応用力学

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A METHOD OF SOLVING EIGENVALUE EQUATIONS IN HILBERT SPACE* [4),8)]

ヒルベルト空間における固有値方程式の一解法

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Synopsis: An effective practical method of solving eigenvalue equations[4),8)] in Hilbert Space,[4)] for example, in problems of rectangular plates,[6),7)] is rationally studied by applying the theories of matrices and determinants, through which much labour and time will be saved. If the equations are given as the transcendetal simultaneous equations,[6),7)] we must further solve them, and if they are given as the linear simultaneous equations, we can immediately get their solutions.

Introduction

Taking the coordinates in the edgewise directions of the given rectangular plate expanding its deflexion into the orthogonal functions, and denoting the matrices of their coeffiicients in the directions of the cordinates with

$$\boldsymbol{C}=[c_m],\qquad \overline{\boldsymbol{C}}=[\bar{c}_n] \quad\cdots\cdots(1)$$

and the matrices of linear transformation from $\overline{\boldsymbol{C}}$ to $\boldsymbol{C}$ and from $\boldsymbol{C}$ to $\overline{\boldsymbol{C}}$ with

$$\boldsymbol{B}=[\beta_{mj}],\qquad \boldsymbol{A}=[\alpha_{nt}] \quad\cdots\cdots(2)$$

respectively, then gives from (1) and (2)

$$\boldsymbol{C}=\boldsymbol{B}\overline{\boldsymbol{C}},\qquad \overline{\boldsymbol{C}}=\boldsymbol{A}\boldsymbol{C} \quad\cdots\cdots(3)$$

that is,

$$c_m=\sum_{j=1}^{\infty}\beta_{mj}\bar{c}_j,\qquad \bar{c}_n=\sum_{t=1}^{\infty}\alpha_{nt}c_t \quad\cdots\cdots(4)$$

From (3), we get

$$\boldsymbol{C}=\boldsymbol{B}\boldsymbol{A}\boldsymbol{C},\qquad \overline{\boldsymbol{C}}=\boldsymbol{A}\boldsymbol{B}\overline{\boldsymbol{C}} \quad\cdots\cdots(5)$$

i.e.,

$$\lambda'\boldsymbol{C}=0,\qquad \lambda\overline{\boldsymbol{C}}=0 \quad\cdots\cdots(6)$$

where

$$\lambda'=[\boldsymbol{E}-\boldsymbol{B}\boldsymbol{A}],\qquad \lambda=[\boldsymbol{E}-\boldsymbol{A}\boldsymbol{B}] \quad\cdots\cdots(7)$$

and

$$\delta\lambda'=\delta(\boldsymbol{E}-\boldsymbol{B}\boldsymbol{A})=0,\qquad \delta\lambda=\delta(\boldsymbol{E}-\boldsymbol{A}\boldsymbol{B})=0 \quad\cdots\cdots(8)$$

that is, referring to from (2) to (8),

$$\begin{vmatrix} 1-\phi_{1,1} & -\phi_{1,3} & -\phi_{1,5} & -\phi_{1,7} & \cdots\cdots \\ -\phi_{3,1} & 1-\phi_{3,3} & -\phi_{3,5} & -\phi_{3,7} & \cdots\cdots \\ -\phi_{5,1} & -\phi_{5,3} & 1-\phi_{5,5} & -\phi_{5,7} & \cdots\cdots \\ \vdots & \vdots & \vdots & & \ddots \end{vmatrix}=0 \quad\cdots\cdots(9)$$

* Completed in March, 1950, except the examples.

(1), (2), ……, (8). Refer to bibliography shown at the end of this paper.

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$$\phi_{i,j}=\sum_{i,j,k=1}^{\infty}\alpha_{ik}\beta_{kj},\qquad i,j,k=1,3,5,\cdots\cdots.$$

For simplicity and generality, we take the matrices:

$$\lambda=[\lambda_{rs}];\qquad \lambda'=[\lambda_{sr}];$$

$$\lim_{r,s\to\infty}\lambda_{rs}\begin{cases}=1, & r=s,\\ =0, & r\neq s,\end{cases}\qquad r,s=1,2,3,\cdots\cdots,\qquad\cdots\cdots\cdots\cdots(10)$$

instead of (7), where the latter is the transposed matrix[4] of the former, and the determinant:

$$|\lambda|=|\lambda'|$$

$$=\begin{vmatrix}\lambda_{11} & \lambda_{12} & \lambda_{13}\cdots\cdots & \lambda_{1n} & \cdots\cdots\\ \lambda_{21} & \lambda_{22} & \lambda_{23}\cdots\cdots & \lambda_{2n} & \cdots\cdots\\ \lambda_{31} & \lambda_{32} & \lambda_{33}\cdots\cdots & & \\ \vdots & \vdots & \vdots\ \ddots & \vdots & \\ \lambda_{n1} & \lambda_{n2} & & \lambda_{nn} & \cdots\cdots\\ \vdots & \vdots & & \vdots & \ddots\end{vmatrix}\qquad\cdots\cdots\cdots\cdots(11)$$

$$=0,\qquad \lim_{r,s\to\infty}\lambda_{rs}\begin{cases}=1, & r=s,\\ =0, & r\neq s.\end{cases}\qquad r,s=1,2,3,\cdots\cdots,$$

instead (8) or (9).

