ON THE BUCKLING STRENGTH OF AN IMPERFECT ELASTIC COLUMN

(Trans. of JSCE, No. 15, Dec. 1952)

*By Dr. Eng., Tomoyasu Yûki, C.E. Member**

I. Introduction

A centrally-loaded straight short-column has generally a buckling stress σ_K greater than the "true elastic limit" σ_E of the material of the column. In this paper, the writer calls such a "short-column" "An Imperfect Elastic Column", and he is to make a proposal based upon a new idea something different from the wellknown "Engesser-Kármán" formula, and the writer's numerous experimental results using several different materials show the favorable supports to his proposal.

Acknowledgement. The writer wishes to express his cordial gratitude to Yutaka TANAKA, Dr. Eng., Honorary Prof. of Tokyô Univ., Member of Japan Academy, for his kind guidance, and to Foundation SAITO HO-ON KAI and to Education Department for his research expense.

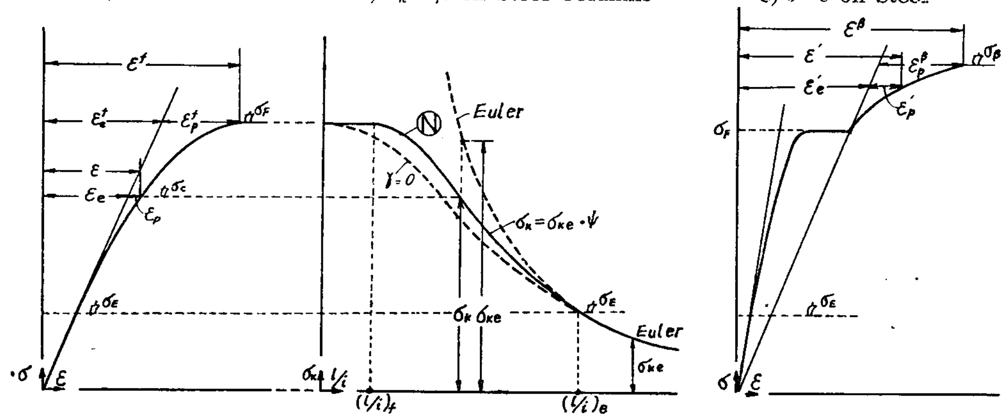
II. New Solution

(1) Fundamental Assumption.

When a straight "short-column" is centrally loaded, and the mean stress (σ= load/gross area) in a cross-section of the column becomes greater than the true elastic limit of the material through the entire length, some elastic constitution of the cross-section turns partially and gradually into the plastic constitution. To the writer's mind, the ratio of the area (A_e) of the remaining elastic parts of a cross-section to its gross area (A_g) may be of a certain value corresponding to the buckling stress. Now, referring to **Fig. 1**, and putting σ_c=any compressive stress, ε=total strain due to σ_c, ε_e=elastic strain due to σ_c, ε_p=plastic strain due to σ_c, σ_E=elastic limit, σ_F=yield point, σ_β=compressive strength, we get the following general equation,

Fig. 1.

a) $\sigma\sim\varepsilon$ on Steel b) $\sigma_K\sim l/i$ on Steel Columns c) $\sigma\sim\varepsilon$ on Steel



* Professor of Shinshu Univ., Civil Eng. Depart.; Faculty of Eng., 500 Wakazato, Nagano City, Honshu, Japan.

$$\mathcal{E}=\mathcal{E}_e+\mathcal{E}_p{}^{1)} \quad \cdots\cdots(1\,a)$$

and $$\mathcal{E}^f=\mathcal{E}_e{}^f+\mathcal{E}_p{}^f \quad \cdots\cdots(1\,b)$$

where $\mathcal{E}^f$, $\mathcal{E}_p{}^f$ and $\mathcal{E}_e{}^f$ denote compressive total strain, plastic strain and elastic strain at σ_F, respectively.

