=c$, and $a_{i,i\pm1}=a$, $i\neq1, n$.

Referring to Eqs. from (21) to (25), we get

for $i>r$,

$$\left.\begin{aligned} &1/m_1=1-cam_2,\\ &1/m_2=1/m_3=\cdots\cdots=1/m_{n-4}=1-a^2/[1-a^2/\{1-a^2/(1-ca)\}],\\ &1/m_{n-3}=1-a^2/\{1-a^2/(1-ca)\}, \qquad 1/m_{n-2}=1-a^2/(1-ca),\\ &1/m_{n-1}=1-ca, \qquad 1/m_n=1; \end{aligned}\right\}\quad\cdots\cdots(29)$$

for $i<r$,

$$\left.\begin{aligned} &1/m'_1=1, \qquad 1/m'_2=1-ca, \qquad 1/m'_3=1-a^2/(1-ca)\\ &1/m'_4=1-a^2/\{1-a^2/(1-ca)\}\\ &1/m'_5=1/m'_6=\cdots\cdots=1/m'_{n-1}=1-a^2/[1-a^2/\{1-a^2/(1-ca)\}]\\ &1/m'_n=1-cam'_{n-1} \end{aligned}\right\}\quad\cdots\cdots(30)$$

and

$$\left.\begin{aligned} &1/z_1=1/z_n=1-cam_2, \qquad 1/z_2=1/z_{n-1}=1/m_2+1/m'_2-1,\\ &1/z_3=1/z_{n-1}=1/m_3+1/m'_3-1,\\ &1/z_4=1/z_5=\cdots\cdots=1/z_{n-4}=1/z_{n-3}=1/m_4+1/m'_4-1. \end{aligned}\right\}\quad\cdots\cdots(31)$$

Now, from Eqs. (13) and (28), gives the solutions.

Examples.

Now, we will practically solve the problems, found in Bibliography, taking the absolute terms as h_i besides $a_{i,i}=1$. All the calculations are so systematically arranged in order just as the matrices that the items, mentioned in Numerical Calculations, can regularly be realized. The integers in the first column show the numbers of the matrices, their decimal fractions those of the rows, and the numbers of the first row at the top those of the columns. The probable errors, which may exist, are due to the calculations, performed by the 20-in. slide rule.

In **Ex. 1**[6], the table of Row No. from (1.1) to (1.17) gives the given equations; that of Row No. from (2.1) to (2.13) m_i, m'_i, and z_r; that of Row No. from (3.1) to

(3.17) the matrix (26); that of Row No. from (4.1) to (4.9) the reciprocation, that is, the solutions of the given equations; that of Row No. (5.1) and (5.9) the two solutions 581.392 M_0^0 and 901.929 M_8^0, calculated from (4.1) and (4.9), respectively; and lastly that of Row No.(6.1) and (6.9) the two solutions, taken out from Bibliography 6), which satisfactorily establish the exactness of (5.1) and (5.9) respectively.