Since eigenvalue equations are frequently given in such a form as (11), their solutions always take much labour and time.[6][7] It must, therefore, be transformed into the other forms, whose solutions can extremely be facilitated.

We will derive the expansion of the most rapidly convergent series from the right-hand side of (11), applying the premultiplier[4][5] to (10).

Up to now, the process of obtaining the triangular matrix or that of lowering the orders of the given determinant has mainly been started from its column or row in the midway except the last.[1][4][5]

In this work, the premultipliers must be so constructed and arranged that it shall continually be applied to from the last column or row up to the principal column or row*, and subsequently, if the principal column or row is situated in the midway, neccessarily to from the first column or row up to the principal column or row, after the process is continued up to that.

Thus, the transformed eigenvalue equation will lastly be obtained, from which, neglecting the terms of higher order, the practical equation can readily be reduced.

Expantion of Eigenvalue Equations

First consider the matrix:

$$\lambda_n=\begin{pmatrix}\lambda_{11} & \lambda_{12}\cdots\cdots\cdots & \lambda_{n1}\\ \lambda_{21} & \lambda_{22}\cdots\cdots\cdots & \lambda_{n2}\\ \vdots & \vdots\ \ddots & \vdots\\ \lambda_{n1} & \lambda_{n2} & \lambda_{nn}\end{pmatrix}\qquad\cdots\cdots\cdots\cdots(12)$$

* For convenience' sake, they mean the main column or row, which contain the elements of large absolute magnitude and have the great influence on the magnitude of the determinant.

and the determinant:

$$|\lambda_n| = \begin{vmatrix} \lambda_{11} & \lambda_{12} \cdots\cdots & \lambda_{n1} \\ \lambda_{21} & \lambda_{22} \cdots\cdots & \lambda_{n2} \\ \vdots & \vdots \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \lambda_{nn} \end{vmatrix} = 0, \quad \cdots\cdots (13)$$

which are made of first n columns and the first n rows of (10) and (11) respectively. In Hilbert space, the nearer the element to the principal column and row is situated, the larger the magnitude it is used to be. Therefore, if n is finite, the greater its magnitude is chosen, the higher the accuracies of the solutions of the eigenvalues are used to become. If n is made infinite, they become quite exact.

Take the premultipliers, their determinants, and the triangular matrix such that

$$E^{t-1}_{n-t+1} = \begin{pmatrix} 1 & 0 \cdots 0 & -\dfrac{\lambda^{t-1}_{1,n-t+1}}{\lambda_{n-t+1,n-t+1}} & 0 \cdots 0 \\ 0 & 1 \cdots 0 & -\dfrac{\lambda^{t-1}_{2,n-t+1}}{\lambda^{t-1}_{n-t+1,n-t+1}} & 0 \cdots 0 \\ \vdots & \vdots \ddots \vdots & \vdots & \vdots \quad \vdots \\ 0 & 0 \cdots 1 & -\dfrac{\lambda^{t-1}_{n-t,n-t+1}}{\lambda^{t-1}_{n-t+1,n-t+1}} & 0 \cdots 0 \\ 0 & 0 \cdots 0 & 1 & 0 \cdots 0 \\ \vdots & \vdots \quad \vdots & \vdots & 1 \cdots 0 \\ \vdots & \vdots \quad \vdots & \vdots & \vdots \ddots \vdots \\ 0 & 0 \cdots 0 & 0 & 0 \cdots 1 \end{pmatrix} \quad \cdots\cdots (14)$$

$$|E^{t-1}_{n-t+1}| = 1 \quad \cdots\cdots (15)$$

$$I^{n-p+h-1}_{h-1} = \begin{pmatrix} 1 \cdots 0 & 0 & 0 \cdots\cdots 0 \\ \vdots \ddots \vdots & \vdots & \vdots \quad \vdots \\ 0 \cdots 1 & 0 & 0 \cdots\cdots 0 \\ 0 \cdots 0 & -\dfrac{\lambda^{n-p+h-1}_{h+1,h}}{\lambda^{n-p+h-1}_{h,h}} & 0 \cdots\cdots 0 \\ 0 \cdots 0 & -\dfrac{\lambda^{n-p+h-1}_{h+2,h}}{\lambda^{n-p+h+1}_{h,h}} & 1 \cdots\cdots 0 \\ \vdots \quad \vdots & \vdots & \vdots \ddots \vdots \\ 0 \cdots 0 & -\dfrac{\lambda^{n-p+h-1}_{p,h}}{\lambda^{n-p+h-1}_{h,h}} & 0 \cdots 1 \cdots 0 \\ 0 \cdots 0 & -\dfrac{\lambda^{n-p+h-1}_{p+1,h}}{\lambda^{n-p+h-1}_{h,h}} & 0 \cdots 1 \cdots 0 \\ \vdots \quad \vdots & \vdots & \vdots \ddots \vdots \\ 0 \cdots 0 & -\dfrac{\lambda^{h-1}_{n,h}}{\lambda^{n-p+h-1}_{h,h}} & 0 \cdots\cdots 1 \end{pmatrix} \quad \cdots\cdots (16)$$

