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偏微分方程式の階差解法 I

DIFFERENCE METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS.

Part I.

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ALL the derivations in this article are due to the exclusive use of the modified Bessel interpolation formula for the function $\phi(x,y)$. Similar calculation may be performed by interpolation formulas of other types, including mixed ones. In this regard it is noted that Collatz-Hidaka's method[2)] of differentiation affords the same results as those derived from the interpolation formula of Stirling type.

Three-dimensional problems may also be treated. The work would require a considerable amount of efforts, even when systematic procedure of calculation is adopted.

1. Interpolation formula. The interpolation formula of modified Bessel type for the function $\phi(x, y)$ is written in the form

$$\begin{aligned}\phi(x,y)=&\sum_{r+s=0}^{n}\sum_{s=0}^{r+s}\rho(u,r)\rho(v,s)\frac{1}{4}\Delta^{2r+2s}_{2rx\;2sy}\{(\overline{r}\,\overline{s})+(\overline{r}\,\overline{s-1})+(\overline{r-1}\,\overline{s})+(\overline{r-1}\,\overline{s-1})\}\\&+\sigma(u,r)\rho(v,s)\frac{1}{2}\Delta^{2r+2s+1}_{(2r+1)x\;2sy}\{(\overline{r}\,\overline{s})+(\overline{r}\,\overline{s-1})\}\\&+\rho(u,r)\sigma(v,s)\frac{1}{2}\Delta^{2r+2s+1}_{2rx\;(2s+1)y}\{(\overline{r}\,\overline{s})+(\overline{r-1}\,\overline{s})\}\\&+\sigma(u,r)\sigma(v,s)\Delta^{2r+2s+2}_{(2r+1)x\;(2s+1)y}(\overline{r}\,\overline{s}),\quad\cdots\cdots(1)\end{aligned}$$

where $u=\dfrac{x-x_0}{h}-\dfrac{1}{2},\qquad v=\dfrac{y-y_0}{k}-\dfrac{1}{2},$

$$\rho(\theta,\nu)=\frac{1}{(2\nu)!}\left(\theta^2-\frac{1}{4}\right)\left(\theta^2-\frac{9}{4}\right)\cdots\left(\theta^2-\frac{\overline{2\nu-1}^2}{4}\right),$$

$$\sigma(\theta,\nu)=\frac{1}{(2\nu+1)!}\,\theta\left(\theta^2-\frac{1}{4}\right)\left(\theta^2-\frac{9}{4}\right)\cdots\left(\theta^2-\frac{\overline{2\nu-1}^2}{4}\right),$$

$$(\overline{r}\,\overline{s})=\phi(x_0-rh,\quad y_0-sk),$$

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$$\Delta^{a+b}_{axby}(\overline{r\ s}) = \sum_{\lambda=0}^{a}\sum_{\mu=0}^{b}\frac{(-)^{\lambda+\mu}\,\underline{|a}\,\underline{|b}}{\underline{|a-\lambda}\,\underline{|b-\mu}\,\underline{|\lambda}\,\underline{|\mu}}(\overline{r-a+\lambda\ \ s-b+\mu}).$$

Interpolation formulas of other types, i. e. those of Newton type, Stirling type, Everett type, etc. can be obtained by repeated application of known interpolation formulas for the function of one variable. Furthermore we can obtain interpolation formulas for the function $\phi(x_1,x_2,\cdots x_k)$ by the similar procedure.

2. Differential equation. As an example, we take the harmonic expression in two dimensions. From the interpolation formula (1), we have, after some amount of calculation,

$$\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)_{\substack{u=0\\v=0}} = \frac{1}{8}\sum_{r+s=0}^{n}\sum_{s=0}^{r+s}(-)^{r+s}\frac{\underline{|2r+2}\,\underline{|2s}}{(2^{2r+2s}\,\underline{|r+1}\,\underline{|s})^2}\left\{1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots+\frac{1}{(2r+1)^2}\right\}\times\left[\frac{1}{h^2}\Delta^{2r+2s+2}_{(2r+2)x2sy}\{(\overline{r+1\ s})+(\overline{r+1\ \ s-1})+(\overline{r\ s})+(\overline{r\ s-1})\}+\frac{1}{k^2}\Delta^{2r+2s+2}_{2sx(2r+2)y}\{(\overline{s\ r+1})+(\overline{s-1\ r+1})+(\overline{s\ r})+(\overline{s-1\ r})\}\right]\cdots\cdots\cdots(2)$$