For concrete, wood, cast iron etc., we may write it better as the following equation,

$$\mathcal{E}^\beta=\mathcal{E}_e{}^\beta+\mathcal{E}_p{}^\beta \quad \cdots\cdots(1\,c)$$

where $\mathcal{E}^\beta$, $\mathcal{E}_p{}^\beta$ and $\mathcal{E}_e{}^\beta$ denote compressive total strain, plastic strain and elastic strain at σ_β, respectively.

In case of $\sigma_K>\sigma_F$ for example for steel, we may write it as the following equation (1 d) shown in **Fig. 1** c) for trial.

$$\mathcal{E}'=\mathcal{E}_e'+\mathcal{E}_p' \quad \cdots\cdots(1\,d)$$

where $\mathcal{E}'$=compressive total strain due to σ_C above σ_F.

$\mathcal{E}_e'$ and $\mathcal{E}_p'$=strain indicated in **Fig. 1** c), due to σ_C above σ_F, respectivery.

Next, putting p as the ratio of $\mathcal{E}_p$ at any compressive stress intensity σ_c to $\mathcal{E}_p{}^f$, we may write generally the formula as

$$p=\mathcal{E}_p/\mathcal{E}_p{}^f \quad \text{or} \quad \mathcal{E}_p/\mathcal{E}_p{}^\beta,\ \mathcal{E}_p'/\mathcal{E}_p{}^\beta. \quad \cdots\cdots(2)$$

Then, on the other hand, the ratio above mentioned of A_e to A_g is to be assumed as a function of p at the buckling stress σ_K, and the writer proposes the following Eq. (3) for trial.

$$A_e/A_g=f(p)=(1-p^\lambda)^\mu \quad \cdots\cdots(3)$$

where λ, μ=experimental exponents.

(2) New Solution

The centrally loaded straight "short-column" remains straight before the buckling, and when the mean stress ($\sigma=P/A_g$) on the cross-section overcomes the true elastic limit σ_E, the elastic parts and the plastic parts may coexist, and for the buckling load, we may merely put as follows:

$$P_K=A_g\cdot\sigma_K=A_e\cdot\sigma_{Ke}+A_p\cdot\sigma_{Kp} \quad \cdots\cdots(4)$$

where P_K=buckling load of a "short-column", σ_{Ke}=elastic stress at buckling, σ_{Kp}=plastic stress at buckling, σ_K=buckling stress, A_g=gross area of cross-section of a column, A_e=sectional area to elastic stress, A_p=sectional area to plastic stress.

Value of σ_{Ke} may be considered to be given from the Euler formula as follows:

$$\sigma_{Ke}=\pi^2E/(l/i)^2 \quad \cdots\cdots(5)$$

where l=length of the column, i=least radius of gyration of the cross-section of the column, measured parallel to the plane of bending, E=modulus of elasticity of the material of the column.

If the value of σ_{Kp} is here considered to be in proportion to σ_{Ke} and $(\mathcal{E}_e/\mathcal{E})^n$, we get the following formula,

$$\sigma_{Kp}=\gamma\cdot\sigma_{Ke}\cdot(\mathcal{E}_e/\mathcal{E})^n. \quad \cdots\cdots(6)$$

Coefficient $\gamma\leqq1$ and constant $n=1$ seem to be adapted in the equation above mentioned.

Setting the values given by Eq.s (3), (5) and (6) in Eq. (4), the formula for P_K becomes,

$$P_K = A_g \cdot f(p) \cdot \sigma_{Ke} + A_g\{1-f(p)\} \cdot \sigma_{Ke}(\varepsilon_e/\varepsilon)\gamma.$$

Then, the formula for σ_K takes the following equation (7).

$$\sigma_K = P_K/A_g = \sigma_{Ke} \cdot \Psi \quad \cdots\cdots (7)$$

where, for abbreviation,

$$\sigma_{Ke} = \pi^2 E/(l/i)^2,$$

$$\Psi = f(p) + \{1-f(p)\}k; \qquad k = \gamma \cdot \frac{\varepsilon_e}{\varepsilon_e + \varepsilon_p},$$

$$f(p) = (1-p^{\lambda})^{\mu}; \quad p = \frac{\varepsilon_p}{\varepsilon_p{}^f}; \quad k^f = \gamma \cdot \frac{\varepsilon_e{}^f}{\varepsilon_e{}^f + \varepsilon_p{}^f},$$