$$|I^{n-p+h-1}_{h-1}| = 1 \quad \cdots\cdots (17)$$

and

$$\tau_n=\prod_{h=1}^{p-1} I_{h-1}^{n-p+h-1}\quad\left(\prod_{t=1}^{n-p} E_{n-t+1}^{t-1}\,\lambda_n\right),\qquad t\leqq(n-p),\qquad t,p=1,2,3,\cdots\cdots,$$

$$=\prod_{h=1}^{p-1} I_{h-1}^{n-p+h-1}\begin{pmatrix}
\lambda_{1,1}^{n-p} & \lambda_{1,2}^{n-p} & \cdots & \lambda_{1,p}^{n-p} & 0 & \cdots & 0 & 0\\
\lambda_{2,1}^{n-p} & \lambda_{2,2}^{n-p} & \cdots & \lambda_{2,p}^{n-p} & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots\\
\lambda_{p,1}^{n-p} & \lambda_{p,2}^{n-p} & \cdots & \lambda_{p,p}^{n-p} & 0 & \cdots & 0 & 0\\
\lambda_{p+1,1}^{n-p-1} & \lambda_{p+1,2}^{n-p-1} & \cdots & \lambda_{p+1,p}^{n-p-1} & \lambda_{p+1,p+1}^{n-p-1} & \cdots & 0 & 0\\
\vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots\\
\lambda_{n-1,1}^{1} & \lambda_{n-1,2}^{1} & \cdots & \lambda_{n-1,p}^{1} & \lambda_{n-1,p+1}^{1} & \cdots & \lambda_{n-1,n-1}^{1} & \vdots\\
\lambda_{n,1}^{0} & \lambda_{n,2}^{0} & \cdots & \lambda_{n,p}^{0} & \lambda_{n,p+1}^{0} & \cdots & \lambda_{n,n-1}^{0} & \lambda_{n,n}^{0}
\end{pmatrix}$$

$$=\begin{pmatrix}
\lambda_{1,1}^{n-p} & \lambda_{1,2}^{n-p} & \cdots & \lambda_{1,p-1}^{n-p} & \lambda_{1,p}^{n-p} & 0 & \cdots & 0 & 0\\
0 & \lambda_{2,2}^{n-p+1} & \cdots & \lambda_{2,p-1}^{n-p+1} & \lambda_{2,p}^{n-p+1} & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots\\
0 & 0 & \cdots & \lambda_{p-1,p-1}^{n-2} & \lambda_{p-1,p}^{n-2} & 0 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & \lambda_{p,p}^{n-1} & 0 & \cdots & 0 & 0\\
\vdots & \vdots & & 0 & \lambda_{p+1,p}^{n-2} & \lambda_{p+1,p+1}^{n-p-1} & \cdots & 0 & 0\\
\vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & \lambda_{n-1,p}^{p} & \lambda_{n-1,p+1}^{1} & \cdots & \lambda_{n-1,n-1}^{1} & 0\\
0 & 0 & \cdots & 0 & \lambda_{n,p}^{p-1} & \lambda_{n,p+1}^{0} & \cdots & \cdots & \lambda_{n,n}^{0}
\end{pmatrix}\quad\cdots(18)$$

where

$$\left.\begin{array}{l}\lambda_{r,s}^{m}=\lambda_{r,s}^{m-1}-\lambda_{r,n-m+1}^{m-1}\,\lambda_{n-m+1,s}^{m-1}/\lambda_{n-m+1,n-m+1}^{m-1},\\ \lambda_{r,s}^{0}=\lambda_{r,s}\end{array}\right\}\quad\cdots\cdots(19a)$$

$$\begin{aligned}\lambda_{r,s}^{m+1}&=\lambda_{r,s}^{m}-\lambda_{r,n-m}^{m}\,\lambda_{n-m,s}^{m}/\lambda_{n-m,n-m}^{m}\\
&=\lambda_{r,s}^{m-1}-(\lambda_{r,n-m}^{m-1}-\lambda_{r,n-m+1}^{m-1}\,\lambda_{n-m+1,n-m}^{m-1}/\lambda_{n-m+1,n-m+1}^{m-1})\\
&\qquad\times(\lambda_{n-m,s}^{m-1}-\lambda_{n-m,n-m+1}^{m-1}\,\lambda_{n-m+1,s}^{m-1}/\lambda_{n-m+1,n-m+1}^{m-1})\\
&\qquad/(\lambda_{n-m,n-m}^{m-1}-\lambda_{n-m,n-m+1}^{m-1}\,\lambda_{n-m+1,n-m}^{m-1}/\lambda_{n-m+1,n-m+1}^{m-1})\\
&\qquad-\lambda_{r,n-m+1}^{m-1}\,\lambda_{n-m+1,s}^{m-1}/\lambda_{n-m+1,n-m+1}^{m-1},\quad\cdots\cdots(19b)\end{aligned}$$