First approximation ($n=0$)

	1	1	
1	-2	-2	1
1	-2	-2	1
	1	1	

$4h^2\nabla^2\phi+0(\Delta^4)$

Second approximation ($n=1$)

		-5	-5		
	-6	45	45	-6	
-5	45	-74	-74	45	-5
-5	45	-74	-74	45	-5
	-6	45	45	-6	
		-5	-5		

$96\,h^2\nabla^2\phi+0(\Delta^6)$

Third approximation ($n=2$)

				259			
				- 2,780	285		
				14,556	- 3,150	285	
				-21,490	14,556	-2,780	259

$23\,040\,h^2\nabla^2\phi+0(\Delta^8)$

Fourth approximation ($n=3$)

					- 3,229				
					40,495	- 3,388			
					-243,500	42,350	- 3,150		
					971,880	-265,734	42,350	- 3,388	
					-1,340,332	971,880	-243,500	40,495	- 3,229

$1\,290\,240\,h^2\nabla^2\phi+0(\Delta^{10})$

An approximation to $\nabla^2\phi$

				- 13	5				
				81	-45				
				44	180				
				-2,940	-420				
- 5	45	-180	420	29,988	1,260	- 420	180	- 45	5
- 23	395	-3,384	31,164	-111,524	29,988	-2,940	44	81	-13
				31,164	420				
				-3,384	- 180				
				395	45				
				- 23	- 5				

$=20\,160\,h^2\nabla^2\phi+0(\Delta^{10})$

In what follows we consider, for convenience' sake, the case when $h=k$. For successive values of n, the harmonic expression (2) becomes as above (four approximations in which $n=0,1,2,3$ respectively).
In the last two schemes, figures are given only in the first quadrant, since others are written down by symmetry.

It is added that if we refer to the interpolation formula of normal Bessel type, then an approximation to the harmonic expression becomes as above (the last scheme). Here the rectangle ▭ shows that the difference equation is to be considered at this lattice-point.

It is to be noted here that difference equation of higher accuracy will naturally be needed for problems having more complicated boundaries, in which case independent inner lattice-points, the unknowns, reduce in number owing to the combination of boundary criteria, so that reduction of the degree of simultaneous equations to be solved will result.

3. Boundary condition. Only straight bounding line parallel to a coordinate axis, y-axis say, will be treated. For more complicated boundary conditions, such as inclined straight boundaries and curved boundaries, it would seem not so easy to formulate general criteria such as those given below.

i) The case when

$$(\phi)_{u=0}=c_0+c_1y. \qquad (c_0,\ c_1 \text{ being constants.}). \cdots\cdots\cdots\cdots(3)$$

From the interpolation formula (1) we have

$$(\phi)_{u=0}=\frac{1}{4}\Delta_{00}+\frac{v}{\underline{|1}}\frac{1}{2}\Delta_{01}+\frac{-\frac{1}{4}}{\underline{|2}}\frac{1}{4}\Delta_{20}+\frac{\left(v^2-\frac{1}{4}\right)}{\underline{|2}}\frac{1}{4}\Delta_{02}$$

$$+\frac{-\frac{1}{4}}{\underline{|2}}\frac{v}{\underline{|1}}\frac{1}{2}\Delta_{21}+\cdots\cdots.$$

To satisfy the boundary condition (3), we may put

$$\frac{1}{4}\Delta_{00}+\frac{v}{\underline{|1}}\frac{1}{2}\Delta_{01}=c_0+c_1y,\ \Delta_{20}=0,\ \Delta_{02}=0,\ \Delta_{21}=0,\ \Delta_{03}=0, \cdots\cdots\cdots(4)$$

This system of equations will afford the required criterion illustrated thus :

Boundary condition $\phi=c_0+c_1y$ for straight bounding line.

