応用力学
偏微分方程式の階差解法 (II)

DIFFERENCE METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS. Part II.

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THIS article gives results derived from the interpolation formula of Stirling type for the function $f(x,y)$. Difference equations here treated are confined to the harmonic and biharmonic equations, the results of which will include those obtained from Collatz-Hidaka's method[2]. Boundary conditions, which are frequently given by linear combination of differential coefficients of the unknown function $f(x,y)$, are considered along a straight line parallel to a coordinate axis, consisting of a low of lattice points. As simpe applications of the results obtained, the torsion-problem of a square cylinder and the vibration of a clamped square plate were treated, and both resulted in fair agreements with their exact values in spite of a little amount of labour.

1. Interpolation formula. The interpolation formula of Stirling type for the function $f(x,y)$ is written in the form

$$\begin{aligned} f(x,y) = \sum_{r+s=0}^{n} \sum_{s=0}^{r+r} \Big[& \varphi(u,r)\varphi(v,s)\Delta^{2r+2s}_{2rx\,2sy}(\overline{r}\,\overline{s}) \\ & + \phi(u,r)\varphi(v,s)\frac{1}{2}\Delta^{2r+2s+1}_{(2r+1)x\,2sy}\left\{(\overline{r+1}\,\overline{s}) + (\overline{r}\,\overline{s})\right\} \\ & + \varphi(u,r)\phi\psi(v,s)\frac{1}{2}\Delta^{2r+2s+1}_{2rx\,(2s+1)y}\left\{(\overline{r}\,\overline{s+1}) + (\overline{r}\,\overline{s})\right\} \\ & + \phi(u,r)\phi(v,s)\frac{1}{4}\Delta^{2r+2r+2}_{(2r+1)x\,(2s+1)y}\left\{(\overline{r+1}\,\overline{s+1}) + (\overline{r+1}\,\overline{s}) + (\overline{r}\,\overline{s+1}) + (\overline{r}\,\overline{s})\right\}\Big] \end{aligned} \qquad (1)$$

where

$$\left.\begin{aligned} & u = \frac{x-x_0}{h}, \quad v = \frac{y-y_0}{k}, \quad (\overline{r}\,\overline{s}) = f(x_0 - rh,\ y_0 - sk), \\ & \varphi(\theta,\nu) = \frac{1}{\underline{|2\nu}}\theta^2(\theta^2-1)(\theta^2-4)\cdots(\theta^2-\overline{\nu-1}^2), \\ & \phi(\theta,\nu) = \frac{1}{\underline{|2\nu+1}}\theta(\theta^2-1)(\theta^2-4)\cdots(\theta^2-\nu^2), \\ & \Delta^{a+b}_{ax\,by}(\overline{r}\,\overline{s}) = \sum_{\lambda=0}^{a}\sum_{u=0}^{b} \frac{(-)^{\lambda+\mu}\underline{|a}\,\underline{|b}}{\underline{|a-\lambda}\,\underline{|b-\mu}\,\underline{|\lambda}\,\underline{|\mu}}(\overline{r-a+\lambda}\ \overline{s-b+\mu}) \end{aligned}\right\} \cdots\cdots\cdots(2)$$

It will be convenient to use abbreviations such as

$$\left.\begin{aligned} & \Delta^{2r+2s}_{2rx\,2sy}(\overline{r}\,\overline{s}) = \Delta_{2r\ 2s}, \quad \Delta^{2r+2s+2}_{(2r+1)x\,(2s+1)y}\left\{(\overline{r+1}\,\overline{s+1}) + (\overline{r+1}\,\overline{s}) + (\overline{r}\,\overline{s+1}) + (\overline{r}\,\overline{s})\right\} \\ & \qquad = \Delta_{2r+1\ 2s+1}, \\ & \Delta^{2r+2s+1}_{(2r+1)x\,2sy}\left\{(\overline{r+1}\,\overline{s}) + (\overline{r}\,\overline{s})\right\} = \Delta^{2r+12s}, \quad \Delta^{2r+2s+1}_{2rx\,(2s+1)y}\left\{(\overline{r}\,\overline{s+1}) + (\overline{r}\,\overline{s})\right\} = \Delta_{2r\ 2s+1}, \end{aligned}\right\} \cdots\cdots(3)$$

when no confusion takes place.

Explanatory form of the above interpolation formula (1) is thus:

$$f(x,y) = (00) + \frac{u}{\underline{|1}}\frac{1}{2}\Delta_x\left\{(\overline{1}0) + (00)\right\} + \frac{v}{\underline{|1}}\frac{1}{2}\Delta_y\left\{(0\overline{1}) + (00)\right\}$$

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$$+\frac{u^2}{\underline{|2}}\Delta^2_{2x}(10)+\frac{u}{\underline{|1}}\frac{v}{\underline{|1}}\frac{1}{4}\Delta^2_{x\,y}\left\{(\bar{1}\bar{1})+(\bar{1}0)+(0\bar{1})+(00)\right\}+\frac{v^2}{\underline{|2}}\Delta^2_{2y}(0\bar{1})$$

$$+\frac{u(u^2-1)}{\underline{|3}}\frac{1}{2}\Delta^3_{3x}\left\{(\bar{2}0)+(\bar{1}0)\right\}+\frac{u^2}{\underline{|2}}\frac{v}{\underline{|1}}\frac{1}{2}\Delta^3_{2x\,y}\left\{(\bar{1}\bar{1})+(\bar{1}0)\right\}$$

$$+\frac{u}{\underline{|1}}\frac{v^2}{\underline{|2}}\frac{1}{2}\Delta^3_{x\,2y}\left\{(\bar{1}\bar{1})+(0\bar{1})\right\}+\frac{v(v^2-1)}{\underline{|3}}\frac{1}{2}\Delta^3_{3y}\left\{(0\bar{2})+(0\bar{1})\right\}$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (4)$$

For practical use, it will be convenient to convert differences in (1) to linear combinations of lattice point values of the function $f(x,y)$, as illustrated below[3] **Table 1.**

Table 1. Difference table for Stirling's formula.