$$p<r,s\leqq(n-m),\qquad r,s,m=1,2,3,\cdots\cdots;$$

$$\begin{aligned}\lambda_{i,j}^{n-p+h}&=\lambda_{i,j}^{n-p+h-1}-\lambda_{i,h}^{n-p+h-1}\,\lambda_{h,j}^{n-p+h-1}/\lambda_{h,h}^{n-p+h-1}\quad\cdots\cdots(20a)\\
&=\lambda_{i,j}^{n-p+h-2}-\lambda_{i,h-1}^{n-p+h-2}\,\lambda_{h-1,j}^{n-p+h-2}/\lambda_{h-1,h-1}^{n-p+h-2}\\
&\qquad-(\lambda_{i,h}^{n-p+h-2}-\lambda_{i,h-1}^{n-p+h-2}\,\lambda_{h-1,h}^{n-p+h-2}/\lambda_{h-1,h-1}^{n-p+h-2})\\
&\qquad\times(\lambda_{h,j}^{n-p+h-2}-\lambda_{h,h-1}^{n-p+h-2}\,\lambda_{h-1,j}^{n-p+h-2}/\lambda_{h-1,h-1}^{n-p+h-2})\\
&\qquad/(\lambda_{h,h}^{n-p+h-2}-\lambda_{h,h-1}^{n-p+h-1}\,\lambda_{h-1,h}^{n-p+h-2}/\lambda_{h-1,h-1}^{n-p+h-2})\quad\cdots\cdots(20b)\end{aligned}$$

$$h<i,j\leqq p,\qquad i,j,h=1,2,3,\cdots\cdots;$$

$$\lambda_{i,j}^{n-i+h}=\lambda_{i,j}^{n-i+h-1}-\lambda_{i,h}^{n-i+h-1}\,\lambda_{h,j}^{n-i+h-1}/\lambda_{h,h}^{n-p+h-1}\quad\cdots\cdots(21)$$

$$h<j\leqq p<i,\qquad i,j,h=1,2,3,\cdots\cdots.$$

Neglecting the terms of higher order, we get from (19a, b)

$$\begin{aligned}\lambda^{m+1}_{r,s} &= \lambda^{m-1}_{r,s} - \lambda^{m-1}_{r,n-m}\,\lambda^{m-1}_{n-m,s}/\lambda^{m-1}_{n-m,n-m} - \lambda^{m-1}_{r,n-m+1}\,\lambda^{m-1}_{n-m+1,s}/\lambda^{m-1}_{n-m+1,n-m+1}\\ &\quad + \{\lambda^{m-1}_{r,n-m}\,\lambda^{m-1}_{n-m,n-m+1}\,\lambda^{m-1}_{n-m+1,s} + \lambda^{m-1}_{r,n-m+1}\,\lambda^{m-1}_{n-m+1,n-m}\,\lambda^{m-1}_{n-m,s}\}\\ &\quad /\lambda^{m-1}_{n-m,n-m}\,\lambda^{m-1}_{n-m+1,n-m+1} \qquad \cdots\cdots (22a)\\ &= \lambda^{m-2}_{r,s} - \lambda^{m-2}_{r,n-m}\,\lambda^{m-2}_{n-m,s}/\lambda^{m-2}_{n-m,n-m} - \lambda^{m-2}_{r,n-m+1}\,\lambda^{m-2}_{n-m+1,s}/\lambda^{m-2}_{n-m+1,n-m+1}\\ &\quad - \lambda^{m-2}_{r,n-m+2}\,\lambda^{m-2}_{n-m+2,s}/\lambda^{m-2}_{n-m+2,n-m+2} + \{\lambda^{m-2}_{s,n-m}\,\lambda^{m-2}_{n-m,n-m+1}\,\lambda^{m-2}_{n-m+1,s}\\ &\quad + \lambda^{m-2}_{r,n-m+1}\,\lambda^{m-2}_{n-m+1,n-m}\,\lambda^{m-2}_{n-m,s}\}/\lambda^{m-2}_{n-m,n-m}\,\lambda^{m-2}_{n-m+1,n-m+1} \qquad \cdots\cdots (22b)\end{aligned}$$

$$p<r,\ s\leqq(n-m), \qquad r,s,m=1,2,3,\cdots\cdots;$$

and similarly from (21a, b)