λ, μ, γ = experimental constants,

$$l/i \geqq \left[(l/i)_f = \pi \sqrt{\frac{Ek^f}{\sigma_F}} \right].$$

In case of $l/i \leqq (l/i)_f$, the writer dares, for the convenience' sake, to place σ_K on the border of σ_F.

(i) When we use materials such as wood, mortar, concrete, cast iron etc., we may as well use σ_β instead of σ_F, and (ii) the strength of the "very short column" may be given by this σ_F or σ_β; in case of $\sigma_K = \sigma_F$, p becomes 1 and so $f(p)=0$, hence, Ψ becomes k^f, in this case, l/i is given by $(l/i)_f$ above mentioned. (iii) in case of a perfect elastic column, however, p becomes 0, and so $f(p)=1$, consequently $\Psi=1$, hence we get $\sigma_K = \sigma_{Ke}$, it coincides with the Euler formula.

Eq. (7) is a new one, presented by the writer, which gives the buckling stress σ_K. Now we wish to make some explanations about the present equation. First, we make clear the stress-strain diagram ($\sigma \sim \varepsilon$) on the material of the column and calculate the value of p at the given buckling stress σ_K, and then calculate the value of Ψ. Then, we get the value of l/i in response to the given buckling stress σ_K, using the value of Ψ to Eq. (7). In short, we may represent the relation $\sigma_K \sim l/i$ of a "short-column" by means of retouchment of the Euler formula, using the value of Ψ corresponded to the given σ_K from the diagram $\sigma \sim \varepsilon$. Curve $\sigma_K \sim l/i$ given by Eq. (7) loses its continuity with the straight line $\sigma_K = \sigma_F$ at the point of $(l/i)_f$, but in practice we consider that the curve is tangent to the straight line. In special case $\gamma=0$, the curve becomes as shown by the broken line in **Fig. 1** and this curve seems particularly very useful in practice. And further, if it is desired to find the σ_K of a reinforced concrete column, the following equation (8) may be adopted.

$$\sigma_K = \sigma_K{}^c(1-\mu') + \mu' \sigma_{Ke}{}^s \quad \cdots\cdots (8)$$

where $\sigma_K{}^c = \sigma_{Ke}{}^c \cdot \Psi$ = buckling stress of concrete column,

$\sigma_{Ke}{}^s = \pi^2 E_s/(l/i)^2$ = buckling stress of steel column,

$\mu' = A_s/A_c$ = ratio of steel area to concrete area of the cross-section of the column,

E_s = modulus of elasticity of steel.

The values of experimental constants λ, μ and γ in the present formula, as the results of our experimental relations $\sigma \sim \varepsilon$ and the strengths of column-test-pieces tested by the writer, are as shown in the following table. Therefore, it is concluded that the value of $f(p)$ is to be $(1-\sqrt{p})^2$ in case of metal columns and $(1-p)^2$ in

TABLE—Values of Experimental Constants λ, μ, γ

No.	Material of Column-test-pieces	Size of Cross-Section of Test-pieces (mm)	Number of Test-pieces	λ	μ	γ
1	Structural Steel (ST. 39)	4×5	66	1/2	2	0.3
2	Carbon Steel	2×5	53	1/2	2	0.3
3	Duralumin	3φ	50	1/2	2	0.3
4	Aluminum	3φ	53	1/2	2	0.3
5	Cast Iron (E.K. 50)[2]	40×60	10	1/2	2	0.3
6	Lead (Merck)	3φ and 5φ	20 and 20	1/2	2	1
7	Japan Cypress	5×7 and 20×20	96 and 34	1	2	0.6
8	Mortar	8×12	43	1	2	0.6
9	Reinforced[3] Concrete	100×100	19	1	2	0.6

case of columns of wood, mortar and concrete. As to the value of γ, we may use in case of metals $\gamma=0.3$, and in case of wood, mortar and concrete $\gamma=0.6$. Exception: In case of lead, $\gamma=1$.