Δ_{00}

1

Δ_{10}

-1	0	1

Δ_{20}

1	-2	1

Δ_{01}

1
0
-1

Δ_{02}

1
-2
1

Δ_{30}

-1	2	0	-2	1

Δ_{40}

1	-4	6	-4	1

Δ_{11}

-1	0	1
0	0	0
1	0	-1

Δ_{21}

1	-2	1
0	0	0
-1	2	-1

Δ_{12}

-1	0	1
2	0	-2
-1	0	1

Δ_{22}

1	-2	1
-2	4	-2
1	-2	1

Δ_{03}

1
-2
0
2
-1

Δ_{04}

1
-4
6
-4
1

Δ_{31}

-1	2	0	-2	1
0	0	0	0	0
1	-2	0	2	-1

Δ_{41}

1	-4	6	-4	1
0	0	0	0	0
-1	4	-6	4	-1

Δ_{32}

-1	2	0	-2	1
2	-4	0	4	-2
-1	2	0	-2	1

Δ_{42}

1	-4	6	-4	1
-2	8	-12	8	-2
1	-4	6	-4	1

Δ_{05}

1
-4
5
0
-5
4
-1

Δ_{06}

1
-6
15
-20
15
-6
1

Δ_{50}

-1	4	-5	0	5	-4	1

Δ_{60}

1	-6	15	-20	15	-6	1

Δ_{13}

-1	0	1
2	0	-2
0	0	0
-2	0	2
1	0	-1

Δ_{23}

1	-2	1
-2	4	-2
0	0	0
2	-4	2
-1	2	-1

Δ_{14}

-1	0	1
4	0	-4
-6	0	6
4	0	-4
-1	0	1

Δ_{24}

1	-2	1
-4	8	-4
6	-12	6
-4	8	-4
1	-2	1

Δ_{33}

-1	2	0	-2	1
2	-4	0	4	-2
0	0	0	0	0
-2	4	0	-4	2
1	-2	0	2	-1

Δ_{43}

1	-4	6	-4	1
-2	8	-12	8	-2
0	0	0	0	0
2	-8	12	-8	2
-1	4	-6	4	-1

Δ_{34}

-1	2	0	-2	1
4	-8	0	8	-4
-6	12	0	-12	6
4	-8	0	8	-4
-1	2	0	-2	1

Δ_{44}

1	-4	6	-4	1
-4	16	-24	16	-4
6	-24	36	-24	6
-4	16	-24	16	-4
1	-4	6	-4	1

Here, for instance, Δ_{21} denotes

$$\Delta_{21}=\Delta^3_{2x\,y}\left\{(\bar{1}\bar{1})+(\bar{1}0)\right\}=(11)-2(01)+(\bar{1}1)-(1\bar{1})+2(0\bar{1})-(\bar{1}\bar{1}).$$

2. Differential coefficients. For convenience sake, differential coefficients are considered at the origin of interpolation coordinates where $u=0$ and $v=0$. We then have, from (1),

$$\frac{\partial f}{\partial x}=\frac{1}{2h}\sum_{r=0}^{n}\frac{(-)^r(\underline{|r})^2}{\underline{|2r+1}}\Delta^{2r+1}_{(2r+1)x}\left\{(\overline{r+1}\,0)+(\bar{r}\,0)\right\},$$

$$\frac{\partial f}{\partial y}=\frac{1}{2k}\sum_{r=0}^{n}\frac{(-)^r(\underline{|r})^2}{\underline{|2r+1}}\Delta^{2r+1}_{(2r+1)y}\left\{(0\,\overline{r+1})+(0\,\bar{r})\right\},$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\underline{|2}}{h^2}\sum_{r=0}^{n}\frac{(-)^r(\underline{|r})^2}{\underline{|2r+2}}\Delta^{2r+2}_{(2r+2)x}(\overline{r+1}\,0),$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{1}{4hk}\sum_{r+s=0}^{u}\sum_{s=0}^{r+s}\frac{(-)^{r+s}(\underline{|r}\,\underline{|s})^2}{\underline{|2r+1}\,\underline{|2s+1}}\Delta^{2r+2s+2}_{(2r+1)x(2s+1)y}\left\{(\overline{r+1}\,\overline{s+1})+(\overline{r+1}\,\bar{s})\right.$$