Ex. 1.[6] *

	Column No.	1	2	3	4	5	6	7	8	9	10
Row No.		M_0^0	M_1^0	M_2^0	M_3^0	M_4^0	M_5^0	M_6^0	M_7^0	M_8^0	M_9^0
1.1	h_1 ‡	1	0.5								
.2	h_2	0.2531	1	0.2469							
.3	h_3		0.2528	1	0.2472						
.4	h_4			0.2527	1	0.2473					
.5	h_5				0.2522	1	0.2478				
.6	h_6					0.2517	1	0.2483			
.7	h_7						0.2507	1	0.2493		
.8	h_8							0.2506	1	0.2494	
.9 *	h_9								0.2500	1	0.2500
2.1	$i > r$										
.2	$a_{i,i+1}\,a_{i+1,i}$	0.1266	0.0624	0.0625	0.0624	0.0624	0.0623	0.0625	0.0624	0.0624	0.0625
.3	$\overline{a}_{i,i+1}\,a_{i+1,i}$	0.1356	0.0669	0.0670	0.0669	0.0669	0.0668	0.0670	0.0669	0.0669	0.0670
.4	$1/m_i$	0.8644	0.9331	0.9330	0.9331	0.9331	0.9332	0.9330	0.9331	0.9331	0.9330
.5	m_i	1.1569	1.0717,	1.0718	1.0717	1.0717	1.0716	1.0718	1.0717	1.0717	1.0718
.6	$i < r$										
.7	$a_{i,i-1}\,a_{i-1,i}$	0	0.1266	0.0624	0.0625	0.0624	0.0624	0.0623	0.0625	0.0624	0.0624
.8	$\overline{a}_{i,i-1}\,a_{i-1,i}$	0	0.1266	0.0714	0.0673	0.0669	0.0669	0.0668	0.0670	0.0669	0.0669
.9	$1/m'_i$	1	0.8734	0.9286	0.9327	0.9331	0.9331	0.9332	0.9330	0.9331	0.9331
.10	m'_i	1	1.1449	1.0769	1.0722	1.0717	1.0717	1.0716	1.0718	1.0717	1.0717
.11	$i = r$										
.12	$1/z_r$	0.8644	0.8065	0.8616	0.8658	0.8662	0.8663	0.8662	0.8661	0.8662	0.8661
.13	z_r	1.1569	1.2400	1.1606	1.1550	1.1544	1.1543	1.1544	1.1545	1.1544	1.1545
		h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9 *	h
3.1	※	+1.1569	−0.5359								
.2		−0.2531	+1.2400	−0.2646							
.3			−0.2894	+1.1606	−0.2649						
.4				−0.27[illegible]	+1.1550	−0.2650					
.5					−0.2704	+1.1544	−0.2655				
.6						−0.2697	+1.1543	−0.2661			
.7							−0.2687	+1.1544	−0.2672		
.8								−0.2685	+1.1545	−0.2673	
.9 *									−0.2680	+1.1544	−0.2680
4.1	$M_0^0 =$ ♀	+1.1568	−0.6200	+0.1640	−0.0435	+0.0115	−0.0031	+0.0008	−0.0002		
.2	$M_1^0 =$	−0.3138	+1.2400	−0.3281	+0.0869	−0.0230	+0.0061	−0.0016	+0.0004	−0.0001	
.3	$M_2^0 =$	+0.0850	−0.3359	+1.1606	−0.3074	+0.0815	−0.0216	+0.0058	−0.0015	+0.0004	−0.0001
.4	$M_3^0 =$	−0.0230	+0.0909	−0.3143	+1.1550	−0.3061	+0.0813	−0.0216	+0.0058	−0.0015	+0.0004
.5	$M_4^0 =$	+0.0062	−0.0246	+0.0850	−0.3122	+1.1544	−0.3065	+0.0816	−0.0218	+0.0058	−0.0016
.6	$M_5^0 =$	−0.0017	+0.0066	−0.0229	+0.0842	−0.3113	+1.1543	−0.3072	+0.0821	−0.0219	+0.0059
.7	$M_6^0 =$	+0.0005	−0.0018	+0.0062	−0.0226	+0.0837	−0.3192	+1.1544	−0.3085	+0.0825	−0.0221
.8	$M_7^0 =$	−0.0001	+0.0005	−0.0017	+0.0061	−0.0225	+0.0833	−0.3100	+1.1545	−0.3084	+0.0827
.9 *	$M_8^0 =$		−0.0001	+0.0004	−0.0016	+0.0060	−0.0223	+0.0831	−0.3094	+1.1544	−0.3094
.10	$M_9^0 =$			−0.0001	+0.0004	−0.0016	+0.0060	−0.0222	+0.0827	−0.3086	+1.1545
.11	$M_{10}^0 =$				−0.0001	−0.0004	−0.0016	+0.0059	−0.0221	+0.0825	−0.3085
.12	$M_{11}^0 =$					−0.0001	+0.0004	−0.0016	+0.0059	−0.0219	+0.0821
.13	$M_{12}^0 =$						−0.0001	+0.0004	−0.0016	+0.0058	−0.0218
.14	$M_{13}^0 =$							−0.0001	+0.0004	−0.0015	+0.0058
.15	$M_{14}^0 =$								−0.0001	+0.0004	−0.0015
.16	$M_{15}^0 =$									−0.0001	+0.0004
		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9 *	a_{10}
5.1	581.392M_0^0 =	+1.1569	−0.3138	+0.0850	−0.0230	+0.0062	−0.0017	+0.0004	−0.0001		
5.9	901.929M_8^0 =		−0.0001	+0.0003	−0.0013	+0.0050	−0.0190	+0.0715	−0.2675	+1	−0.2675
6.1	581.392M_0^0 =⊕	+1.15691	−0.31381	+0.08504	−0.02303	+0.00622	−0.00168	+0.00044	−0.00012	+0.00003	−0.00001
6.9	901.929M_8^0 =	+0.00005	−0.00010	+0.00036	−0.00134	+0.00504	−0.01900	+0.07144	−0.26748	+1	−0.26748