$$\begin{aligned}\lambda^{n-p+h}_{i,j} &= \lambda^{n-p+h-2}_{i,j} - \lambda^{n-p+h-2}_{i,h}\,\lambda^{n-p+h-2}_{h,j}/\lambda^{n-p+h-2}_{h,h} - \lambda^{n-p+h-2}_{i,h-1}\,\lambda^{n-p+h-2}_{h-1,j}/\lambda^{n-p+h-2}_{h-1,h-1}\\ &\quad + \{\lambda^{n-p+h-2}_{i,h}\,\lambda^{n-p+h-2}_{h,h-1}\,\lambda^{n-p+h-2}_{h-1,j} + \lambda^{n-p+h-2}_{i,h-1}\,\lambda^{n-p+h-2}_{h-1,h}\,\lambda^{n-p+h-2}_{h,j}\}\\ &\quad /\lambda^{n-p+h-2}_{h-1,h-1}\,\lambda^{n-p+h-2}_{h,h} \qquad \cdots\cdots (23a)\\ &= \lambda^{n-p+h-3}_{i,j} - \lambda^{n-p+h-3}_{i,h}\,\lambda^{n-p+h-3}_{h,j}/\lambda^{n-p+h-3}_{h,h} - \lambda^{n-p+h-3}_{i,h-1}\,\lambda^{n-p+h-3}_{h-1,j}/\lambda^{n-p+h-3}_{h-1,h-1}\\ &\quad - \lambda^{n-p+h-3}_{i,h-2}\,\lambda^{n-p+h-3}_{h-2,j}/\lambda^{n-p+h-3}_{h-2,n-2} + \{\lambda^{n-p+h-3}_{i,h}\,\lambda^{n-p+h-3}_{h,h-1}\,\lambda^{n-p+h-3}_{h-1,j}\\ &\quad + \lambda^{n-p+h-3}_{i,h-1}\,\lambda^{n-p+h-3}_{h-1,h}\,\lambda^{n-p+h-3}_{h,j}\}/\lambda^{n-p+h-3}_{h,h}\,\lambda^{n-p+h-3}_{h-1,h-1} \qquad \cdots\cdots (23b)\end{aligned}$$

$$h<i,j\leqq p, \qquad i,j,h=1,2,3,\cdots\cdots.$$

From (13) to (18), gives

$$\begin{aligned}|\tau_n| &= |\lambda_n|\\ &= \lambda^{n-p}_{1,1}\,\lambda^{n-p+1}_{2,2}\cdots\cdots\lambda^{n-1}_{p,p}\,\lambda^{n-p-1}_{p+1,p+1}\,\lambda^{n-p-2}_{p+2,p+2}\cdots\cdots\lambda^{1}_{n-1,n-1}\,\lambda_{n,n}\\ &= 0, \qquad \cdots\cdots (24)\end{aligned}$$

from which, since

$$\lambda^{n-p+r-1}_{r,r} \neq 0, \qquad r\neq p \qquad \cdots\cdots (25)$$

$$\lambda^{n-r}_{r,r} \neq 0, \qquad n-p<r \qquad \cdots\cdots (26)$$

must clearly hold true, we get

$$\lambda^{n-1}_{p,p} = 0 \qquad \cdots\cdots (27)$$

that is, the transformed eigenvalue equation.

The Plactical Eigenvalue Equations

Firstly, if the first column and the first row of the eigenvalue equation (13) are the principal,* then putting

$$n=4, \qquad p=1, \qquad m=2, \qquad r=s=1 \qquad \cdots\cdots (28)$$

into (22b) and referring to (27), we get

$$\begin{aligned}\lambda^{3}_{1,1} &= \lambda_{1,1} - \lambda_{1,2}\lambda_{2,1}/\lambda_{2,2} - \lambda_{1,3}\lambda_{3,1}/\lambda_{3,3} - \lambda_{1,4}\lambda_{4,1}/\lambda_{4,4} + \{\lambda_{1,2}\lambda_{2,3}\lambda_{3,1} + \lambda_{1,3}\lambda_{3,2}\lambda_{2,1}\}/\lambda_{2,2}\lambda_{3,3}\\ &= 0. \qquad \cdots\cdots (29)\end{aligned}$$

Neglecting the terms of higher order, we obtain

$$\lambda^{3}_{1,1} = \lambda_{1,1} - \lambda_{1,2}\lambda_{2,1}/\lambda_{2,2} - \lambda_{1,3}\lambda_{3,1}/\lambda_{3,3} = 0 \qquad \cdots\cdots (30)$$

$$\lambda^{3}_{1,1} = \lambda_{1,1} - \lambda_{1,2}\lambda_{2,1}/\lambda_{2,2} = 0 \qquad \cdots\cdots (31)$$

$$\lambda^{3}_{1,1} \approx \lambda_{1,1} \approx 1 \qquad \cdots\cdots (32)$$

whose accuracies are dependent on the degrees of the approximation.

When the original eigenvalue equations are transcendental, their eigenvalues are generally obtained from (29). Referring to from (29) to (32), we can readily

* For convenience' sake, they mean the main column or row, which contain the elements of large absolute magnitude and have the great influence on the magnitude of the determinant.

understand the reason why $\Delta_1=0$ gives the very good approximate solutions, with respect to $\Delta_2=0$, $\Delta_3=0$ or $\Delta_4=0$ respectively, in bibliography (7).