$$\left.+(\bar{r}\,\overline{s+1})+(\bar{r}\,\bar{s})\right\},$$

$$\left.\begin{aligned}
\frac{\partial^2 f}{\partial y^2}&=\frac{\underline{|2}}{k^2}\sum_{r=0}^{n}\frac{(-)^r(\underline{|r})^2}{\underline{|2r+2}}\varDelta^{2r+2}_{(2r+2)y}(0\;\overline{r+1}),\\
\frac{\partial^3 f}{\partial x^3}&=\frac{\underline{|3}}{2h^3}\sum_{r=0}^{n}\frac{(-)^r(\underline{|r+1})^2}{\underline{|2r+3}}\left\{\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(r+1)^2}\right\}\varDelta^{2r+3}_{(2r+3)x}\left\{(\overline{r+2}\,0)+(\overline{r+1}\,0)\right\},\\
\frac{\partial^3 f}{\partial x^2\partial y}&=\frac{1}{h^2k}\sum_{r+s=0}^{n}\sum_{s=0}^{r+s}\frac{(-)^{r+s}(\underline{|r}\,\underline{|s})^2}{\underline{|2r+2}\,\underline{|2s+1}}\varDelta^{2r+2s+3}_{(2r+2)x\,(2s+1)y}\left\{(\overline{r+1}\;\overline{s+1})+(\overline{r+1}\;\overline{s})\right\}\\
\frac{\partial^3 f}{\partial x\partial y^2}&=\frac{1}{hk^2}\sum_{r+s=0}^{n}\sum_{s=0}^{r+s}\frac{(-)^{r+s}(\underline{|r}\,\underline{|s})^2}{\underline{|2r+1}\,\underline{|2s+2}}\varDelta^{2r+2s+3}_{(2r+1)x(2s+2)y}\left\{(\overline{r+1}\;\overline{s+1})+(\overline{r}\;\overline{s+1})\right\},\\
\frac{\partial^3 f}{\partial y^3}&=\frac{\underline{|3}}{2k^3}\sum_{r=0}^{n}\frac{(-)^r(\underline{|r+1})^2}{\underline{|2r+3}}\left\{\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(r+1)^2}\right\}\varDelta^{2r+3}_{(2r+3)y}\left\{(0\;\overline{r+2})+(0\;\overline{r+1})\right\},\\
\frac{\partial^4 f}{\partial x^4}&=\frac{\underline{|4}}{h^4}\sum_{r=0}^{n}\frac{(-)^r(\underline{|r+1})^2}{\underline{|2r+4}}\left\{\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(r+1)^2}\right\}\varDelta^{2r+4}_{(2r+4)x}(\overline{r+2}\;0),\\
\frac{\partial^4 f}{\partial x^2\partial y^2}&=\frac{\underline{|2}\,\underline{|2}}{h^2k^2}\sum_{r+s=0}^{n}\sum_{s=0}^{r+s}\frac{(-)^{r+s}(\underline{|r}\,\underline{|s})^2}{\underline{|2r+2}\,\underline{|2s+2}}\varDelta^{2r+2s+4}_{(2r+2)x\,(2s+2)y}(\overline{r+1}\;\overline{s+1}),\\
\frac{\partial^4 f}{\partial y^4}&=\frac{\underline{|4}}{k^4}\sum_{r=0}^{n}\frac{(-)^r(\underline{|r+1})^2}{\underline{|2+4}}\left\{\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(r+1)^2}\right\}\varDelta^{2r+4}_{(2r+4)y}(0\;\overline{r+2}).
\end{aligned}\right\}\cdots(5)$$

Differential coefficients of higher order may also be obtained if required.

The above expressions are also useful for curvilinear coordinates. For instance in the case of symmetrical strain of a solid of revolution, its coordinates being (ρ, z), we have for the differential coefficient $\dfrac{\partial}{\partial\rho}\left(\dfrac{1}{\rho}\dfrac{\partial f}{\partial\rho}\right)$

$$\left[\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial f}{\partial\rho}\right)\right]_{\substack{\rho=\rho_0\\ z=z_0}}=\frac{2}{h^2\rho_0}\sum_{r=0}^{n}\frac{(-)^r(\underline{|r})^2}{\underline{|2r+2}}\varDelta^{2r+2}_{(2r+2)\rho}(\overline{r+1}\;0)$$
$$-\frac{1}{2h\rho_0^2}\sum_{r=0}^{n}\frac{(-)^r(\underline{|r})^2}{\underline{|2r+1}}\varDelta^{2r+1}_{(2r+1)\rho}\left\{(\overline{r+1}\,0)+(\overline{r}\,0)\right\}.$$

3. Differential equations. Partial differential equations here given are those of harmonic and biharmonic types. In what follows we for convenience sake take $h=k$.

Then we have for the harmonic expression

$$\nabla^2 f=\frac{2}{h^2}\sum_{r=0}^{n}\frac{(-)^r(\underline{|r})^2}{\underline{|2r+2}}\left\{\varDelta^{2r+2}_{(2r+2)x}(\overline{r+1}\;0)+\varDelta^{2r+2}_{(2r+2)y}(0\;\overline{r+1})\right\}.\quad\cdots\cdots\cdots(6)$$

With the aid of difference table, **Table 1,** we develop differences into lattice point values, and then we have for respective values of n the following schematic expressions **Figs. 1–7.**

Fig. 1.
First approximation ($n=1$).

	1	
1	-4	1
	1	

$=h^2\nabla^2 f+O(\varDelta^4)$

Fig. 2.
Second approximation ($n=2$).

		-1		
	0	16	0	
-1	16	-60	16	-1
	0	16	0	
		-1		

$=12h^2\nabla^2 f+O(\varDelta^6)$

Fig. 3.
Third approximation ($n=3$).
(First quadrant only.)

2			
-27	0		
270	0	0	
-980	270	-27	2

$=180h^2\nabla^2 f+O(\varDelta^8)$

Fig. 4.
Fourth approximation ($n=4$).
(First quadrant only.)

-9				
128	0			
-1 008	0	0		
8 064	0	0	0	
-28 700	8 064	-1 008	128	-9

$= 5\,040 h^2 \nabla^2 f + O(\Delta^{10})$

Fig. 5.
Fifth approximation ($n=5$).
(First quadrant only.)

8					
-125	0				
1 000	0	0			
-6 000	0	0	0		
42 000	0	0	0	0	
-147 532	42 000	-6 000	1 000	-125	8

$= 25\,200 h^2 \nabla^2 f + O(\Delta^{12})$

Fig. 6.
Sixth approximation ($n=6$).
(First quadrant only.)

-50						
864	0					
-7 425	0	0				
44 000	0	0	0			
-222 750	0	0	0	0		
1 425 600	0	0	0	0	0	
-4 960 956	1 425 600	-222 750	44 000	-7 425	864	-50

$= 831\,600 h^2 \nabla^2 f + O(\Delta^{14})$

Fig. 7.
Seventh approximation ($n=7$).
(First quadrant only.)