* For simplicity, only the left half of the table is shown from it center symmetry.
⊕ Five decimal places ane taken besides H = 0 for the coefficients in Bibliography 6).
‡ Eq. (1). ※ (26). ♀ Eqs. (13) or (28)·h.

From **Ex. 2** and **3**, we can readily get the respective solutions, when the proper values of n are inserted into them. If $n \geqq 12$ exists, the solutions are practically exact[11) 14)], and if $4 \leqq n < 11$, they are approximate, whose accuracies are dependent on the magnitude of n. Furthermore, if the coefficients of the given equations are very close to those of **Ex. 2** and **3**, their approximate solutions can also de obtained from them, when n is properly taken. Thus, we can simply get the approximate solutions of **Ex. 1**, if $n=17$ is inserted into **Ex. 2.** The process of getting the solutions is as follows.

Generally, if a set of n linear simultaneous equations are given, take out the elements, situated in the first $n/2$ rows and the first n columns, from the table, and rotate them around the center of the elements of the first n rows and the first n columns by 180 degrees, then lastly we get the solutions of x_1, x_2, ... and x_n.

Ex. 2. Given $a_{i,i}=1$, $a_{i,i\pm1}=0.25$ except $a_{1,2}=a_{n,n-1}=0.5$, and $10<n$, find the solutions

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}
$x_1=$	+1.155	−0.619	+0.166	−0.044	+0.012	−0.003	+0.001								
$x_2=$	−0.309	+1.238	−0.332	+0.089	−0.024	+0.006	−0.002								
$x_3=$	+0.089	−0.332	+1.161	−0.311	+0.083	−0.022	+0.006	−0.002							
$x_4=$	−0.022	+0.089	−0.311	+1.155	−0.310	+0.083	−0.022	+0.006	−0.002						
$x_5=$	+0.006	−0.024	+0.083	−0.310	+1.155	−0.310	+0.083	−0.022	+0.006	−0.002					
$x_6=$	−0.002	+0.006	−0.022	+0.083	−0.310	+1.155	−0.310	+0.083	−0.022	+0.006	−0.002				
$x_7=$		−0.002	+0.006	−0.022	+0.083	−0.310	+1.155	−0.310	+0.083	−0.022	+0.006	+0.002			
$x_8=$			−0.002	+0.006	−0.022	+0.083	−0.310	+1.155	−0.310	+0.083	−0.022	−0.006	−0.002		
$x_9=$				−0.002	+0.006	−0.022	+0.083	−0.310	+1.155	−0.310	+0.083	+0.022	+0.006	−0.002	
.	·······(The same coefficients are in succession) ·······················														
.	··														
. ※	···														

※ For simplicity, only the first half of the equations are shown from its center symmstry.