Inner lattice-points　　Outer lattice-points

$(\bar{2}3)$	$(\bar{1}3)$	(03)	$-(03)+2\{C_0+C_1(3k+y_0)\}$	$-(\bar{1}3)+2\{C_0+C_1(3k+y_0)\}$	$-(\bar{2}3)+2\{C_0+C_1(3k+y_0)\}$
$(\bar{2}2)$	$(\bar{1}2)$	(02)	$-(02)+2\{C_0+C_1(2k+y_0)\}$	$-(\bar{1}2)+2\{C_0+C_1(2k+y_0)\}$	$-(\bar{2}2)+2\{C_0+C_1(2k+y_0)\}$
$(\bar{2}1)$	$(\bar{1}1)$	(01)	$-(01)+2\{C_0+C_1(k+y_0)\}$	$-(\bar{1}1)+2\{C_0+C_1(k+y_0)\}$	$-(\bar{2}1)+2\{C_0+C_1(k+y_0)\}$
$(\bar{2}0)$	$(\bar{1}0)$	(00)	$-(00)+2\{C_0+C_1y_0\}$	$-(\bar{1}0)+2\{C_0+C_1y_0\}$	$-(\bar{2}0)+2\{C_0+C_1y_0\}$
$(\bar{2}\bar{1})$	$(\bar{1}\bar{1})$	$(0\bar{1})$	$-(0\bar{1})+2\{C_0+C_1(k+y_0)\}$	$-(\bar{1}\bar{1})+2\{C_0+C_1(-k+y_0)\}$	$-(\bar{2}\bar{1})+2\{C_0+C_1(k+y_0)\}$
$(\bar{2}\bar{2})$	$(\bar{1}\bar{2})$	$(0\bar{2})$	$-(0\bar{2})+2\{C_0+C_1(2k+y_0)\}$	$-(\bar{1}\bar{2})+2\{C_0+C_1(-2k+y_0)\}$	$-(\bar{2}\bar{2})+2\{C_0+C_1(-2k+y_0)\}$

↑
Straight bounding line parallel to y-axis

General expression of the criterion obtained above will be

$$(rs)=-(\overline{r-1}\,s)+2\{c_0+c_1(sk+y_0)\}, \cdots(5)$$

where $r=1,2,3,\cdots$ and $s=0,\pm1,\pm2,\cdots\cdots$; (rs) denoting outer lattice-points, and $(\overline{r-1}\,s)$ inner lattice-points. It is noted that, when boundary condition is in general of the form

$$(\phi)_{u=0}=\sum_{m=0}^{\infty}c_my^m,$$

the resulting criterion would be expected to be

$$(rs)=-(\overline{r-1}\,s)+2\sum_{m=0}^{\infty}c_m(sk+y_0)^m.$$

ii) The case when

$$\left(\frac{\partial\phi}{\partial x}\right)_{u=0}=c_0+c_1y+c_2y^2+\cdots\cdots\cdots. \qquad \cdots\cdots\cdots\cdots(6)$$

The required criterion will be written in the form

$$(rs)=(\overline{r-1}\,s)+(2r-1)h\{c_0+c_1(sk+y_0)+c_2(sk+y_0)^2+\cdots\}. \cdots\cdots\cdots(7)$$

iii) The case when

$$\left(\frac{\partial^2\phi}{\partial x^2}\right)_{u=0}=c_0+c_1y+c_2y^2+\cdots\cdots\cdots. \qquad \cdots\cdots\cdots\cdots(8)$$

The required criterion will be written in the form

$$(rs)=(1s)+(0s)-(\overline{r-1}\,s)+r(r-1)h^2\{c_0+c_1(sk+y_0)+c_2(sk+y_0)^2+\cdots\}. \cdots(9)$$

iv) The case when

$$\left(\frac{\partial^2\phi}{\partial y^2}\right)_{u=0}=c_0+c_1y+c_2y^2. \qquad \cdots\cdots(10)$$