900							
-17 150	0						
160 524	0	0					
-1 003 275	0	0	0				
4 904 900	0	0	0	0			
-22 072 050	0	0	0	0	0		
132 432 300	0	0	0	0	0	0	
-457 624 596	132 432 300	-22 072 050	4 904 900	-1 003 275	160 524	-17 150	900

$= 75\,675\,600 h^2 \nabla^2 f + O(\Delta^{16})$

As to the last five approximations to $\nabla^2 f$ (**Figs. 3–7**), weights in the first quadrant only are given, since those in the remaining quadrants can be written down by symmetry.

In the same way we can obtain difference equations for the biharmonic expression $\nabla^4 f$ for respective values of n as follows (**Figs. 8–14**).

Fig. 8.
First approximation ($n=1$).

		1		
	2	-8	2	
1	-8	20	-8	1
	2	-8	2	
		1		

$= h^4 \nabla^4 f + O(\Delta^6)$

Fig. 9.
Second approximation ($n=2$).

			-1			
		-1	14	-1		
	-1	20	-77	20	-1	
-1	14	-77	184	-77	14	-1
	-1	20	-77	20	-1	
		-1	14	-1		
			-1			

$= 6 h^4 \nabla^4 f + O(\Delta^8)$

Fig. 10.
Third approximation ($n=3$).
(First quadrant only)

21				
-320	16			
2 520	-256	10		
-11 456	3 040	-256	16	
25 660	-11 456	2 520	-320	21

$= 720 h^4 \nabla^4 f + O(\Delta^{10})$

Fig. 11. Fourth approximation ($n=4$).
(First quadrant only)

-82					
1 369	-54				
-11 442	880	-28			
67 352	-7 980	546	-28		
-272 384	73 248	-7 980	880	-54	
623 028	-272 384	67 352	-11 442	1 369	-82

$$= 15\,120 h^4 \nabla^4 f + O(\Delta^{12})$$

Fig. 12. Fifth approximation ($n=5$).
(First quadrant only.)

5 269						
-96 624	3 168					
871 794	-55 440	1 485				
-5 316 080	498 960	-28 512	1 232			
26 095 905	-3 373 920	307 692	-28 512	1 485		
-97 473 024	26 444 880	-3 373 920	498 960	-55 440	3 168	
220 269 896	-97 473 024	26 095 905	-5 316 080	871 794	-96 624	5 269

$$= 4\,989\,600 h^4 \nabla^4 f + O(\Delta^{14})$$

Fig. 13. Sixth approximation ($n=6$)
(First quadrant only ; weights being rounded off.)

-213							
4 276	-120						
-41 937	2 290	-53					
272 038	-21 759	1 065	-40				
-1 354 810	142 963	-11 111	882	-40			
5 865 536	-802 222	87 699	-11 111	1 065	-53		
-20 682 103	5 646 032	-802 222	142 963	-21 759	2 290	-120	
45 322 310	-20 682 103	5 865 536	-1 354 810	272 038	-41 937	4 276	-213

$$= 10^6 h^4 \nabla^4 f + O(\Delta^{16})$$

Fig. 14. Seventh approximation ($n=7$).
(First quadrant only ; weights being rounded off.)

44								
-965	24							
10 288	-493	10						
-71 797	5 041	-216	7					
375 351	-34 478	2 328	-162	6				
-1 619 765	184 691	-17 495	1 925	-162	7			
6 392 646	-909 206	112 778	-17 495	2 328	-216	10		
-21 576 477	5 916 826	-909 206	184 691	-34 478	5 041	-493	24	
47 995 161	-21 576 477	6 392 646	-1 619 765	375 351	-71 797	10 288	-965	44

$$= 10^6 h^4 \nabla^4 f + O(\Delta^{18})$$

It is noted here that the first four approximations (**Figs. 8-11**) have been given by Prof. Hidaka otherwise[2], and so we may infer that the known Collatz-Hidaka's method for differentiation is nothing but a special case derivable from our interpolation method. The process of the former method seems to be troublesome, whilst the latter one can afford the same results with much less labour, in which systematic procedure of calculation is available.

The above difference equations might at times be improved by taking certain higher differences into account. For instance, the above second approximation (**Fig. 9**) can be improved by adding the term in Δ_{44}. The result becomes thus **Fig. 15.**

Fig. 15. **'Improved' second approximation to $\nabla^4 f$.**

			-12			
	1	-16	174	-16	1	
	-16	256	-948	256	-16	
-12	174	-948	2244	-948	174	-12
	-16	256	-948	256	-16	
	1	-16	174	-16	1	
			-12			

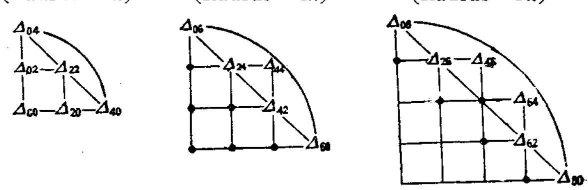
$= 72h^4\nabla^4 f + O(3h)$

The idea of forming such difference equations comes from the following consideration. Difference equation at the origin where $u=0$ and $v=0$ is influenced by surrounding lattice points. According as a lattice point is situated at a remote distance from the origin, the effect of the lattice to the difference equation will become faint. In this connection, we may consider a cricle for respective difference equations. Lattice points within the circle have to be taken into account, and those beyond the circle are of no importance for the difference equation of the approximation considered. We call this circle 'influence-circle'. Influence-circles for respective approximation are given in the following **Fig. 16.**

An attempt was made to use such difference equations in treating an eigenvalue problem, that is the vibration of a square plate with four clamped edges. The results obtained confirmed a certain degree of effectiveness of such difference equations, compared with those of ordinary type. This is only a trial and much more extensive work is necessary to see the true utility of such equations.