Now, when the original eigenvalue equations are linear as the secular equations, then the first root shall only be gotten from (29). and the p-th root shall generally be gotten from

$$\lambda_{p,p}^{n-1}=0, \qquad p=2,3,4,\cdots\cdots, \qquad \cdots\cdots (33)$$

when the p-th column and the p-th row are the principal.

Therefore, if the second column and the second row of (13) are the principal, then, putting

$$n=5, \qquad p=2, \qquad h=1, \qquad i=j=2 \qquad \cdots\cdots (34)$$

into (23b), we obtain

$$\lambda_{2,2}^{4}=\lambda_{2,2}-\lambda_{2,1}\lambda_{1,2}/\lambda_{1,1}-\lambda_{2,3}\lambda_{3,2}/\lambda_{3,3}-\lambda_{2,4}\lambda_{4,2}/\lambda_{4,4}-\lambda_{2,5}\lambda_{5,2}/\lambda_{5,5}$$
$$+\{\lambda_{2,3}\lambda_{3,4}\lambda_{4,2}+\lambda_{2,4}\lambda_{4,3}\lambda_{3,2}\}/\lambda_{3,3}\lambda_{4,4}=0 \qquad \cdots\cdots (35)$$

Neglecting the terms of higher order, gives from (35)

$$\lambda_{2,2}^{4}=\lambda_{2,2}-\lambda_{2,3}\lambda_{3,2}/\lambda_{3,3}-\lambda_{2,4}\lambda_{4,2}/\lambda_{4,4}=0 \qquad \cdots\cdots (36)$$
$$\lambda_{2,2}^{4}=\lambda_{2,2}-\lambda_{2,3}\lambda_{3,2}/\lambda_{3,3}=0 \qquad \cdots\cdots (37)$$
$$\lambda_{2,2}^{4}\approx\lambda_{2,2}\approx 0 \qquad \cdots\cdots (38)$$

whose accuracies are dependent on the degrees of approximation.

The Practical Eigenvalue Equations for Problems of Rectangular Plates

Now, if we come back to Eq. (9), that is, the case where the transcendental eigenvalue equations are given in problems of rectangular plates, then, referring to from (28) to (32), gives

$$1-\Sigma\alpha_{1i}\beta_{i1}-\Sigma\alpha_{1i}\beta_{i3}\Sigma\alpha_{3i}\beta_{i1}/(1-\Sigma\alpha_{3i}\beta_{i3})-\Sigma\alpha_{1i}\beta_{i5}\Sigma\alpha_{5i}\beta_{i1}/(1-\Sigma\alpha_{5i}\beta_{i5})$$
$$-\Sigma\alpha_{1i}\beta_{i7}\Sigma\alpha_{7i}\beta_{i1}/(1-\Sigma\alpha_{7i}\beta_{i7})+\{\Sigma\alpha_{1i}\beta_{i3}\Sigma\alpha_{3i}\beta_{i5}\Sigma\alpha_{5i}\beta_{i1}$$
$$+\Sigma\alpha_{1i}\beta_{i5}\Sigma\alpha_{5i}\beta_{i3}\Sigma\alpha_{3i}\beta_{i1}\}/(1-\Sigma\alpha_{3i}\beta_{i3})(1-\Sigma\alpha_{5i}\beta_{i5})$$
$$=0, \qquad i=1,3,5.7 \qquad \cdots\cdots (39)$$

$$1-\Sigma\alpha_{1i}\beta_{i1}-\Sigma\alpha_{1i}\beta_{i3}\Sigma\alpha_{3i}\beta_{i1}/(1-\Sigma\alpha_{3i}\beta_{i3})-\Sigma\alpha_{1i}\beta_{i5}\Sigma\alpha_{5i}\beta_{i1}/(1-\Sigma\alpha_{5i}\beta_{i5})$$
$$=0, \qquad i=1,3,5 \qquad \cdots\cdots (40)$$

$$1-\Sigma\alpha_{1i}\beta_{i1}-\Sigma\alpha_{1i}\beta_{i3}\Sigma\alpha_{3i}\beta_{i1}/(1-\Sigma\alpha_{3i}\beta_{i3})$$
$$=0, \qquad i=1,3 \qquad \cdots\cdots (41)$$

$$1-\Sigma\alpha_{1i}\beta_{i1}\approx 0 \qquad \cdots\cdots (42)$$

For the quadratic plates with crosswise symmetrical boundary conditions, we obtain, referring to from (1) to (9),

$$C=A^2C=\overline{C}, \qquad A=[\alpha_{mi}]=B \qquad \cdots\cdots (43)$$

and

$$\begin{vmatrix} 1\pm\alpha_{1,1} & \pm\alpha_{1,3} & \pm\alpha_{1,5} & \cdots \\ \pm\alpha_{3,1} & 1\pm\alpha_{3,3} & \pm\alpha_{3,5} & \cdots \\ \pm\alpha_{5,1} & \pm\alpha_{5,3} & 1\pm\alpha_{5,5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}=0 \qquad \cdots\cdots (44)$$