Having calculated coefficients Ψ with these experimental constants, curves $\sigma_K \sim l/i$ may be drawn as shown in **Fig. 2–9.** Marks (●) in the figures show the writer's experimental values tested by the compression testing apparatus specially designed by him as shown in **Fig. 10.** Curves Ⓝ represent Eq. (7), and curves ◌ show the cases $\gamma=0$. Curves Ⓔ represent the Engesser-Kármán formula.[1] Curves ①, ②, ③ in **Fig. 8** represent $\gamma=0.6$, 0.2, 0 respectively. The proportion of the sand mortar mixture shown in **Fig. 8** is c/s=1/2 and $W/(c+s)=9\%$ by weight. The age at his tests is 21 months.

Fig. 2. $\sigma \sim \varepsilon$ Diagrams on St. 39 and $\sigma_K \sim l/i$ on St. 39 Columns

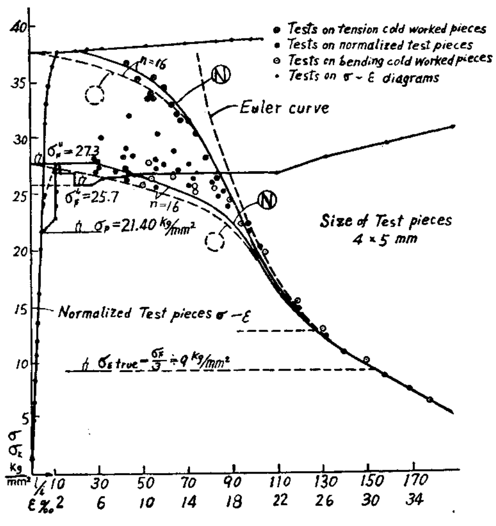