Fig. 16. 'Influence-circles' for Stirling's formula

16a. First approximation (Radius$=2h$)　16b. Second approximation (Radius$=3h$)　16c. Third approximation (Radius$=4h$)



4. **Boundary conditions.** Boundary conditions are frequently given by linear combination of differential coefficients on boundary curves. When these curves are of straight line parallel to a coordinate axis and the boundary conditions can be expanded into power series of one variable, criteria for expressing boundary conditions will be obtained in elegant forms.

a. We first take the boundary condition

$$[f(x,y)]_{x=x_0}=c_0+c_1y+c_2y^2 \quad (c_0,\ c_1,\ c_2 \text{ being constants}) \quad \cdots\cdots\cdots\cdots (7)$$

along a straight line parallel to the axis of y. We have, from the interpolation formula (1),

$$[f(x,y)]_{x=x_0}=\Delta_{00}+\frac{v}{\underline{|1}}\frac{1}{2}\Delta_{01}+\frac{v^2}{\underline{|2}}\Delta_{02}+\frac{v(v^2-1)}{\underline{|3}}\frac{1}{2}\Delta_{03}+\frac{v^2(v^2-1)}{\underline{|4}}\Delta_{04}$$
$$+\frac{v(v^2-1)(v^2-4)}{\underline{|5}}\Delta_{05}+\ \cdots\cdots\cdots,\quad \cdots\cdots\cdots\cdots (8)$$

To satisfy the boundary condition (7), we may put

$$\Delta_{00}+\frac{v}{\lfloor 1}\frac{1}{2}\Delta_{01}+\frac{v^2}{\lfloor 2}\Delta_{02}=c_0+c_1y+c_2y^2,\ \Delta_{03}=0,\ \Delta_{04}=0,\ \Delta_{05}=0,\ \Delta_{06}=0,\ \cdots\cdots \quad (9)$$

This system of equations will afford a criterion, which is written in the form

$$(0\ s)=c_0+c_1(sk+y_0)+c_2(sk+y_0)^2, \quad \cdots\cdots (10)$$

where $(0\,s)$ denotes a boundary lattice point, and $s=0, \pm1, \pm2, \cdots\cdots$. The above criterion (10) is illustrated in the following **Fig. 17**.

Fig. 17. Boundary condition $[f(x,y)]_{x=x_0}=c_0+c_1y+c_2y^2$.

Inner lattice points	Boundary lattice points		Outer lattice points
$(\bar{1}3)$	$(03)=$	$C_0+(3k+y_0)C_1+(3k+y_0)^2C_2$	(13)
$(\bar{1}2)$	$(02)=$	$C_0+(2k+y_0)C_1+(2k+y_0)^2C_2$	(12)
$(\bar{1}1)$	$(01)=$	$C_0+(k+y_0)C_1+(k+y_0)^2C_2$	(11)
$(\bar{1}0)$	$(00)=$	$C_0+y_0C_1+y_0^2C_2$	(10)
$(\bar{1}\bar{1})$	$(01)=$	$C_0+(-k+y_0)C_1+(-k+y_0)^2C_2$	$(1\bar{1})$
$(\bar{1}\bar{2})$	$(02)=$	$C_0+(-2k+y_0)C_1+(-2k+y_0)^2C_2$	$(1\bar{2})$
$(\bar{1}\bar{3})$	$(03)=$	$C_0+(-3k+y_0)C_1+(-3k+y_0)^2C_2$	$(1\bar{3})$

Boundary line parallel to y-axis

Furthermore it might be inferred that, when boundary condition is in general of the form

$$[f(x,y)]_{x=x_0}=c_0+c_1y+c_2y^2+\cdots\cdots=\sum_{m=0}^{\infty}c_my^m, \quad \cdots\cdots (11)$$

the corresponding criterion will become

$$(0\,s)=c_0+c_1(sk+y_0)+c_2(sk+y_0)^2+\cdots\cdots=\sum_{m=0}^{\infty}c_m(sk+y_0)^m. \quad \cdots\cdots (12)$$

b. In this way we can obtain **Table 2**, which affords criteria for respective bound a-ry conditions:

Table 2. Criteria for boundary conditions.

No.	*Boundary condition*	*Corresponding criterion*
1.	$[f(x,y)]_{x=x_0}=c_0+c_1y+c_2y^2+\cdots$	$(0\,s)=c_0+c_1(sk+y_0)+c_2(sk+y_0)^2+\cdots\cdots$
2.	$\left(\frac{\partial f}{\partial x}\right)_{x=x_0}=c_0+c_1y+c_2y^2+\cdots$	$(rs)-(\bar{r}s)=2rh\{c_0+c_1(sk+y_0)+c_2(sk+y_0)^2+\cdots\}$
3.	$\left(\frac{\partial^2 f}{\partial x^2}\right)_{x=x_0}=c_0+c_1y+c_2y^2+\cdots$	$(rs)-2(0\,s)+(\bar{r}s)=r^2h^2\{c_0+c_1(sk+y_0)+c_2(sk+y_0)^2+\cdots\}$
4.	$\left(\frac{\partial f}{\partial y}\right)_{x=x_0}=c_0+c_1y+c_2y^2$	$(0\,s)-(00)=sk\left\{c_0+\left(\frac{sk}{2}+y_0\right)c_1+\left(\frac{(sk)^2}{3}+sky_0+y_0^2\right)c_2\right\}$
5.	$\left(\frac{\partial^2 f}{\partial y^2}\right)_{x=x_0}=c_0+c_1y+c_2y^2$	$(0s)-(00)=s\{(01)-(00)\}+\frac{s(s-1)}{2}k^2\left[c_0+\left\{\frac{(s+1)k}{3}+y_0\right\}c_1+\left\{\frac{(s^2+s+1)k^2}{6}+\frac{2(s+1)}{3}ky_0+y_0^2\right\}c_2\right]$
6.	$\left(\frac{\partial^2 f}{\partial x\partial y}\right)_{x=x_0}=c_0+c_1y+c_2y^2+$	$(rs)-(\bar{r}s)=(r0)-(\bar{r}\,0)+2rshk\left\{c_0+\left(\frac{sk}{2}+y_0\right)c_1+\left(\frac{(sk)^2}{3}+sky_0+y_0^2\right)c_2\right\}$

c. From the above crieria it is easy to write down those for the boundary conditions

$$\Big[f(x,y)\Big]_{x=x_0}=0 \quad \text{and} \quad \left[\frac{\partial f(x,y)}{\partial x}\right]_{x=x_0}=0,$$

which sometimes take place in mathematical physics. These become respectively as is given in **Figs. 18** and **19**.