Ex. 3 Given $a_{i,i}=1$, $a_{i,i\pm1}=0.25$, and $10<n$, find the solutions

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}	h_{13}	h_{14}	h_{15}
$x_1=$	+1.072	−0.287	+0.077	−0.021	+0.006	−0.002									
$x_2=$	−0.287	+1.149	−0.308	+0.083	−0.022	+0.006	−0.002								
$x_3=$	+0.077	−0.308	+1.154	−0.309	+0.083	−0.022	+0.006	−0.002							
$x_4=$	−0.021	+0.083	−0.309	+1.154	−0.309	+0.083	−0.022	+0.006	−0.002						
$x_5=$	+0.006	−0.022	+0.083	−0.309	+1.154	−0.309	+0.083	−0.022	+0.006	−0.002					
$x_6=$	−0.002	+0.006	−0.022	+0.083	−0.309	+1.154	−0.309	+0.083	−0.022	+0.006	−0.002				
$x_7=$		−0.002	+0.006	−0.022	+0.083	−0.309	+1.154	−0.309	+0.083	−0.022	+0.006	−0.002			
$x_8=$			−0.002	+0.006	−0.022	+0.083	−0.309	+1.154	−0.309	+0.083	−0.022	+0.006	−0.002		
$x_9=$				−0.002	+0.006	−0.022	+0.083	−0.309	+1.154	−0.309	+0.083	−0.022	+0.006	−0.002	
.	·····(The same coefficients are in succession) ·························														
.	··														
. ※	···														

※ For simplicity, only the first half of the equations are shown from its center symmstry.

For examples, if $n=4$ is inserted into **Ex. 2** and **3** respectively, we get

$$\left.\begin{aligned} x_1&=+1.155\,h_1-0.619\,h_2+0.166\,h_3-0.044\,h_4\\ x_2&=-0.309\,h_1+1.238\,h_2-0.332\,h_3+0.089\,h_4\\ x_3&=+0.089\,h_1-0.332\,h_2+1.238\,h_3-0.309\,h_4\\ x_4&=-0.044\,h_1+0.166\,h_2-0.619\,h_3+1.155\,h_4 \end{aligned}\right\} \qquad \cdots\cdots(32)$$

and

$$\left.\begin{aligned} x_1&=+1.072\,h_1-0.287\,h_2+0.077\,h_3-0.021\,h_4\\ x_2&=-0.287\,h_1+1.149\,h_2-0.308\,h_3+0.083\,h_4\\ x_3&=+0.083\,h_1-0.308\,h_2+1.149\,h_3-0.287\,h_4\\ x_4&=-0.021\,h_1+0.077\,h_2-0.287\,h_3+1.072\,h_4 \end{aligned}\right\} \qquad \cdots\cdots(33)$$

respectively, while the true solutions[14)11)]:

$$\left.\begin{aligned} x_1&=+1.155\,h_1-0.622\,h_2+0.178\,h_3-0.044\,h_4\\ x_2&=-0.311\,h_1+1.244\,h_2-0.356\,h_3+0.089\,h_4\\ x_3&=+0.089\,h_1-0.356\,h_2+1.244\,h_3-0.311\,h_4\\ x_4&=-0.044\,h_1+0.178\,h_2-0.622\,h_3+1.155\,h_4 \end{aligned}\right\} \qquad \cdots\cdots(34)$$

and

$$\left.\begin{aligned} x_1&=+1.072\,h_1-0.287\,h_2+0.077\,h_3-0.019\,h_4\\ x_2&=-0.287\,h_1+1.148\,h_2-0.306\,h_3+0.077\,h_4\\ x_3&=+0.077\,h_1-0.306\,h_2+1.148\,h_3-0.287\,h_4\\ x_4&=-0.019\,h_1+0.077\,h_2-0.287\,h_3+1.072\,h_4 \end{aligned}\right\} \qquad \cdots\cdots(35)$$

establish the effectiveness of the former, even when $n=4$ respectively.

Conclusion

1. The exactness, simplicity, and speediness of the new method can satisfactorily be verified.

2. The effectiveness of the results of **Ex.2, Ex. 3**, and the special case $a_{1,2}=a_{n,n-1}=c$ and $a_{i,i\pm1}=a$, $i\neq1, n$, can also be proved.

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