Fig. 3. Tests on Cast Iron Columns

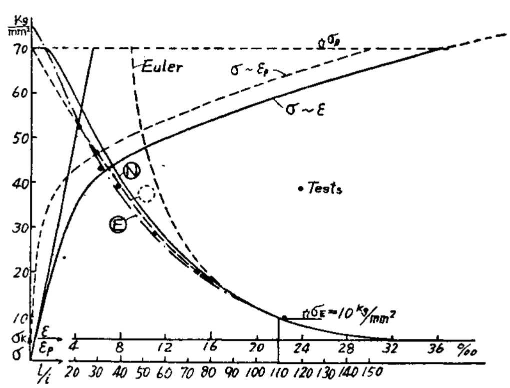


Fig. 4. Tests on Aluminium Columns

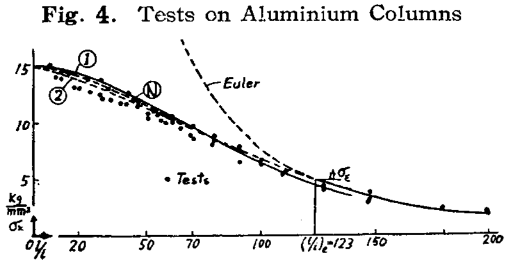


Fig. 5 a. $\sigma \sim \varepsilon$ on Lead

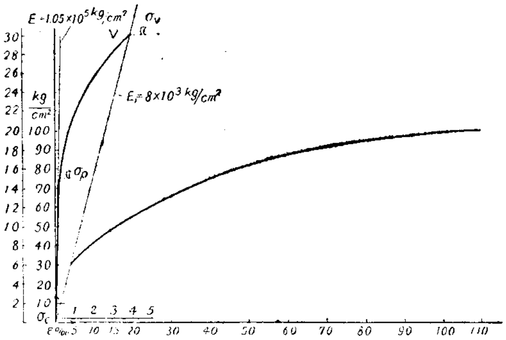


Fig. 5 b. Tests on Lead Columns

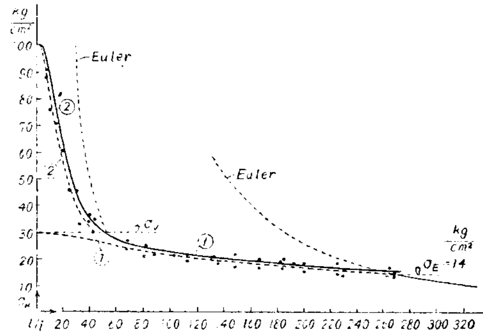


Fig. 6. Tests on Duralumin Columns

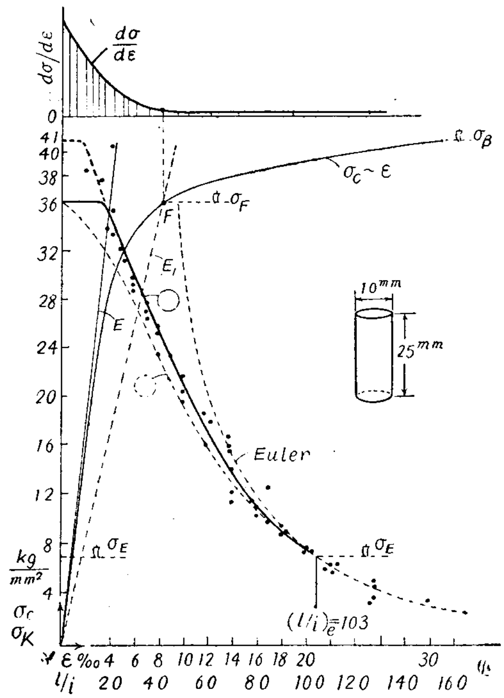


Fig. 7. Tests on Japan Cypress Columns

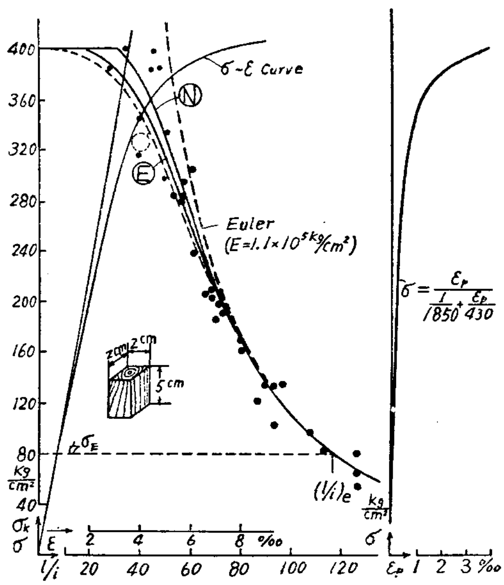


Fig. 8. Tests on Mortar Columns

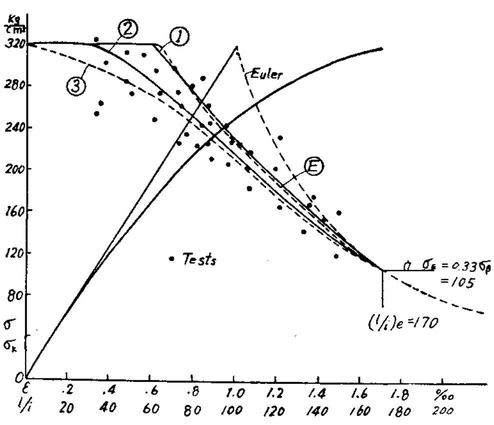


Fig. 9 a. Tests on Mortar & Reinforced Concrete Columns by O. Baumann

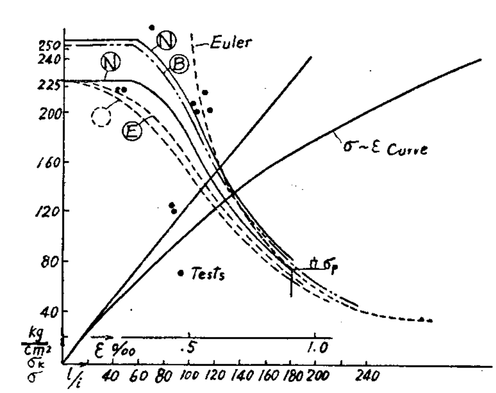


Fig. 11～13 show the views of the column-test-pieces after they are tested.