Fig. 18. Boundary condition

$[f(x,y)]_{x=x_0}=0$

$(\bar{2}3)$	$(\bar{1}3)$	0	(13)	(23)
$(\bar{2}2)$	$(\bar{1}2)$	0	(12)	(22)
$(\bar{2}1)$	$(\bar{1}1)$	0	(11)	(21)
$(\bar{2}0)$	$(\bar{1}0)$	0	(10)	(20)
$(\bar{2}\bar{1})$	$(\bar{1}\bar{1})$	0	$(1\bar{1})$	$(2\bar{1})$
$(\bar{2}\bar{2})$	$(\bar{1}\bar{2})$	0	$(1\bar{2})$	$(2\bar{2})$
$(\bar{2}\bar{3})$	$(\bar{1}\bar{3})$	0	$(1\bar{3})$	$(2\bar{3})$

Bounding line

Fig. 19. Boundary condition

$\left[\frac{\partial f(x,y)}{\partial x}\right]_{x=x_0}=0$

$(\bar{2}3)$	$(\bar{1}3)$	(03)	$(\bar{1}3)$	$(\bar{2}3)$
$(\bar{2}2)$	$(\bar{1}2)$	(02)	$(\bar{1}2)$	$(\bar{2}2)$
$(\bar{2}1)$	$(\bar{1}1)$	(01)	$(\bar{1}1)$	$(\bar{2}1)$
$(\bar{2}0)$	$(\bar{1}0)$	(00)	$(\bar{1}0)$	$(\bar{2}0)$
$(\bar{2}\bar{1})$	$(\bar{1}\bar{1})$	$(0\bar{1})$	$(\bar{1}\bar{1})$	$(\bar{2}\bar{1})$
$(\bar{2}\bar{2})$	$(\bar{1}\bar{2})$	$(0\bar{2})$	$(\bar{1}\bar{2})$	$(\bar{2}\bar{2})$
$(\bar{2}\bar{3})$	$(\bar{1}\bar{3})$	$(0\bar{3})$	$(\bar{1}\bar{3})$	$(\bar{2}\bar{3})$

Bounding line

When the above two conditions have to hold simultaneously, we have by superposition **Fig. 20**, which is a known criterion[4].

Fig. 20 Combined conditions.

$f=0,\ \frac{\partial f}{\partial x}=0$

$(\bar{2}3)$	$(\bar{1}3)$	0	$(\bar{1}3)$	$(\bar{2}3)$
$(\bar{2}2)$	$(\bar{1}2)$	0	$(\bar{1}2)$	$(\bar{2}2)$
$(\bar{2}1)$	$(\bar{1}1)$	0	$(\bar{1}1)$	$(\bar{2}1)$
$(\bar{2}0)$	$(\bar{1}0)$	0	$(\bar{1}0)$	$(\bar{2}0)$
$(\bar{2}\bar{1})$	$(\bar{1}\bar{1})$	0	$(\bar{1}\bar{1})$	$(\bar{2}\bar{1})$
$(\bar{2}\bar{2})$	$(\bar{1}\bar{2})$	0	$(\bar{1}\bar{2})$	$(\bar{2}\bar{2})$
$(\bar{2}\bar{3})$	$(\bar{1}\bar{3})$	0	$(\bar{1}\bar{3})$	$(\bar{2}\bar{3})$

Bounding line

Fig. 21. Single condition.

$f=0$

$(\bar{2}3)$	$(\bar{1}3)$	(03)	$-(03)$	$-(\bar{1}3)$
$(\bar{2}2)$	$(\bar{1}2)$	(02)	$-(02)$	$-(\bar{1}2)$
$(\bar{2}1)$	$(\bar{1}1)$	(01)	$-(01)$	$-(\bar{1}1)$
(20)	$(\bar{1}0)$	(00)	$-(00)$	$-(\bar{1}0)$
$(\bar{2}\bar{1})$	$(\bar{1}\bar{1})$	$(0\bar{1})$	$-(0\bar{1})$	$-(\bar{1}\bar{1})$
$(\bar{2}\bar{2})$	$(\bar{1}\bar{2})$	$(0\bar{2})$	$-(0\bar{2})$	$-(\bar{1}\bar{2})$
$(\bar{2}\bar{3})$	$(\bar{1}\bar{3})$	$(0\bar{3})$	$-(0\bar{3})$	$-(\bar{1}\bar{3})$

Bounding line

If the boundary condition is $[f(x,y)]_{x=x_0}=0$ alone, it will be recommended that reference is made to the result derived from the modified Bessel interpolation formula, and the resulting criterion is expressed in the schematic form as is given in **Fig. 21**[5].