Therefore, from (41), referring to from (28) to (32), gives

$$1\pm\alpha_{1,1}-\alpha_{1,3}\alpha_{3,1}/(1\pm\alpha_{3,3})-\alpha_{1,5}\alpha_{5,1}/(1\pm\alpha_{5,5})-\alpha_{1,7}\alpha_{7,1}/(1\pm\alpha_{7,7})$$
$$\pm\{\alpha_{1,3}\alpha_{3,5}\alpha_{5,1}+\alpha_{1,5}\alpha_{5,3}\alpha_{3,1}\}/(1\pm\alpha_{3,3})(1\pm\alpha_{5,5})=0 \qquad \cdots\cdots (45)$$

$$1\pm\alpha_{1,1}-\alpha_{1,3}\alpha_{3,1}/(1\pm\alpha_{3,3})-\alpha_{1,5}\alpha_{5,1}/(1\pm\alpha_{5,5})=0 \quad \cdots\cdots(46)$$

$$1\pm\alpha_{1,1}-\alpha_{1,3}\alpha_{3,1}/(1\pm\alpha_{3,3})=0 \quad \cdots\cdots(47)$$

$$1\pm\alpha_{1,1}\approx 0 \quad \cdots\cdots(48)$$

Numerical Solutions of Eigenvalue Equations

The further process of numerical solution of the simplified eigenvalue equations, obtained from the original by applying Eq. (29), (30), ……, or (48), is generally divided into the following two classes:

1. If the given eigenvalue equations are linear and simultaneous, then we can readily solve them by the iteration method.[8]

2. If the given eigenvalue equations are transcendental and simultaneous, then we must solve them by (a) the Method of "Regula Falsi,"[8] (b) the Newton-Raphson Method,[8] (c) the Lagrangian Interpolation Method,[8] etc.[8]

Examples

In order to testify the accuracies and the usefulness of the derived theories and the practical eigenvalue equations, the three examples, found in bibliography (7), (9), and (3), will be given as in the following:

Ex. 1. To solve the eigenvalue equation:[7]

$$\begin{vmatrix} 1+K_{11} & K_{13} & K_{15} & K_{17} \\ K_{31} & 1+K_{33} & K_{35} & K_{37} \\ K_{51} & K_{53} & 1+K_{55} & K_{57} \\ K_{71} & K_{73} & K_{75} & 1+K_{77} \end{vmatrix} =0 \quad \cdots\cdots(49)$$

where

$$K_{ns}=\frac{8\,\mu ns}{\pi G_n}\,\frac{0.49\,n^2s^2+1.7\,\mu^2}{(n^2+s^2)^2-\mu^2}, \qquad n, s=1, 3, 5, \cdots\cdots, \quad \cdots\cdots(50)$$

$$G=\sqrt{n^2+\mu}\,(0.7\,n^2-\mu)^2\,\mathrm{Tgh}\frac{\pi}{2}\sqrt{n^2+\mu} - \sqrt{n^2-\mu}\,(0.7\,n^2-\mu)\,\mathrm{Tgh}\frac{\pi}{2}\sqrt{n^2-\mu}\,. \quad \cdots\cdots(51)$$

Applying Eq. (45) and (46) to (49), and utilizing the Lagrangian Interpolation method, we get

$$\mu=0.720\,494 \quad \cdots\cdots(52)$$

$$\mu=0.720\,515 \quad \cdots\cdots(53)$$

which are more accurate than the exactly estimated values:

$$\mu=0.720\,503 \quad \text{for} \quad \Delta_3{}^{7)} \quad \cdots\cdots(54)$$

$$\mu=0.720\,555 \quad \text{for} \quad \Delta_2{}^{7)} \quad \cdots\cdots(55)$$

respectively.

Ex. 2. To solve the eigenvalue equation, obtained by W. Ritz in the vibration problem of the rectangular plate:[9]

$$\begin{vmatrix} 13.95-\lambda & -32.08 & +18.60 & +32.08 & -37.20 & +18.6 \\ -16.04 & 411.8-\lambda & 120.0 & -133.6 & +166.8 & +140 \\ +18.60 & -240.0 & 1686-\lambda & -218.0 & -1134 & +330 \\ +16.04 & -133.6 & +109.0 & 2945-\lambda & -424 & +179 \\ -18.60 & +166.8 & -567 & -424.0 & 6303-\lambda & -1437 \\ +18.60 & +280 & -330 & +358 & -2874 & 13674-\lambda \end{vmatrix} =0 \quad \cdots\cdots(56)$$

Applying (29), (30), (35) and (36) to (56), we get

$$\lambda_1 = 12.493 \qquad \cdots\cdots (57)$$

$$\lambda_1 \doteq 12.45 \qquad \cdots\cdots (58)$$

and

$$\lambda_2 = 379.2 \qquad \cdots\cdots (59)$$

$$\lambda_2 = 382.8 \qquad \cdots\cdots (60)$$

respectively, for which the exact eigenvalues: $\lambda_1 = 12.4712$ and $\lambda_2 = 378.038$ are calculated by the iteration method.[8)]