Fig. 9 b. Tests on Mortar & Reinforced Concrete Columns by O. Baumann

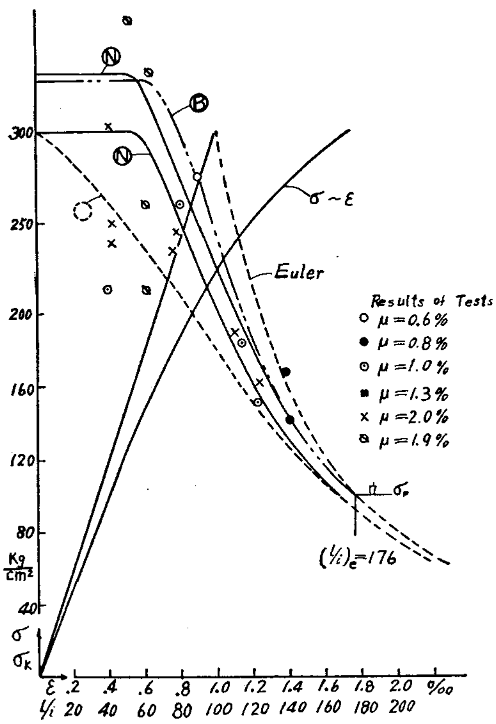


Fig. 10. Compression Testing Apparatus

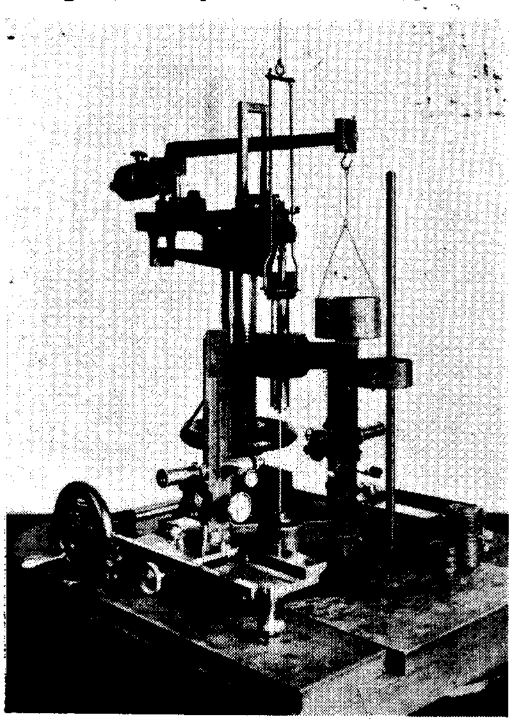


Fig. 11 Views of Stiuctual Steel Columns (4×5 mm) after Test.

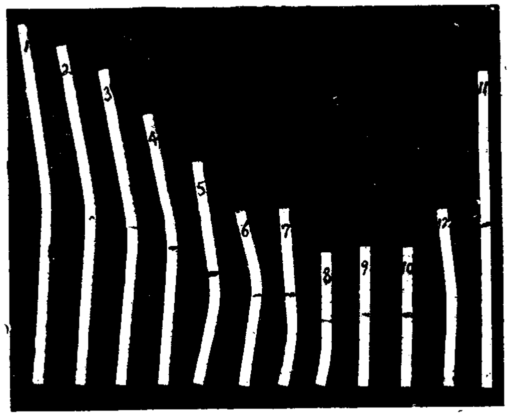


Fig. 12. Views of Aluminum Columns (3 mm φ) after Test.