5. Torsion-problem of a square cylinder. As a first application of the above derivations, together with those given in the preceding article[6], we take the torsion-problem of a prism whose cross-section is a square. The problem can be reduced to solve the differential equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+2=0, \quad \cdots\cdots\cdots (13)$$

which holds in the domain $-1<x<1$, $-1<y<1$, with the condition

$$f=0 \quad \cdots\cdots\cdots (14)$$

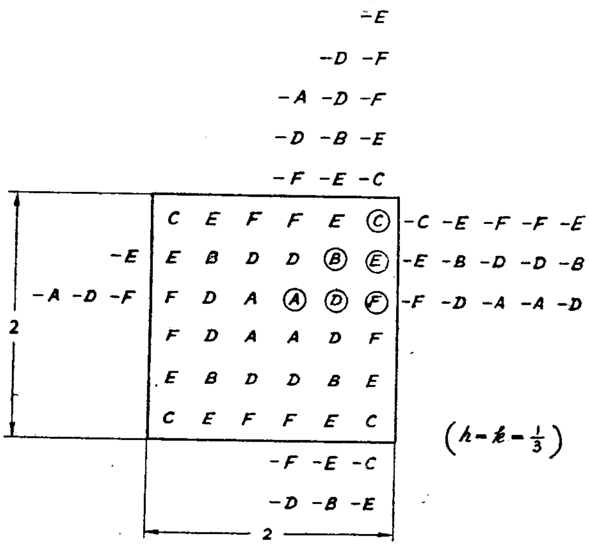
on the boundaries where $x=\pm 1$ and $y\pm 1$. The moment or torque M necessary to produce the twist τ per unit length of the prism is given by

$$M=2\mu\tau\iint f(x, y)\, dx\, dy, \qquad (15)$$

μ being the modulus of rigidity of the prism; and the torsional rigidity, C say, is given by $C=M/\tau$. The problem has been solved analytically, the required torque being $2.2492\,\mu\tau$.

If the domain within which the differential equation (13) holds be divided into 6×6 equidistant small divisions, lattice point arrangement satisfying the given boundary condition (14) is given in **Fig. 22.** It is added that an alternative criterion expressing the cited boundary condition (14) has been described by Prof. Hidaka[2], without giving its derivation, which is much similar to our criterion, **Fig. 21.**

Fig. 22. Lattice point arrangement satisfying boundasy condition $f=0$.



Difference equation by which the differential equation (13) is replaced is here of the type of fifth approximation (cf. **Fig. 5**). This equation is considered at six points marked A, B, C, D, E and F in **Fig. 22.** We then have the six equations,

A	B	C	D	E	F	h^2
−31 774	0	0	36 117	0	−5 875	25 200
0	−73 766	0	36 117	47 867	0	25 200
0	0	−115 758	0	47 867	−5 875	25 200
36 117	36 117	0	−105 540	−5 875	47 867	50 400
0	47 867	47 867	−5.875	−189 524	36 117	50 400
−5 875	0	−5 875	47 867	36 117	−147 532	50 400

$\qquad (16)$

This system of equations can at once be reduced to the third order, and the solutions become

$$\left.\begin{array}{lll} A=0.571\,248, & B=0.371\,684, & C=0.083\,763, \\ D=0.457\,849, & E=0.168\,833, & F=0.201\,756, \end{array}\right\} \qquad (17)$$

where $h=\dfrac{1}{3}$ has been taken into account. Thus we have the lattice point values of the torsion-function **Fig. 23.**

Now a tentative rule for the mechanical cubature suitable for the arrangement of lattice point values given above is thus **Fig. 24**[7].

Fig. 23. Lattice point values of the torsion-function.
(First quadrant only.)

-0.57125					
-0.45785	-0.37168				
-0.20176	-0.16883	-0.08376			
0.20176	0.16883	0.08376	-0.08376		
0.45785	0.37168	0.16883	-0.16883	-0.37168	
0.57125	0.45785	0.20176	-0.20176	-0.45785	-0.57125

Fig. 24. Weight-table for the mechanical cubature.
(First quadrant only.)

$= 504\,627\,200\,\frac{I}{hk}$

3 961 043					
-33 469 638	1 844 403				
142 003 066	-2 718 507	18 111 522			
258 474 069	529 614 492	495 352 572	18 111 522		
829 417 134	294 406 332	529 614 492	-2 718 507	1 844 403	
257 410 728	829 417 134	258 474 069	142 003 066	-33 469 638	3 961 043

Then by the multiplications of corresponding values in **Figs. 23** and **24**, and by the addition of the results, we have for the evaluation of the torsional rigidity

$$C=2\mu\iint f(x,y)\,dx\,dy=2\mu I=2\mu\frac{4hk}{504\;627\;200}\,1\;305\;925\;000=2.3004\,\mu. \qquad\cdots\cdots\cdots(18)$$

Since the corresponding true value is $C=2.2492\,\mu$, the error entailed by the above result is

$$E=\frac{2.3004-2.2492}{2.2492}=0.0227=2.27\%. \qquad\cdots\cdots\cdots(19)$$

More accurate result may be obtained if required, by taking the division h and k finer and employing more accurate difference equation. In fact, when the domain is divided into 8×8 finer squares, and the difference equation is of the seventh approximation (**Fig. 7**), I obtained the result $C=2.2855$, the error entailed reducing to 1.61%.

6. Transverse vibration of a clamped square plate. As a second application let us take the transverse vibration of a square plate clamped at four edges. The problem is to find eigenvalue of the differential equation

$$\frac{\partial^4 f}{\partial x^4}+2\frac{\partial^4 f}{\partial x^2\partial y^2}+\frac{\partial^4 f}{\partial y^4}+\lambda f=0, \qquad\cdots\cdots\cdots(20)$$