The required criterion will be written in the form

$$\begin{aligned}(rs)=&-(\overline{r-1}\,s)-(s-1)\{(\overline{r-1}\,0)+(r\,0)\}+s\{(\overline{r-1}\,1)+(r\,1)\}\\&+s(s-1)k^2\left[c_0+c_1\left\{\frac{1}{3}(s+1)k+y_0\right\}+c_2\left\{\left(\frac{1}{3}(s+1)k+y_0\right)^2\right.\right.\\&\left.\left.+\frac{k^2}{18}(s^2-s+1)\right\}\right]. \qquad \cdots\cdots(11)\end{aligned}$$

v) The case when

$$\left(\frac{\partial^2\phi}{\partial x\partial y}\right)_{u=0}=c. \qquad \cdots\cdots(12)$$

The required criterion will be written in the form

$$(rs)=(\overline{r-1}\,s)-(\overline{r-1}\,0)+(r\,0)+(2r-1)schk. \qquad \cdots\cdots(13)$$

vi) As a simple example of combined boundary conditions of the above criteria (11) and (13), we take the case where stress-components are free from traction along a straight bounding line parallel to the axis of y. In this case we have

$$X_x=\frac{\partial^2\phi}{\partial y^2}=0, \quad -X_y=\frac{\partial^2\phi}{\partial x\partial y}=0,$$

ϕ denoting Airy's stress-function. Then from (11) and (13), we obtain the accompanying scheme.

Boundary conditions $\dfrac{\partial^2\phi}{\partial y^2}=0$ and $\dfrac{\partial^2\phi}{\partial x\partial y}=0$.

Inner lattice-points Outer lattice-points

$-2(\bar{2}0)+3(\bar{2}1)$	$-2(\bar{1}0)+3(\bar{1}1)$	$-2(00)+3(01)$	$-3(00)+3(01)+(10)$	$-3(\bar{1}0)+3(\bar{1}1)+(20)$	$-3(\bar{2}0)+3(\bar{2}1)+(30)$
$-(\bar{2}0)+2(\bar{2}1)$	$-(\bar{1}0)+2(\bar{1}1)$	$-(00)+2(01)$	$-2(00)+2(01)+(10)$	$-2(\bar{1}0)+2(\bar{1}1)+(20)$	$-2(\bar{2}0)+2(\bar{2}1)+(30)$
$(\bar{2}1)$	$(\bar{1}1)$	(01)	$-(00)+(01)+(10)$	$-(\bar{1}0)+(\bar{1}1)+(20)$	$-(\bar{2}0)+(\bar{2}1)+(30)$

$(\bar{2}0)$	$(\bar{1}0)$	(00)	(10)	(20)	(30)
$2(\bar{2}0)-(\bar{2}1)$	$2(\bar{1}0)-(\bar{1}1)$	$2(00)-(01)$	$(00)-(01)+(10)$	$(\bar{1}0)-(\bar{1}1)+(20)$	$(\bar{2}0)-(\bar{2}1)+(30)$
$3(\bar{2}0)-2(\bar{2}1)$	$3(\bar{1}0)-2(\bar{1}1)$	$3(00)-2(01)$	$2(00)-2(01)+(10)$	$2(\bar{1}0)-2(\bar{1}1)+(20)$	$2(\bar{2}0)-2(\bar{2}1)+(30)$

↑

Bounding line parallel to y-axis

Similar calculations due to the interpolation formula of Stirling type are now in progress, together with their applications to technological problems of practical importance.

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NOTES :

1) 500, Wakazato, Nagano, Honshu, Japan.

2) K. Hidaka, Numerical Integration, vol. ii, 1936 (in Japanese) ; L. Collatz, Das Differenzenverfahren mit höherer Approximation für lineare Differentialgleichungen, Schriften des mathematischen Seminärs und des Instituts für angewandte Mathematik der Universität Berlin, Bd. 3, Heft 1, Berlin, 1935.

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