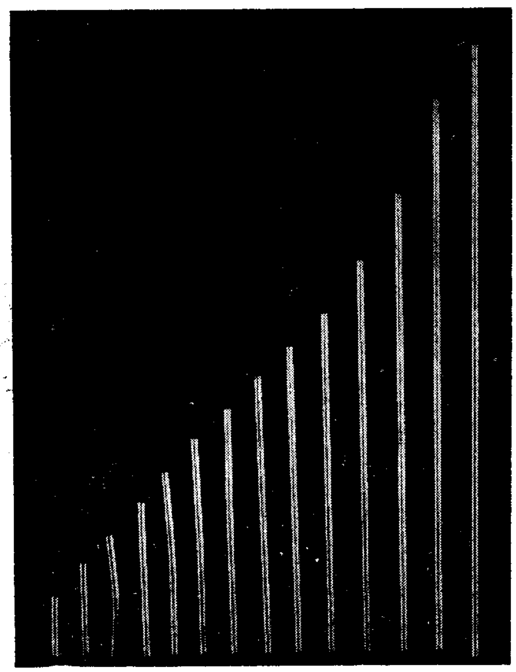
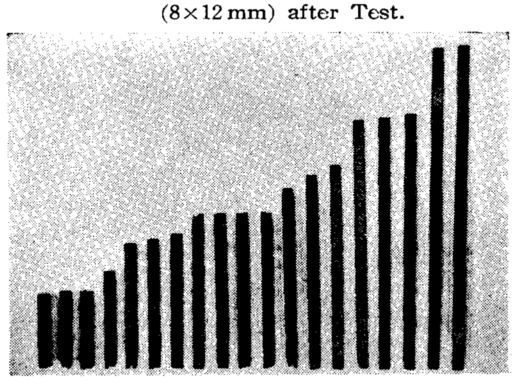


Fig. 13. Views of Mortar Columns (8×12 mm) after Test.



III. Summary of Tests

A brief summary of the more important results of the tests given below:

(1) Relation $\sigma \sim \varepsilon_p$ is generally measured using a simple compression test piece which has its height less than 2.5 times its diameter. This relation may be approximately represented by a hyperbolic equation of 1 or 2 degrees like σ or $\sigma^2=\varepsilon_p/(a+b\varepsilon_p)$, wherein a,b=exp. constants. Exception: For steel, the result of $\sigma_t \sim \varepsilon_p$ test will be in practice used as that of $\sigma \sim \varepsilon_p$ test, and its true elastic limit σ_E seems approximate to $\sigma_F/3$. Its curve $\sigma \sim \varepsilon_p$ may be treated as a parabola of the high degree with its origin at the true elastic limit.

(2) Curves $\sigma_K \sim l/i$ given by Eq. (7) using the experimental constants shown in the above Table may appreciate an exactness of Eq. (7) by comparising of the results of his tests on columns as shown in the figures.

References

1) Bericht über die II Internationale Tagung für Brückenbau und Hochbau Wien. S. 282 & 222. (1928)

2) M. Roš: International Association for Bridge and Structual Engineering. 1st Congress, Paris. (1932)

3) Oskar Baumann: Die Knickung der Eisenbeton-Säulen Eidg. M.A.d. E.T.H. Zürich, Bericht Nr. 89. (1934)

4) T. YÛKI: Sendai—Saito Ho-on Kai, Monographs No. 23. (1944) (Japanese)

昭和27年12月1日 印刷 土木学会論文集
昭和27年12月15日 発行 第 15 号

定価 60円

著者 結城朝恭
編集兼発行者 東京都千代田区大手町2丁目4番地 中川一美
印刷者 東京都港区赤坂溜池5番地 大沼正吉
印刷所 東京都港区赤坂溜池5番地 株式会社 技報堂

東京中央局区内 千代田区大手町2丁目4番地 電話 和田倉(20)3945番
発行所 社団法人 土木学会 振替東京16828番