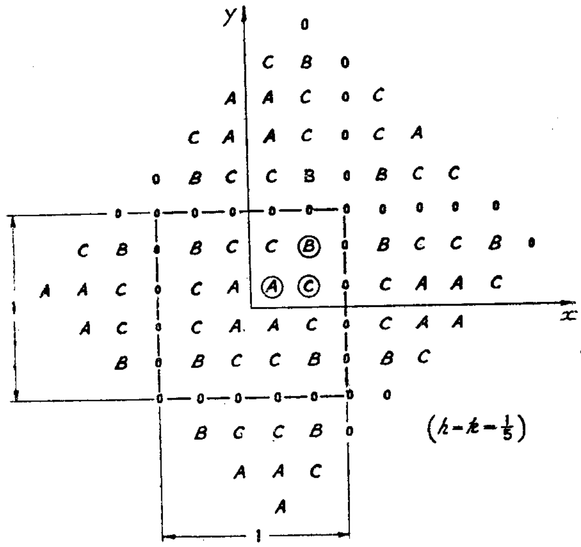
with the boundary conditions

$$\left.\begin{aligned} &f=\frac{\partial f}{\partial x}=0 \quad \text{along} \quad x=\pm\frac{1}{2},\\ \text{and}\quad &f=\frac{\partial f}{\partial y}=0 \quad \text{along} \quad y=\pm\frac{1}{2}. \end{aligned}\right\} \qquad\cdots\cdots\cdots(21)$$

Let for instance the domain be divided into 5×5 equal small divisions, so that $h=k=\frac{1}{5}$. We shall concern only with the fundamental mode of vibration, and then the behaviour of the vibration is symmetrical in respect to four axes. The boundary conditions (21) are replaced by **Fig. 25** (Cf. **Fig. 20.**).

Difference equation here employed is of the type of fourth approximation to $\nabla^4 f$, which was given in **Fig. 11.** This equation is considered at three points marked with circle in **Fig. 25.** We then have the three equations

Fig. 25. **Lattice point arrangement satisfying given boundary conditions.**



$$\left.\begin{aligned}(153\,810-\lambda_0)A+59\,430B-298\,130C=0,\\ 59\,430A+(735\,010-\lambda_0)B-443\,430C=0,\\ -149\,065A-221\,715B+(459\,925-\lambda_0)C=0,\end{aligned}\right\}\qquad(22)$$

where, for shortness,

$$\lambda_0=15\,120\,h^4\lambda \qquad \text{with} \quad h=\frac{1}{5}.$$

By eliminating the unknowns from the above equations (22), we get a secular equation of third order for λ_0. On solving this equation we obtain

$$\frac{\lambda}{\pi^4}=\frac{\lambda_0}{15\,120h^4\pi^4}=13.2502. \qquad(23)$$

Since the corresponding exact value by Prof. S. Tomotika is $\frac{\lambda}{\pi^4}=13.2948$[8], the error entailed by our result is -0.335%. We thus see that the result obtained is fairly satisfactory, despite taht the simultaneous equations treated are only of third order.

If the third approximation to $\nabla^4 f$ (**Fig. 10**) is employed, the error entailed by the result amounts to -2.15%, which was also given by Prof. Hidaka[9].

7. On the quest for accuracy. In conclusion, it should be noted that some investigators[10], holding "engineering" outlook, have insisted upon the lowest approximation to partial differential equation considered, whereas others, who have "mathematical" outlook, have taken the alternative attitude that is to employ approximations containing higher differences.

In this regard, it is of some interest to quote here the opinion of Sir R. V. Southwell, who would seem at first to hold the mathematical outlook and later to convert it to the engineering one. But, it can be stated at least that both of the two examples given by him are not rather appropriate to the exemplification of the matter[11].

Moreover, a paper by L. Fox[12] seems to have gained a great success in solving differential equations by employing higher difference approxiamations, in spite of the contention of Sir Southwell.[13]
Thus the contention seems not to have come to an end.

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References

1) 500 Wakazato, Nagano City, Honshū, Japan.

2) K. Hidaka, Numercical Integration vol. ii, 1936 (in Japanese); L. Collatz, Das Differenzen verfahren mit höherer Approximation für lineare Differentialgleichungen, Shriften des mathematischen Seminärs und des Instituts für angewandte Mathematik der Universität Berlin, Bd. 3, Heft 1, Berlin, 1935.

3) I have prepared such 'difference table' up to the 20th order, which is very useful in performing calculations. Some of first differences only are given here.

4) Loc. cit. 2), K. Hidaka.

5) Refesence is made to my preceding article, 'Difference Method for Partial Differential Equations, Part I, Trans. of JSCE, Sept. 1952.

6) Loc. cit. 5).

7) My work concerning the mechanical cubature, biquadrature, etc. will be published in the near future.

8) S. Tomotika, On the Transverse Vibration of a Square Plate with Four Clamped Edges, Report of the Aeronautical Research Institute, Tōkyō University, No. 10 (1935).

9) K. Hidaka, loc. cit. 2), pp. 240—243.

10) Cf., for instance, Numerical Methods of Analysis in Engineering, arranged and edited by L. E. Grinter. New York, 1949.

11) R. V. Southwell, Relaxation Methods in Engineering Science, Oxford, 1940. pp. 184—185; and loc cit. 10), pp. 71—72, Example 1.

In fact the integral $\mathbf{I}=\int_0^1 \mathbf{x} \sin^2\pi\mathbf{x}d\mathbf{x}$ is evaluated to its exact value 0.25 by repeated use of Simpson's rule, whilst the interpolation of the function $\mathbf{y}=\frac{1}{1+\mathbf{x}^2}$ at its *middle* point becomes closer with higher differences; and it would be a matter of course that in general the polynomial representation goes against its exact value at *extremities* with an increased order of polynomial.

(17 Nov. 1952.)

12) L. Fox, Some Improvements in the Use of Relaxation Methods for the Solution of Ordinary and Partial Differential Equations, Proc. Roy. Soc., A No. 1020, vol. 190 (1947).

In spite of his clear success, this work would be open to question in the inevitable employment of the process of extrapolation at boundary lattice points, and the accuracy of the results is largely governed by the ambiguous extrapolation.

13) Loc. cit. 10), Chapter IV.