Ex. 3. To solve the eigenvalue equation:[3)]

$$\begin{vmatrix} \frac{\lambda(1+\beta^2)^2}{\beta^2} & \frac{4}{9} & 0 & 0 & 0 & \frac{8}{45} \\ \frac{4}{9} & \frac{16\lambda(1+\beta^2)^2}{\beta^2} & -\frac{4}{5} & -\frac{4}{5} & \frac{36}{25} & 0 \\ 0 & -\frac{4}{5} & \frac{\lambda(1+9\beta^2)^2}{\beta^2} & 0 & 0 & -\frac{24}{75} \\ 0 & -\frac{4}{5} & 0 & \frac{\lambda(9+\beta^2)^2}{\beta^2} & 0 & \frac{24}{21} \\ 0 & \frac{36}{25} & 0 & 0 & \frac{\lambda(9+9\beta^2)^2}{\beta^2} & -\frac{72}{35} \\ \frac{8}{45} & 0 & -\frac{24}{75} & \frac{24}{21} & -\frac{72}{35} & \frac{\lambda(16+4\beta^2)^2}{\beta^2} \end{vmatrix} = 0 \qquad \cdots (61)$$

Applying (35) to (61), we imediately obtain

$$\frac{16\,\lambda(1+\beta^2)^2}{\beta^2} - \left(\frac{4}{9}\right)^2 \frac{\beta^2}{\lambda(1+\beta^2)^2} - \left(\frac{4}{5}\right)^2 \frac{\beta^2}{\lambda(1+9\,\beta^2)^2}$$
$$- \left(\frac{4}{5}\right)^2 \frac{\beta^2}{\lambda(9+\beta^2)^2} - \left(\frac{36}{25}\right)^2 \frac{\beta^2}{\lambda(9+9\,\beta^2)^2} = 0$$

from which gives

$$\lambda^2 = \frac{\beta^4}{81(1+\beta^2)^4}\left[1 + \frac{81}{625} + \frac{81}{25}\left(\frac{1+\beta^2}{1+9\beta^2}\right)^2 + \frac{81}{25}\left(\frac{1+\beta^2}{9+\beta^2}\right)^2\right] \qquad \cdots\cdots (62)$$

that is, the very equation (i), given in bibliography (3).

Ex. 4. To solve the eigenvalue equation:[10)]

$$\begin{vmatrix} 892 - 72\,\xi & -1\,133 & 243 \\ 1\,416 & -18\,382 + 360\,\xi & 13\,697 \\ 135 & -609 & 41\,584 - 360\,\xi \end{vmatrix} = 0 \qquad \cdots\cdots (63)$$

Referring to (29) and (31), we obtain from (63)

$$\xi_1 = 10.88 \qquad \cdots\cdots (64)$$

$$\xi_1 = 10.85 \qquad \cdots\cdots (65)$$

respectively by the iteration method[8)], for which the exact eigenvalue: $\xi_1 = 10.872$ is determined on checking.

Conclusion

It is satisfactorily verified that, for all the sufficient accuracies, the solution of eigenvalue equations in Hilbert sqace can rationally be facilitated, and furthermore, much labour and time can be saved. Originally, it comes out from the successful realization of the idea, that the process of deriving the triangular matrix from the given matrix or the process of lowering the order of the given determinant in Hilbert space must successively be applied to from its last column or row up to its principal column and row and then from its first column or row up to them, though it is usually not the case in text-books of mathematics.

The same formulae will also be applied to the approximate estimations of the determinants with the converging principal column and row.

Bibliography

(1) G. Kowalewski, Einfuerung in die Determinantentheorie, Berlin, 1909.

(2) R. Courant and D. Hibert, Die Grundlagen der Methode der mathematischen Physik, Berlin, Bd. I, 1931.

(3) S. Timoshenko, Theory of Elastic Stability, New York, 1936, p. 395, Eq. (g).

(4) M. Fujiwara, Matrices and Determinants, Tokyo, 1937.

(5) R.A. Frazer, W.J. Duncan and A.R. Collar, Elementary Matrices, Cambridge at the University Press, 1938.

(6) S. Iguchi, Die Eigenwertprobleme fuer die elastische rechteckige Platte, the Memoirs of the Faculty of Engineering, Hokkaido University, Jaqan, Vol. 4, No. 4, Aug., 1938.

(7) S. Iguchi, Die Eigenschwingungen und Klangfiguren der vierseitig freien rechteckigen Platte, the Memoirs of the Faculty of Engineering, Hokkaido University. Japan, Vol. 6, No. 5, April, 1942.

(8) H. Margenau and G.M. Murphy, the Mathematics of Physics and Chemistry, Yale University Press, 1948.

(9) K.Hidaka, Methods of Numerical Integration, Tokyo, 1948, Vol. 2, p. 210, Eq. (13.12).

(10) T. Yoshimachi, Theories and Calculations of Steel Bridges, Sapporo, Japan, 1952, p. 172, Ex.

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